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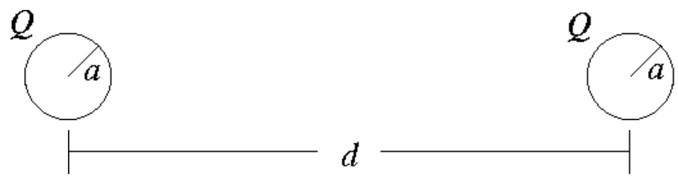
MIT 8.02 Spring 2002
Assignment #1 Solutions

Problem 1.1

Relative strengths of gravitational and electrostatic forces.



The dust grains have diameter 50 μm, and thus have a radius $a = 25\ \mu\text{m} = 2.5 \times 10^{-5}$ m. They have mass density $\rho = 2.5\ \text{gm/cm}^3 = 2.5 \times 10^3\ \text{kg/m}^3$ and charge $Q = -ne$ (e being the magnitude of the electron's charge). The mass of each grain is then

$$m = \frac{4}{3}\pi a^3 \rho \cong 1.6 \times 10^{-10}\ \text{kg} \quad .$$

The gravitational attraction is

$$F_G = \frac{Gm^2}{d^2} \quad .$$

The electrostatic repulsion is

$$F_C = \frac{n^2 e^2}{4\pi\epsilon_0 d^2} \quad .$$

There would be no net force if $F_C = F_G$, i.e.

$$n = \sqrt{4\pi\epsilon_0 G}\frac{m}{e} \cong 0.09 < 1 \quad .$$

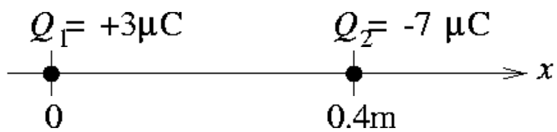
Thus a mere single extra electron on each grain would prevent the grains from colliding.

For comparison, each grain contains $m/m_p = (1.6 \times 10^{-10})/(1.67 \times 10^{-27}) \approx 10^{17}$ nucleons (protons + neutrons). For a neutral grain, we have one electron for each proton, or one for every two nucleons, if # protons = # neutrons. Thus the total number of electrons is $\cong 5 \times 10^{16}$.

Problem 1.2

Electric field along the line passing through two point charges.

(*Note: In these solutions, as in the text, we will always denote vector quantities such as the electric field by boldface font:* **E**. *In lectures, and probably in your own handwritten work, vectors are more commonly written with an arrow over the top:* $\vec{E}$.)



(a) Let us first determine the electric field due to Q_1 alone. From the expression for the electric field due to a point charge,

$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{r^2}\hat{r} \quad .$$

If we are at position x on the x-axis, $\mathbf{r} = x\hat{x}$, $r = \sqrt{x^2} = |x|$, and $\hat{r} = \mathbf{r}/|\mathbf{r}| = (x/|x|)\hat{x}$, and

$$\begin{aligned} \mathbf{E}_1 &= \frac{Q_1}{4\pi\epsilon_0}\frac{1}{|x|^2}\frac{x}{|x|}\hat{x} \\ &= \frac{Q_1}{4\pi\epsilon_0}\frac{x}{|x|^3}\hat{x} \quad . \end{aligned}$$

Note that $\mathbf{E}_1$ is in the $\hat{x}$ direction for $x > 0$ and in the $-\hat{x}$ direction for $x < 0$, as we would expect, since the charge is located at $x = 0$. Similarly,

$$\mathbf{E}_2 = \frac{Q_2}{4\pi\epsilon_0}\frac{x - 0.4}{|x - 0.4|^3}\hat{x}$$

(variables always measured in SI units unless otherwise noted). Note that $\mathbf{E}_2$ changes direction depending on whether we are to the left or right of the charge at $x = 0.4$ m. The total $\mathbf{E}$ is the sum of $\mathbf{E}_1$ and $\mathbf{E}_2$:

$$\mathbf{E} = \frac{10^{-6}}{4\pi\epsilon_0}\left[3\frac{x}{|x|^3} - 7\frac{x - 0.4}{|x - 0.4|^3}\right]\hat{x} \quad .$$

This expression is good for $-\infty < x < \infty$, but it is worthwhile writing it out as follows:

$$\mathbf{E} = \frac{10^{-6}}{4\pi\epsilon_0}\begin{cases} (-3/x^2 + 7/(x - 0.4)^2)\,\hat{x} & -\infty < x < 0 \\ (+3/x^2 + 7/(x - 0.4)^2)\,\hat{x} & 0 < x < 0.4 \\ (+3/x^2 - 7/(x - 0.4)^2)\,\hat{x} & 0.4 < x < \infty \quad . \end{cases}$$

Note the sign changes as one passes over the positions of the two charges and the field due to that charge reverses.

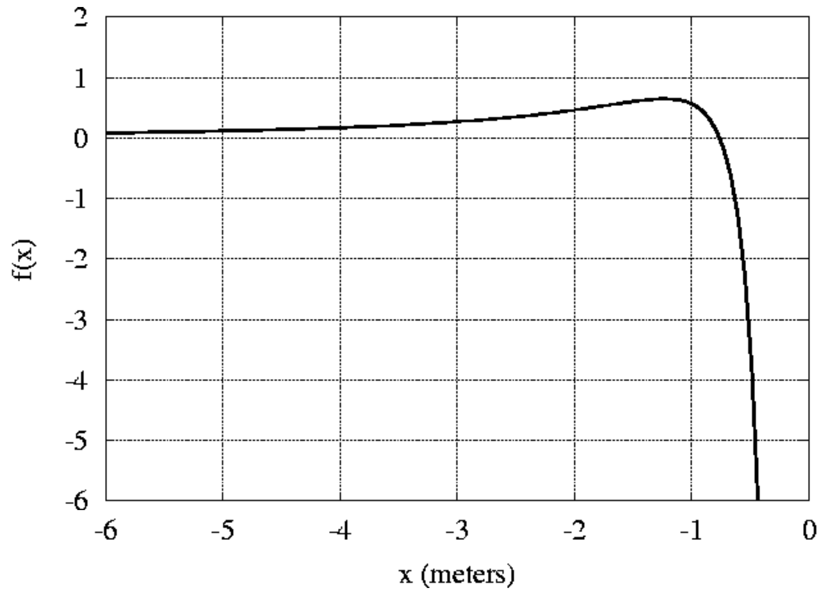
(b) For $x < 0$,

$$\mathbf{E}(x) = \hat{x}\frac{10^{-6}}{4\pi\epsilon_0} f(x) \quad ,$$

where

$$f(x) = -\frac{3}{x^2} + \frac{7}{(x-0.4)^2} \quad .$$

(See graph at right.)

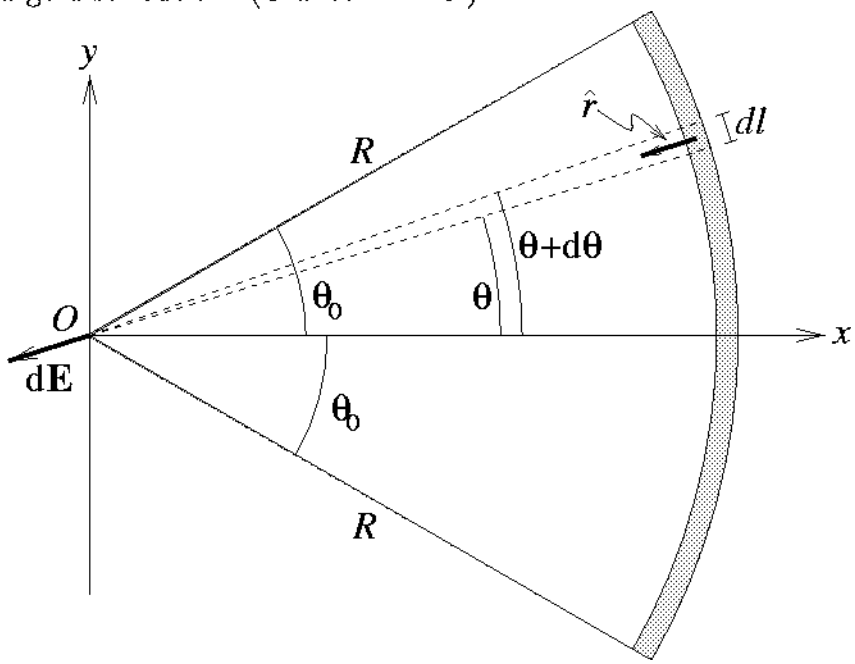


This function has a zero at $x = -0.758$ m, and a maximum at $x = -1.225$ m, with a value there of 0.652 m^{-2}. As we move toward $x = 0$, $f(x)$ blows up to $-\infty$, because of the positive charge at $x = 0$. As we move toward $x = -\infty$, $f(x)$ goes asymptotically to $+4/x^2$, which simply means that far enough away ($|x| \gg 0.4$) the field looks like that due to a point charge of charge $Q_1 + Q_2 = -4$ μC.

(c) There are no other zeros of $\mathbf{E}(x)$ except at $x = -0.758$ m. A glance at our expression for $\mathbf{E}(x)$ above clearly shows that there can be no zeros in the range $0 < x < 0.4$ (sum of two positive terms can never be zero). You might think that we could get a zero in the range $0.4 < x < \infty$, but the negative charge (which is of greater magnitude than the positive charge) is closer throughout this range, and the electric field is therefore always in the $-\hat{x}$ direction. Solving for the zeros of the expression for $\mathbf{E}(x)$ in the $0.4 < x < \infty$ range will give zeros that are outside of that range and therefore unphysical.

Problem 1.3

Continuous charge distribution. (Giancoli 21-49.)



We first find the electric field $d\mathbf{E}$ at point O from a portion of the arc of length dl and located as shown in the diagram. This little element carries charge $dq = \lambda\, dl$. If dl subtends an angle $d\theta$ (see diagram), then we have $dl = R\, d\theta$. Now,

$$d\mathbf{E} = \frac{dq}{4\pi\epsilon_0 R^2}\hat{r} \quad ,$$

where $\hat{r}$ is a unit vector pointing <u>from</u> the little element <u>towards</u> O. If the little piece of arc is located at an angular position θ as shown in the diagram, then $\hat{r} = -\hat{x}\cos\theta - \hat{y}\sin\theta$. Thus,

$$d\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0 R}\left[-\hat{x}\cos\theta\, d\theta - \hat{y}\sin\theta\, d\theta\right] \quad .$$

To get the total $\mathbf{E}$, we integrate $d\mathbf{E}$ from $\theta = -\theta_0$ to $\theta = \theta_0$, noting that

$$\int_{-\theta_0}^{\theta_0} \cos\theta\, d\theta = \sin\theta|_{-\theta_0}^{\theta_0} \quad = 2\sin\theta_0 \quad ;$$

$$\int_{-\theta_0}^{\theta_0} \sin\theta\, d\theta = -\cos\theta|_{-\theta_0}^{\theta_0} \quad = 0 \quad .$$

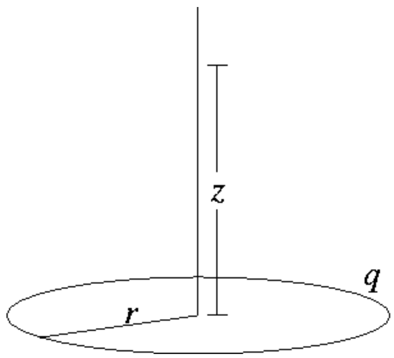
Thus

$$\mathbf{E} = -\hat{x}\frac{\lambda\sin\theta_0}{2\pi\epsilon_0 R} \quad .$$

(The astute problem solver would have concluded from the outset, based on the symmetry of the system, that $\mathbf{E}$ at O could not conceivably have any y-component, and would only have bothered to calculate E_x explicitly.)

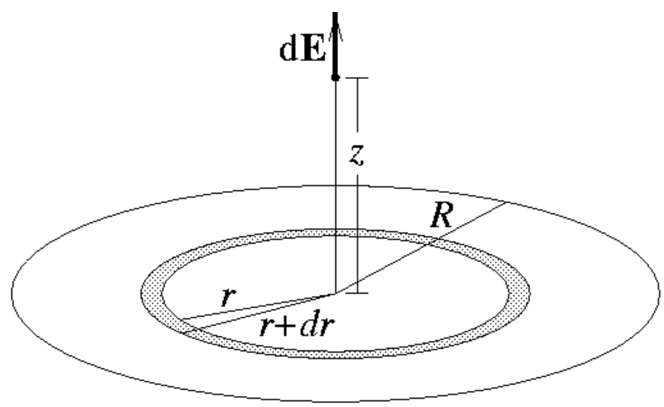
Problem 1.4

E-field of a uniformly charged disk.



This problem requires an expression for the electric field on the axis of a ring of radius r carrying a charge q (see Giancoli Example 21-9):

$$\mathbf{E}_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{qz}{(z^2+r^2)^{3/2}}\hat{z} \quad .$$



The problem at hand is a disk, not a ring, but we may break the disk up into many rings and add up the field due to each, using the above expression for $\mathbf{E}_{\text{ring}}$. The area of the shaded region in the diagram at right is $2\pi r\,dr$, and thus its charge is $\sigma(2\pi r\,dr) = (Q/\pi R^2)(2\pi r\,dr)$. Using our expression above, the electric field $d\mathbf{E}$ due to the shaded area is

$$d\mathbf{E} = \hat{z}\frac{1}{4\pi\epsilon_0}\frac{dq\;z}{(z^2+r^2)^{3/2}} = \hat{z}\frac{Qz}{2\pi\epsilon_0 R^2}\frac{r\,dr}{(z^2+r^2)^{3/2}} \quad .$$

(a)

$$\mathbf{E} = \int d\mathbf{E} = \hat{z}\frac{Qz}{2\pi\epsilon_0 R^2}\int_0^R \frac{r\,dr}{(z^2+r^2)^{3/2}} = \hat{z}\frac{Qz}{2\pi\epsilon_0 R^2}\left[\frac{-1}{(z^2+r^2)^{1/2}}\right]\Bigg|_0^R$$

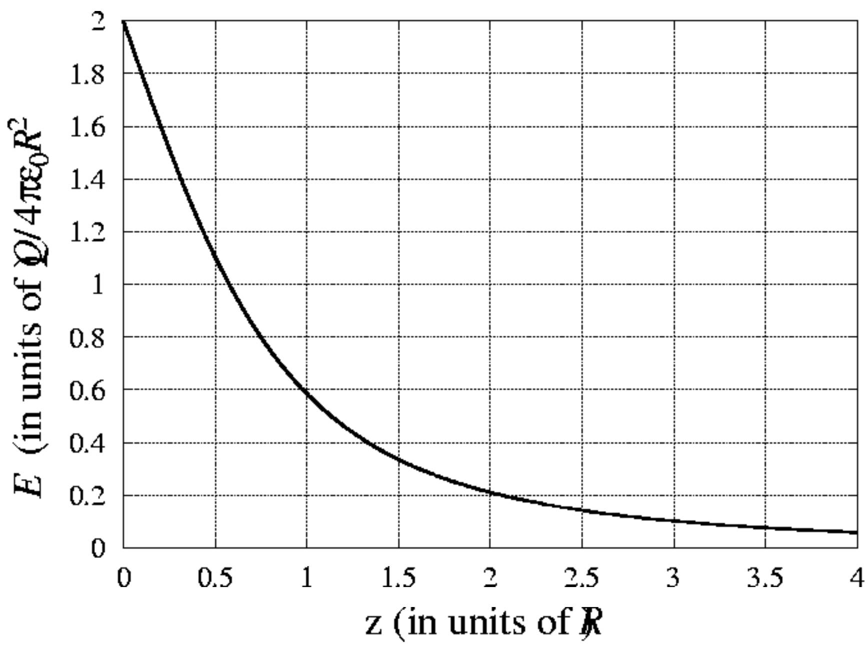
$$\mathbf{E}(z) = \hat{z}\frac{Qz}{2\pi\epsilon_0 R^2}\left[\frac{1}{|z|} - \frac{1}{\sqrt{z^2+R^2}}\right]$$

(The integral above is easily done using the substitution $s = \sqrt{z^2+r^2}$, whereby $r\,dr = s\,ds$. This leaves us with the simple integral $\int s^{-2}\,ds$.)

(b) For $z > 0$, we can write this as

$$\frac{\mathbf{E}(z)}{(Q/4\pi\epsilon_0 R^2)} = \hat{z}(2)\left[1 - \frac{z/R}{\sqrt{1+(z/R)^2}}\right] \quad .$$

We use this expression for the plot on the following page.



(c) We can understand the shape of this curve for small and large z by using a Taylor series expansion. From Giancoli Appendix A-3,

$$f(u) = f(a) + \left.\frac{df}{du}\right|_a (u - a) + \cdots$$

Let's expand $(1 + u)^n$ about $u = 0$ using this formalism. First,

$$\frac{d}{du}(1 + u)^n = n(1 + u)^{n-1} \quad ,$$

so that

$$\begin{aligned} (1 + u)^n &= \left.(1 + u)^n\right|_{u=0} + \left.n(1 + u)^{n-1}\right|_{u=0} (u - 0) + \cdots \\ &= 1 + nu + \cdots \end{aligned}$$

This is the binomial expansion of Giancoli Appendix A-2. It is a good approximation as long as u is $\ll 1$. For case (i), $z^2 \ll R^2$, we have

$$\mathbf{E}(z) = \hat{z}\frac{Q}{2\pi\epsilon_0 R^2}\left[1 - \frac{z/R}{\sqrt{1 + (z/R)^2}}\right]$$

but $(1 + (z/R)^2)^{-1/2} \simeq 1 - (1/2)(z/R)^2$ from above, so

$$\mathbf{E}(z) = \hat{z}\frac{Q}{2\pi\epsilon_0 R^2}\left[1 - (z/R)\left(1 - (1/2)(z/R)^2\right)\right]$$

$$\mathbf{E}(z) = \hat{z}\frac{Q}{2\pi\epsilon_0 R^2}, \quad z^2 \ll R^2 \quad \text{(leading term only.)}$$

Note if we had kept the next term, we would have an initial slope of -2 near $z = 0$, in keeping with our plot above. For case (ii), $z^2 \gg R^2$, we have

$$\mathbf{E}(z) = \hat{z}\frac{Q}{2\pi\epsilon_0 R^2}\left[1 - \frac{1}{\sqrt{(R/z)^2+1}}\right]$$

but $(1+(R/z)^2)^{-1/2} \simeq 1 - (1/2)(R/z)^2$ for $(R/z)^2 \ll 1$, and

$$\mathbf{E}(z) \simeq \hat{z}\frac{Q}{2\pi\epsilon_0 R^2}\left[1 - \left(1 - (1/2)(R/z)^2\right)\right]$$

$$\mathbf{E}(z) \simeq \hat{z}\frac{Q}{4\pi\epsilon_0}\frac{1}{z^2}, \qquad z^2 \gg R^2 \quad .$$

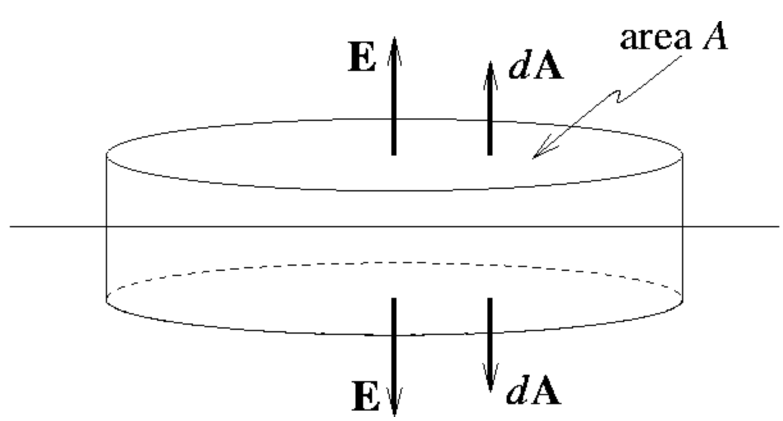
(d) Clearly the above case (ii) expression looks like Coulomb's law for a point charge Q.

(e) If we are very close to the disk ($z \ll R$), it looks like an infinite plane with surface charge density σ. The field due to an infinite plane has the same magnitude above and below the plane, but with opposite directions (see diagram below). Applying Gauss's law to the pillbox shown below, we have

$$\begin{aligned} \oint \mathbf{E}\cdot d\mathbf{A} &= Q_{\text{encl}}/\epsilon_0 \\ EA + EA &= \sigma A/\epsilon_0 \implies E = \sigma/2\epsilon_0 \quad . \end{aligned}$$

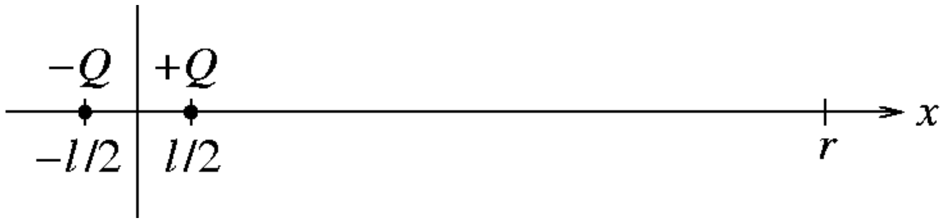
(We grudgingly adopt the convenient but somewhat abstract notation $d\mathbf{A}$, to be consistent with Giancoli. We would prefer to use the more intuitive notation $\hat{n}\, dS$.)

Since $\sigma = Q/\pi R^2$, this expression is exactly as in c(i) above.



Problem 1.5

Electric Dipole. (Giancoli 21-65.)



Let the charges be located on the x-axis, with the positive charge $(+Q)$ at $x = l/2$ and the negative charge $(-Q)$ at $x = -l/2$, as shown in the diagram above. From the expression for the electric field due to a point charge, the electric field on the x-axis at $x = r$ (> 0) due to the positive charge is

$$\mathbf{E}_+ = \frac{1}{4\pi\epsilon_0}\frac{(+Q)}{(r - l/2)^2}\hat{x}$$

and the electric field there due to the negative charge is

$$\mathbf{E}_- = \frac{1}{4\pi\epsilon_0}\frac{(-Q)}{(r + l/2)^2}\hat{x}$$

The total electric field is

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{Q}{4\pi\epsilon_0 r^2}\left[\frac{1}{(1 - l/2r)^2} - \frac{1}{(1 + l/2r)^2}\right]\hat{x} \quad .$$

Since $l/2r \ll 1$, we may to a good approximation expand the two terms in square brackets using the binomial series expansion (as in the previous problem), and retain only the first-order term in $l/2r$:

$$(1 \pm l/2r)^{-2} \simeq 1 \mp l/r \quad (\text{for } l/2r \ll 1) \quad .$$

This gives us

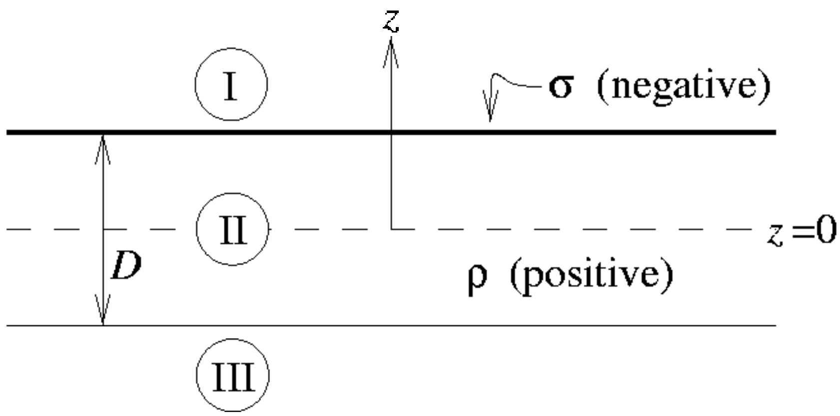
$$\mathbf{E} \simeq \frac{Q}{4\pi\epsilon_0 r^2}\left[(1 + l/r) - (1 - l/r)\right]\hat{x} = \frac{2Ql}{4\pi\epsilon_0 r^3}\hat{x} = \frac{2p}{4\pi\epsilon_0 r^3}\hat{x} \quad .$$

The magnitude checks out with the problem statement. $\mathbf{E}$ points in the positive x-direction, as we would expect, since the positive charge is a bit closer than the negative. Notice that the net E-field is proportional to $1/r^3$ whereas the E-field due to each charge separately falls off as $1/r^2$.

Problem 1.6

Gauss's law and the superposition principle.

Let's choose the z-axis perpendicular to the slab and sheet, with $z = 0$ in the middle of the sheet:



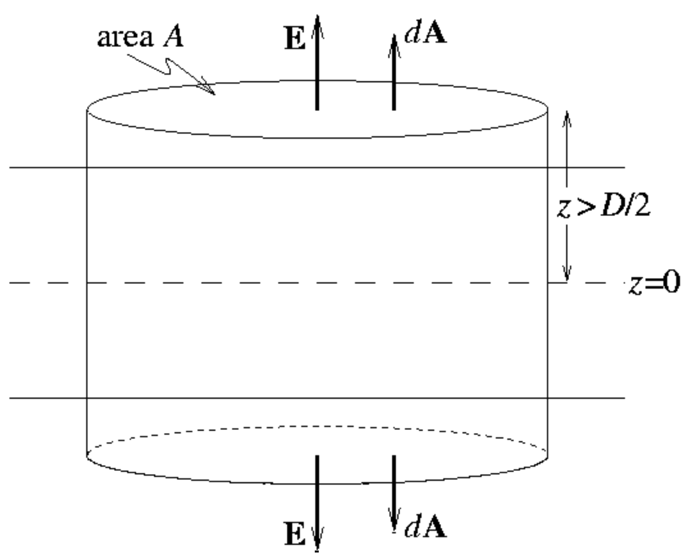
We will find different expressions for $\mathbf{E}$ in each of the three regions shown:

$$\begin{aligned} \text{Region I} &: \ z > D/2 \\ \text{Region II} &: \ -D/2 < z < D/2 \\ \text{Region III} &: \ z < -D/2 \ . \end{aligned}$$

The best way to approach this problem is to make use of the superposition principle and symmetry arguments, i.e., to calculate the electric field $\mathbf{E}_{\text{sheet}}$ of the sheet of charge <u>alone</u>, then to calculate the electric field $\mathbf{E}_{\text{slab}}$ of the slab of charge <u>alone</u>, and then to add vectorially the results. From above, we have

$$\begin{aligned} \text{Region I} &: \ \mathbf{E}_{\text{sheet}} = +\frac{\sigma}{2\epsilon_0}\hat{z} \quad (\text{independent of } z) \\ \text{Region II \& III} &: \ \mathbf{E}_{\text{sheet}} = -\frac{\sigma}{2\epsilon_0}\hat{z} \ . \end{aligned}$$

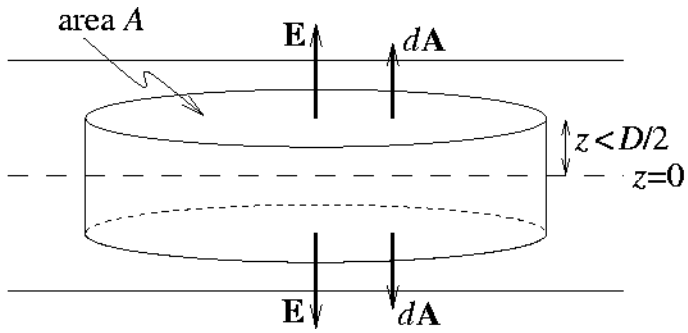
What about $\mathbf{E}_{\text{slab}}$? Consider the following Gaussian pillbox: the top is a distance $z > D/2$ above the $z = 0$ plane, and the bottom is *the same distance* <u>below</u> the $z = 0$ plane. By symmetry, the electric field on the top of the pillbox has exactly the same magnitude, but is oppositely directed from the $\mathbf{E}$ field on the bottom of the pillbox. The charge Q enclosed by the pillbox is ρDA, so



$$\oint \mathbf{E} \cdot d\mathbf{A} = Q_{\text{encl}}/\epsilon_0$$

becomes $2EA = \rho DA/\epsilon_0$, or $E = \rho D/2\epsilon_0$, independent of z.

That gives us the field <u>outside</u> the slab, but what about inside? Consider a pillbox similar to the previous, except now $z < D/2$. The total height of this pillbox is $2z$, so that the charge contained inside is now $\rho(2z)A$, and



$$\oint \mathbf{E} \cdot d\mathbf{A} = Q_{\rm encl}/\epsilon_0 \implies 2EA = 2z\rho A/\epsilon_0 \quad .$$

Notice that again we used a symmetry argument by carefully choosing the pillbox to have top and bottom the *same distance* from $z = 0$. Thus *inside* the slab, $E = \rho z/\epsilon_0$. This depends on z, as it should!

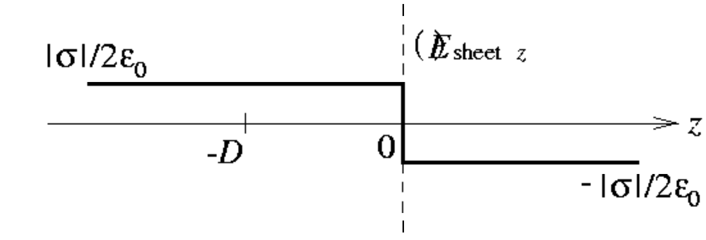
For the slab, then,

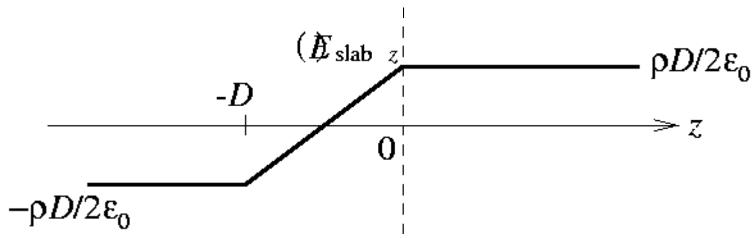
$$\begin{aligned}
\text{Region I} \quad &: \quad \mathbf{E}_{\rm slab} = +\frac{\rho D}{2\epsilon_0}\hat{z} \\
\text{Region II} \quad &: \quad \mathbf{E}_{\rm slab} = +\frac{\rho z}{\epsilon_0}\hat{z} \\
\text{Region III} \quad &: \quad \mathbf{E}_{\rm slab} = -\frac{\rho D}{2\epsilon_0}\hat{z} \quad .
\end{aligned}$$

In summary, we have

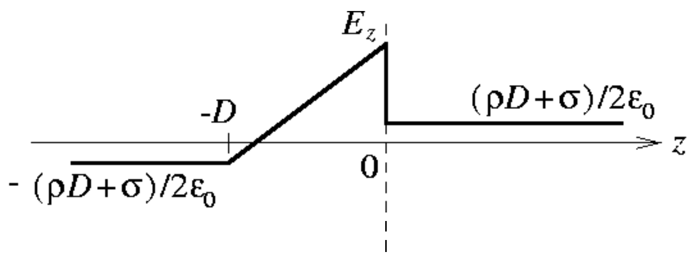
$$\begin{aligned}
\textbf{(a)}\ \text{Region I} \ \ (z > D/2) \quad &: \quad \mathbf{E} = \frac{(\rho D + \sigma)}{2\epsilon_0}\hat{z} \\
\textbf{(c)}\ \text{Region II} \ \ (-D/2 < z < D/2) \quad &: \quad \mathbf{E} = \left[\frac{\rho z}{\epsilon_0} - \frac{\sigma}{2\epsilon_0}\right]\hat{z} \\
\textbf{(b)}\ \text{Region III} \ \ (z < -D/2) \quad &: \quad \mathbf{E} = -\frac{(\rho D + \sigma)}{2\epsilon_0}\hat{z} \quad .
\end{aligned}$$

(d) It is instructive to plot $\mathbf{E}_{\text{sheet}}$ and $\mathbf{E}_{\text{slab}}$ separately (remembering that $\sigma < 0$).





Together, they look like this (assuming $\rho D > |\sigma|$):

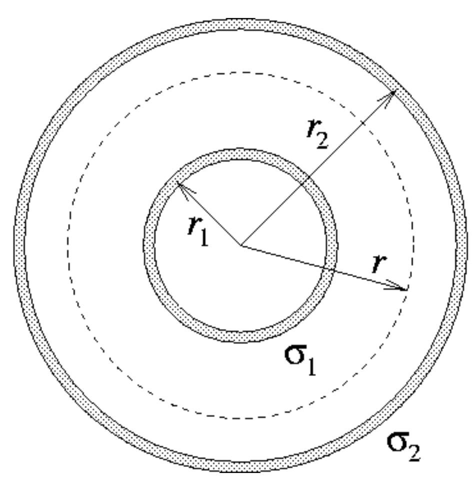


Note that the field across this and <u>any</u> sheet of charge is discontinuous, with a jump of magnitude σ/ϵ_0.

Problem 1.7

Two spherical charged shells. (Giancoli 22-21.)

From the symmetry of the system, we may conclude that the electric field is entirely in the radial direction, and is a function of r alone. We take as a Gaussian surface a spherical shell concentric with the charged shells, and with radius r in the region within which we wish to determine the electric field. Gauss's law gives



$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{A} &= Q_{\text{encl}}/\epsilon_0 \\ \Longrightarrow 4\pi r^2 E &= Q_{\text{encl}}/\epsilon_0 \quad . \end{aligned}$$

(a) In the region where $r < r_1$, $Q_{\rm encl} = 0$, and thus $\mathbf{E} = 0$ there.

(b) In the region where $r_1 < r < r_2$, $Q_{\rm encl} = 4\pi r_1^2 \sigma_1$, so there

$$\mathbf{E} = \frac{\sigma_1}{\epsilon_0}\left(\frac{r_1}{r}\right)^2 \hat{r} \quad .$$

(c) In the region where $r > r_2$, $Q_{\rm encl} = 4\pi(r_1^2\sigma_1 + r_2^2\sigma_2)$, giving

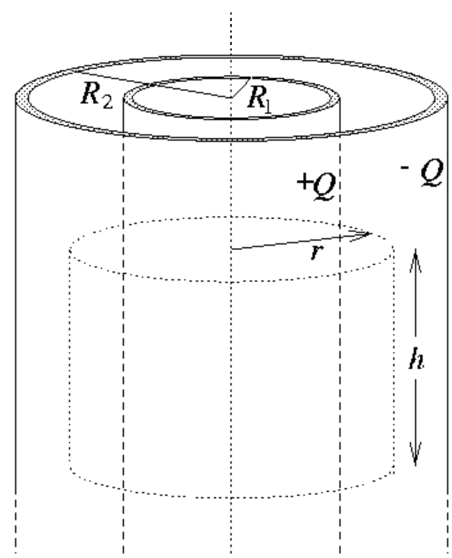
$$\mathbf{E} = \frac{(r_1^2\sigma_1 + r_2^2\sigma_2)}{\epsilon_0 r^2}\hat{r} \quad .$$

(d) We will have $\mathbf{E} = 0$ for $r > r_2$ if $r_1^2\sigma_1 = -r_2^2\sigma_2$. This amounts to having equal and opposite charges on the two shells.

(e) $\mathbf{E} = 0$ for $r_1 < r < r_2$ is only possible if $\sigma_1 = 0$, regardless of the value of σ_2.

Problem 1.8

Two concentric charged cylinders. (Giancoli 22-29.)



For $L \gg R_1, R_2$, we may model the system as being of infinite length to a good approximation. We then conclude from the symmetry of the system that the electric field is directed radially outward, perpendicular to the axis of the cylinders, and is a function only of the perpendicular distance r from the axis. To apply Gauss's law, we consider a cylindrical surface coaxial with the charged shells and of length $h \ll L$, with radius r in the region in which we wish to determine the electric field (see diagram at right). $\mathbf{E}$ is (approximately) perpendicular to the endcaps, so the only contribution to the flux integral comes from the sides of the Gaussian cylinder. Gauss's law gives

$$\oint \mathbf{E}\cdot d\mathbf{A} = Q_{\rm encl}/\epsilon_0 \Longrightarrow 2\pi r h E = Q_{\rm encl}/\epsilon_0 \quad .$$

(a) For $r < R_1$, $Q_{\rm encl} = 0$ and we conclude that $\mathbf{E} = 0$ in this region.

(b) For $R_1 < r < R_2$, we have $Q_{\rm encl} = +Qh/L$, and Gauss's law gives us

$$\mathbf{E}(r) = \frac{Q}{2\pi\epsilon_0 L r}\hat{r}$$

for this region. (Note that the meaning of $\hat{r}$ differs between this problem and the previous problem: here it is intended to indicate the direction perpendicularly away from the cylinders' axis)

(c) For $r > R_2$, $Q_{\rm encl} = (+Q - Q)h/L = 0$, giving $\mathbf{E} = 0$.

(d) The electron will experience an inwardly-directed electrostatic force with magnitude given by

$$F = eE = \frac{eQ}{\pi\epsilon_0 L(R_1 + R_2)} \quad .$$

This force will provide the centripetal acceleration for the maintenance of the circular orbit, so that

$$F = m_{\rm e} a_{\rm c} = \frac{m_{\rm e} v^2}{(R_1 + R_2)/2} \quad .$$

Equating these two expressions for F and recalling that kinetic energy is given by $(KE) = mv^2/2$, we obtain

$$(KE)_{\rm e} = \frac{eQ}{4\pi\epsilon_0 L} \quad .$$

END