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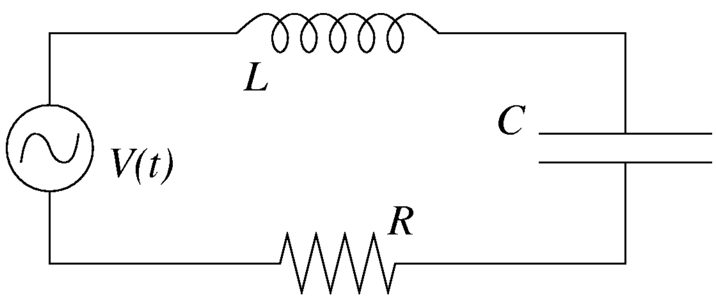
MIT 8.02 Spring 2002
Assignment #8 Solutions

Problem 8.1
An LRC circuit.



$$\begin{aligned} L &= 15\,\text{mH} \\ R &= 80\,\Omega \\ C &= 5\,\mu\text{F} \\ V(t) &= V_0 \sin(\omega t) \\ V_0 &= 40\,\text{V} \end{aligned}$$

(a) As discussed in lecture (and in Giancoli Section 31-6, p. 780), in an *LRC* circuit, the driving frequency at which the current reaches a maximum (resonance) is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.015)(5 \times 10^{-6})}} = 3651.5\,\text{rad/s} \quad .$$

(b) From Giancoli Section 31-5 (pp. 776-779), we have

$$I(t) = I_0 \sin(\omega t - \phi) = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \sin(\omega t - \phi) \quad .$$

(Note: Giancoli adopts the convention $t = 0$ when the current in the circuit $I(t) = 0$. In this problem, our convention is to take $t = 0$ when the source voltage $V(t) = 0$. So, to apply Giancoli's results to our situation, we shift the origin in time according to $\omega t_{\text{Giancoli}} = \omega t_{\text{us}} - \phi$.)

Thus we have

ω	ω (rad/s)	$\frac{1}{\omega C}$ (Ω)	ωL (Ω)	$\sqrt{R^2 + (\omega L - 1/\omega C)^2}$ (Ω)	I_0 (A)
$0.25\ \omega_0$	913	219	13.7	220	0.18
ω_0	3651	54.8	54.8	80	0.50
$4\ \omega_0$	14606	13.7	219	220	0.18

(c) For $\omega = \omega_0$, $\omega L = 1/\omega C$, and thus the phase angle ϕ between the peak current and peak source voltage is zero (see Giancoli Equation (31-10a), p. 778). This gives

$$I(t) = \frac{V_0}{R} \sin \omega_0 t \quad .$$

Since $I = dQ/dt$, we have for the charge on the capacitor

$$Q(t) = -\frac{V_0}{\omega_0 R}\cos\omega_0 t \quad .$$

(The "integration constant" must be zero: our solution for the circuit behavior has assumed that the voltage across the capacitor, and hence the charge on the capacitor, is purely sinusoidal in time.) So,

$$\begin{aligned}
U_C(t) &= \frac{1}{2}\frac{Q^2}{C} = \frac{V_0^2}{2\omega_0^2 R^2 C}\cos^2(\omega_0 t) = \frac{V_0^2 L}{2R^2}\cos^2(\omega_0 t) \\
&= (0.0019)\cos^2(\omega_0 t) \quad \text{(Joules)}, \\
I(t) &= \frac{1}{2}LI^2 = \frac{V_0^2 L}{2R^2}\sin^2(\omega_0 t) \\
&= (0.0019)\sin^2(\omega_0 t) \quad \text{(Joules)}.
\end{aligned}$$

Problem 8.2

Average power dissipated in an LRC circuit. (Giancoli 31-20)

(In the opinion of the solution author, the instantaneous power $P = IV$ of this problem should be called the "power delivered by the power supply", not the "power dissipated in the circuit".)

From $I = I_0\sin(\omega t)$ and $V = V_0\sin(\omega t + \phi)$, we have

$$P = IV = I_0 V_0 \sin(\omega t)\sin(\omega t + \phi) \quad .$$

Next we make use of the trigonometric identity

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad ,$$

giving

$$P = I_0 V_0[\sin^2(\omega t)\cos\phi + \sin(\omega t)\cos(\omega t)\sin\phi] \quad .$$

The average of $\sin^2(\omega t)$ over one period is $\frac{1}{2}$, while the average of $\sin(\omega t)\cos(\omega t)$ is zero, so we have

$$\overline{P} = \frac{1}{2}I_0 V_0\cos\phi \quad ,$$

as advertised.

Problem 8.3

Width of resonance peak. (Giancoli 31-30)

(Note: to be consistent with previous notation, the problem statement should really ask for the difference between the two frequencies where $I_0 = \frac{1}{2} I_{0,\max}$.)

The results of problem 8.1(b) suggest that driving frequencies $\omega = \alpha\omega_0$ and $\omega = \alpha^{-1}\omega_0$ will give the same peak current I_0. We can see this plainly if we use $\omega_0 = 1/\sqrt{LC}$ to recast the expression for I_0:

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \frac{V_0}{R\sqrt{1 + \frac{L}{R^2 C}\left(\omega/\omega_0 - \omega_o/\omega\right)^2}} \quad . \tag{1}$$

Now, let ω_+ and ω_- be the two driving frequencies on either side of the resonance peak that give $I_0 = \frac{1}{2} I_{0,\max} = \frac{1}{2} V_0/R$. In view of (??), they will evidently be related to ω_0 by $\omega_+ = \alpha\omega_0$ and $\omega_- = \alpha^{-1}\omega_0$ for some α. If the resonance peak is sharp, $I_0(\omega)$ falls of very quickly on either side of ω_0, so ω_+ and ω_- must be relatively close to ω_0. Thus we can take $\alpha = 1 + \delta$, with $\delta \ll 1$. This gives $\omega_+ = (1+\delta)\omega_0$, $\omega_- = (1+\delta)^{-1}\omega_0 \approx (1-\delta)\omega_0$, and $\Delta\omega \equiv \omega_+ - \omega_- \approx 2\delta\omega_0$. For $\omega =$ either ω_+ or ω_-,

$$\left(\frac{\omega_\pm}{\omega_0} - \frac{\omega_0}{\omega_\pm}\right)^2 \approx \left((1 \pm \delta) - \frac{1}{(1 \pm \delta)}\right)^2 \approx (\pm 2\delta)^2 = 4\delta^2 \approx \frac{(\Delta\omega)^2}{\omega_0^2} = LC(\Delta\omega)^2 \quad . \tag{2}$$

We can now find $\Delta\omega$ by plugging (??) into (??):

$$I_0 = \frac{1}{2} I_{0,\max} \quad \Longrightarrow \quad \frac{V_0}{R\sqrt{1 + \frac{L^2}{R^2}(\Delta\omega)^2}} \approx \frac{V_0}{2R} \quad .$$

This solves easily to give

$$\Delta\omega \approx \frac{\sqrt{3}R}{L} \quad ,$$

as we were to show.

Problem 8.4

Traveling waves on a string.

(a) For any traveling wave of the form

$$y = a\sin(kx - \omega t) \quad ,$$

a is the amplitude, k is the wavenumber, $\lambda = 2\pi/k$ is the wavelength, ω is the frequency in radians per second ($f = \omega/2\pi$ is the frequency in Hertz), $T = 2\pi/\omega$ is the period, and $v = \omega/k$ is the speed of the wave. For our wave,

$$y = 0.4\sin[\pi(0.5x - 200t)] \quad (x,\ y \text{ in cm},\ t \text{ in sec})$$

and we have

$$\begin{aligned} a &= 0.4\,\mathrm{cm} \\ k &= 0.5\pi = 1.57\,\mathrm{cm}^{-1} \\ \lambda &= 2\pi/k = 4\,\mathrm{cm} \\ \omega &= 200\pi = 628\,\mathrm{rad/s} \\ (f &= \omega/2\pi = 100\,\mathrm{Hz}) \\ T &= 2\pi/\omega = 0.01\,\mathrm{sec} \\ v &= \omega/k = 400\,\mathrm{cm/s} \quad . \end{aligned}$$

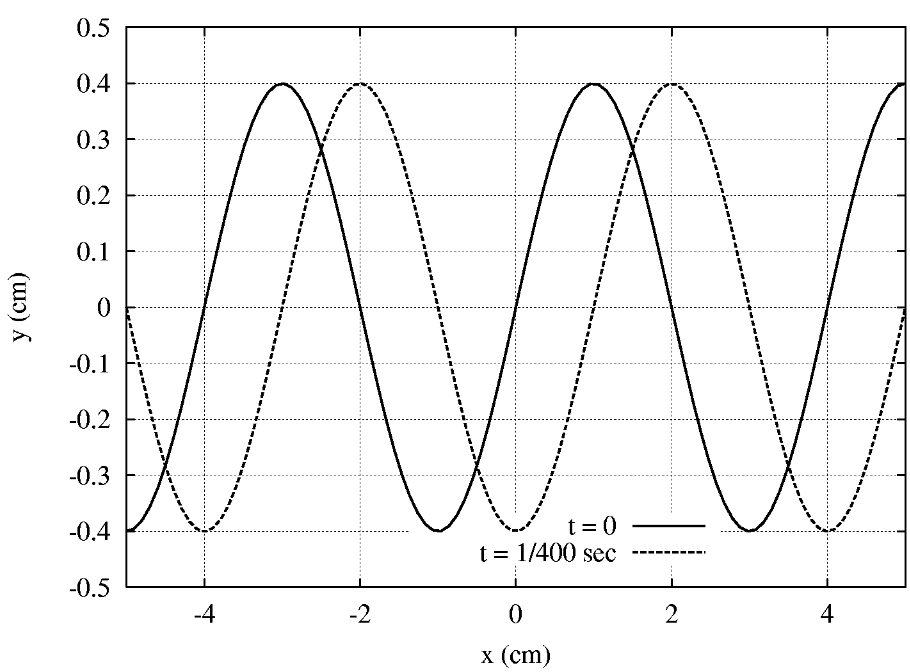
(b) At $t = 0$, the equation of the wave is

$$y(x,t)|_{t=0} = 0.5\sin(1.57x) \quad .$$

At $t = 1/400\,\mathrm{sec}$ ($= T/4$), the equation is

$$y(x,t)|_{t=1/400} = 0.5\sin(1.57x - \pi/2) = -0.5\cos(1.57x) \quad .$$

Following is a plot of y vs. x for these two times:



(c) The transverse (y-direction) speed of a point on our string is

$$\frac{dy}{dt} = a\omega \cos(kx - \omega t) \quad ,$$

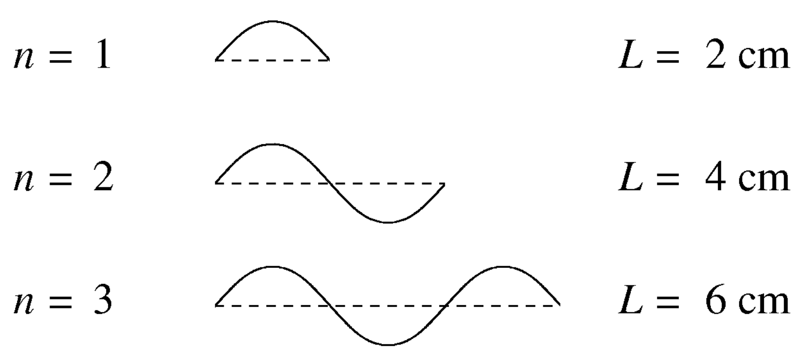
with maximum value

$$\left.\frac{dy}{dt}\right|_{\max} = a\omega = 251\,\mathrm{cm/s} \quad .$$

(d) If we clamp the string at two points a distance L cm apart and observe a standing wave of the same wavelength as above, then L must be an integer number of half-wavelengths:

$$L = n\lambda/2 \qquad (n = 1, 2, 3, 4, \ldots) \quad .$$

For our value of $\lambda = 4\,\mathrm{cm}$, the first three possibilities are:



If L is to be strictly *less than* 10 cm, there are 4 possible values for L: 2 cm, 4 cm, 6 cm, and 8 cm.

Problem 8.5

Standing waves on a string.

(a) A standing wave of the form

$$y = a \sin(kx) \cos(\omega t)$$

has the same wavelength, etc. as given in 8.3(a) above. Thus, for

$$y = 0.3 \sin(3x) \cos(1200t)$$

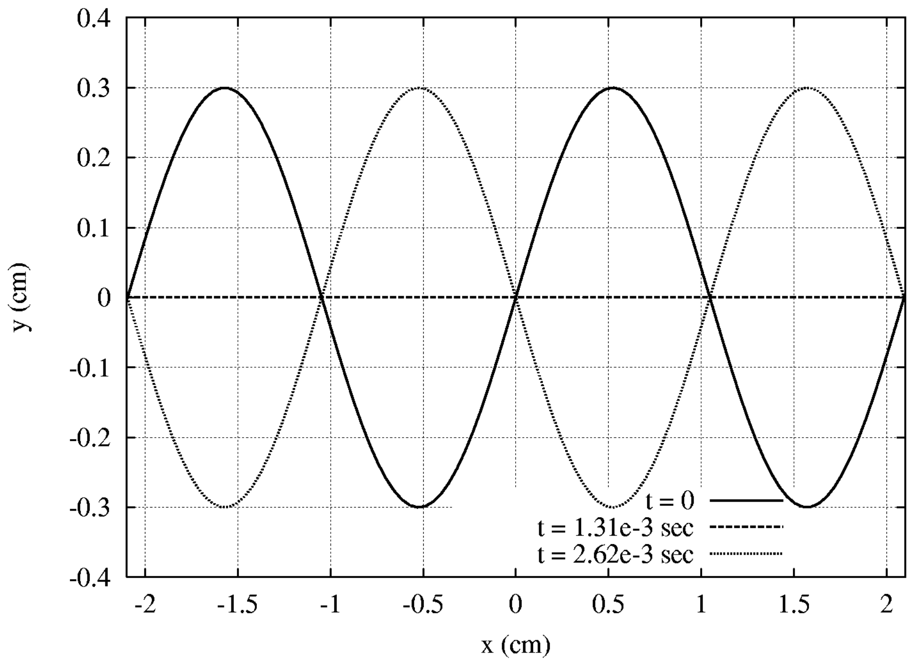
with x, y in centimeters and t in seconds, we have

$$\begin{aligned}
k &= 3\,\mathrm{cm}^{-1} \\
\lambda &= 2\pi/k = 2.1\,\mathrm{cm} \\
\omega &= 1200\,\mathrm{rad/s} \\
(f &= \omega/2\pi = 191\,\mathrm{Hz}) \\
T &= 2\pi/\omega = 5.24 \times 10^{-3}\,\mathrm{s} \quad .
\end{aligned}$$

(b) The times of interest are $t = 0$, $t = 1.31 \times 10^{-3}\,\mathrm{s} = T/4$, and $t = 2.62 \times 10^{-3}\,\mathrm{s} = T/2$:

$$\begin{aligned} y(x,t)|_{t=0} &= 0.3\sin(3x) \\ y(x,t)|_{t=1.31\times10^{-3}} &= 0.3\sin(3x)\cos(\pi/2) \equiv 0 \\ y(x,t)|_{t=2.62\times10^{-3}} &= 0.3\sin(3x)\cos(\pi) = -0.3\sin(3x) \quad . \end{aligned}$$

Following is a plot of y vs. x for these three times:



Compare the evolution of this standing wave with the evolution of the traveling wave of problem 8.3.

(c) As in 8.3(c), the maximum transverse speed is

$$\left.\frac{dy}{dt}\right|_{\max} = a\omega = 360\,\mathrm{cm/s} \quad .$$

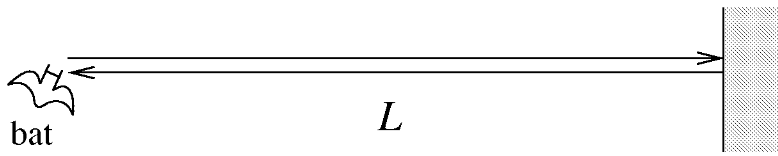
(d) One could interpret this question in one of two ways. If by "speed of propagation" we mean the so-called phase speed, then it is

$$v = \omega/k = 400\,\mathrm{cm/s} \quad .$$

If on the other hand we mean the speed with which this *particular* waveform (standing wave) travels along the string, then it is zero. (See plot under (b) above.)

Problem 8.6

Distance sensing with sound.



(a) The time T between the emission and return of the bat's pulse to a wall L meters away is $T = 2L/v_\mathrm{a}$, where v_a is the speed of sound in air. Thus a distance uncertainty ΔL corresponds to a time uncertainty $\Delta T = 2\Delta L/v_\mathrm{a}$. For $\Delta L = \pm 0.2\,\mathrm{m}$ and $v_\mathrm{a} = 344\,\mathrm{m/s}$ (1 atm at 20°C), we have

$$\Delta T = \pm 1.2 \times 10^{-3}\,\mathrm{sec} \quad .$$

(b) Suppose the bat in our methane-filled cave sends out a pulse which covers a distance L_actual. The bat will receive the reflected pulse after a time $T = 2L_\mathrm{actual}/v_\mathrm{m}$, where v_m is the speed of sound in methane. If the bat perceives that he is in air, he will interpret this time delay as being due to an apparent distance

$$L_\mathrm{apparent} = \frac{v_\mathrm{a}T}{2} = \frac{v_\mathrm{a}}{v_\mathrm{m}} L_\mathrm{actual} \quad .$$

For $v_\mathrm{a} = 344\,\mathrm{m/s}$ and $v_\mathrm{m} = 432\,\mathrm{m/s}$, this gives

$$L_\mathrm{apparent} = 0.8 L_\mathrm{actual} \quad .$$

So the bat will perceive things as being closer than they actually are by a factor of 0.8, because the signals return faster than he expects.

Problem 8.7

Design a flute.

(a) When the system oscillates in the fundamental mode, the length of the tube (open open) is half the wavelength, so the length of the flute should be $L = \lambda/2$. Wavelength is related to frequency f (in Hertz) and sound speed v_a by $f\lambda = v_a$, or $\lambda = v_a/f$, so that $L = v_a/2f$. For $f = 261.7\,\text{Hz}$ and $v_a = 344\,\text{m/s}$,

$$L = 0.657\,\text{meters} \quad .$$

(b) For a modern equal-tempered scale, the frequency of any given note is $\sqrt[12]{2} \simeq 1.0595$ times the frequency of the note one half-step below (note frequencies may also be found tabulated). This allows us to calculate the necessary effective tube lengths for each note from $L = v_a/2f$, and hence the corresponding key spacings ΔL:

Note	f (Hz)	L (m)	ΔL (m)
C	261.7	0.657	(0)
$C\sharp/D\flat$	277.3	0.620	0.037
D	293.7	0.586	0.034
$D\sharp/E\flat$	311.2	0.553	0.033
E	329.7	0.522	0.031
F	349.3	0.492	0.030
$F\sharp/G\flat$	370.1	0.465	0.027
G	392.1	0.439	0.026
$G\sharp/A\flat$	415.4	0.414	0.025
A	440.1	0.391	0.023
$A\sharp/B\flat$	466.3	0.369	0.022
B	494.0	0.348	0.021
C	523.4	0.329	0.019

(Note: the somewhat jumpy progression of ΔL's is due to round-off error.)

END