The Falling Magnet

Consider the situation in which a permanent magnet is located on the vertical axis of a stationary, conducting, non-magnetic ring, and is constrained to move along that axis. The magnetic dipole moment of the magnet is also constrained to be vertical, parallel to the axis of the ring. The magnet is released from rest at $t = 0$, and falls under gravity toward the conducting ring. Eddy currents arise in the ring because of the changing magnetic flux as the magnet falls toward the ring, and the sense of these currents will be such as to repel the magnet from below. After the magnet passes through the ring, the eddy currents will reverse direction, now attracting the magnet from above. We formulate the dynamics of the problem.

Let our magnet have dipole moment $M_o\hat{\mathbf{z}}$. Suppose the circular ring has radius a, resistance R, and inductance L. The equation of motion of the magnet is

$$m\frac{d^2z}{dt^2} = -mg + M_o\frac{dB_z}{dz} \tag{1}$$

where B_z is the field due the current I in the ring, positive in counterclockwise direction as viewed from above. The expression for B_z is

$$B_z = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}} \tag{2}$$

so that equation (1) is

$$m\frac{d^2z}{dt^2} = -mg - \frac{3M_o\mu_o I a^2}{2}\frac{z}{(a^2+z^2)^{5/2}} \tag{3}$$

Ohm's Law and Faraday's Law applied to the ring give

$$\oint \mathbf{E}\cdot\mathbf{dl} = IR = -\frac{d}{dt}\int \mathbf{B}\cdot\mathbf{dA} = -\frac{d}{dt}\int \mathbf{B}_{dipole}\cdot\mathbf{dA} - L\frac{dI}{dt} \tag{4}$$

To determine the magnetic flux through the ring due to the dipole field , we calculate the flux through a spherical cap of radius $\sqrt{a^2+z^2}$ with an opening angle given θ given by $\sin\theta = a/\sqrt{a^2+z^2}$ (this is the same as the flux through the ring because $\nabla\cdot\mathbf{B}=0$). The flux through a spherical cap only involves the radial component of the dipole field, and our expression for the flux is easily seen to be

$$\int \mathbf{B}_{dipole}\cdot\mathbf{dA} = \int \frac{\mu_o}{2\pi}\frac{M_o\cos\theta}{r^3}2\pi r^2\sin\theta d\theta = \frac{\mu_o M_o}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}} \tag{5}$$

Inserting (5) into (4) yields

$$IR = -L\frac{dI}{dt} + \frac{3\mu_o a^2 M_o}{2}\frac{z}{\left(a^2+z^2\right)^{5/2}}\frac{dz}{dt} \tag{6}$$

Equations (3) and (6) are the coupled ordinary differential equations which determine the dynamics of the situation. If we multiply (3) by $v = \frac{dz}{dt}$ and (6) by I, after some algebra, we find that

$$\frac{d}{dt}\left[\ \tfrac{1}{2}mv^2 + mgz + \tfrac{1}{2}LI^2\ \right] = -I^2R \tag{7}$$

This equation expresses the conservation of energy for the falling magnet plus the magnetic field of the ring.

We now put these equations into dimensionless form. We measure all distances in terms of the distance *a*, and all times in terms of the time $\sqrt{a/g}$. Let

$$z' = \frac{z}{a} \qquad t' = \frac{t}{\sqrt{a/g}} \qquad I' = \frac{I}{I_o}, \quad where\ I_o = \frac{mga^2}{\mu_o M_o} \tag{8}$$

The time $\sqrt{a/g}$ is roughly the time it would take the magnet to fall under the influence of gravity through a distance *a* starting from rest. The current I_o is roughly the current in the ring that is required to produce a force sufficient to offset gravity when the magnet is a distance *a* above the ring. We introduce the three dimensionless parameters

$$\alpha = \frac{R}{L}\sqrt{\frac{a}{g}} \qquad \beta = \frac{\mu_o M_o}{LI_o a} = \frac{(\mu_o M_o)^2}{Lmga^3} \qquad \lambda = \frac{L}{\mu_o a} \tag{9}$$

Note that we can write the reference current I_o in terms of these parameters as

$$\frac{I_o a^2}{M_o} = \frac{mga^4}{\mu_o M_o^{\ 2}} = \frac{1}{\lambda\beta} \tag{10}$$

The parameters have the following physical meanings. The quantity α is the ratio of the free fall time to the inductive time constant. If α is very large, inductive effects are negligible. The quantity β is roughly the ratio of the current due to induction alone to the reference current I_O, in the case that the resistance is zero. The quantity λ is the ratio of the inductance of the ring to its minimum possible value, $\mu_o a$. If we define the speed $v' = dz'/dt'$, then we can write three coupled first-order ordinary differential equations for the triplet (z', v', I'), as

$$dz'/dt' = v' \tag{11}$$

$$\frac{dv'}{dt'} = -1 - \frac{3}{2}\frac{z'}{(1+z'^2)^{5/2}}I' \tag{12}$$

$$\frac{dI'}{dt'} = -\alpha I' + \frac{3}{2}\frac{\beta z'}{(1+z'^2)^{5/2}}\frac{dz'}{dt'} \tag{13}$$

We show a numerical solution to these equations in Figure 1. The initial conditions (z', v', I') for this solution plotted are (2,0,0), and the values of (α, β) are (1,32). In Figure 1 we plot the position as a function of time and the current as a function of time using our dimensionless parameters. The behavior of these solutions is what we expect. When the magnet reaches a distance of about *a* above the ring, it slows down because of the increasing current in the ring, which repels the magnet. As the current increases, energy is stored in the magnetic field of the ring, and some of that energy is returned to the magnet as it "bounces" slightly at about $t' = 3.5$. The magnet then starts to fall again. As it passes through the ring, the current reverses direction (with a slight time lag, because of the inductance of the ring), now attracting the magnet from above, which again slows the magnet. Finally the magnet falls far enough that the current in the ring becomes small, and the magnet is again in free fall. For other choices of values of (α,β) (e.g., (0,32)), the magnet will actually levitate above the ring, since there is no dissipation in the system when $\alpha = 0$.

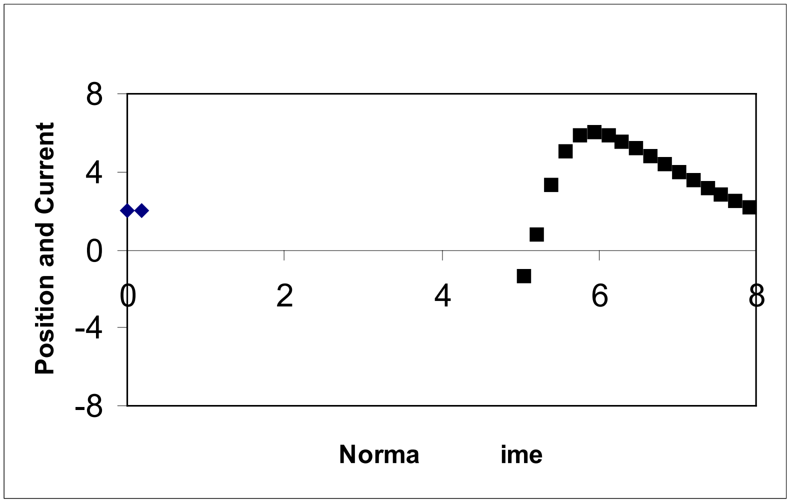

Figure 1: The normalized position z' of the falling magnet above the ring and the normalized current I' in the ring as a function of normalized time t', for values of the parameters (α,β) equal to (1,32).

How much freedom do we have in choosing the absolute value of the current once we have solved our dimensionless equations, and how does that freedom affect the topology of the magnetic field lines? The absolute current is given by $I = I_o I'$ (cf. equation (8)). One measure of the shape of the total field is the ratio of the field at the center of the ring due to the ring to the field at the center of the ring due to the magnet when the magnet is a distance *a* above the ring. Clearly when this ratio varies the overall shape of the total field must vary. It can be shown using equation (9) that this ratio is to within numerical factors given by $I' / \lambda\beta$. That is, the overall shape of the magnetic field topology is totally determined once we make the one remaining choice of the dimensionless constant λ, defined in equation (9), which up to this point we have not chosen (we need only pick values of α and β to solve our dimensionless equations). Once that choice is made, we have no additional freedom to affect the field topology. In all of the subsequent calculations, we have chosen λ to be 2, which is the smallest value that is physically reasonable.

Finally, the treatment above refers to a falling magnet and a stationary ring, but all the dynamics are the same for a falling ring and a stationary magnet, since it is only the relative velocity of the two that affects the dynamics. The falling ring animations use the same set of equations as the falling magnetic animations.