8.962 Lecture 10
March 12, 2018

MATHEMATICAL CONCEPTS and DEFINITIONS

Basic Notation

$\cup$ — $A \cup B$ denotes the union of sets A and B

$\cap$ — $A \cap B$ denotes the intersection of the sets A and B

$\subset$ — $A \subset B$ denotes that A is a subset of B. (May or may not mean proper subset.)

$-$ — $B - A$ denotes the complement in B of the set A

$\in$ — $p \in A$ denotes that p is an element of A

$\{|\}$ — $\{p \in A|Q\}$ denotes the set consisting of those elements p of the set A which satisfy condition Q

$\times$ — Cartesian product; $A \times B$ is the set $\{(a,b)|a \in A \text{ and } b \in B\}$

$\emptyset$ — the empty set

$\mathbb{R}$ — the set of real numbers

$\mathbb{R}^n$ — the set of n-tuples of real numbers

$\mathbb{C}$ — the set of complex numbers

$\mathbb{C}^n$ — the set of n-tuples of complex numbers

$:\rightarrow$ — $f : A \rightarrow B$ denotes that f is a map from the set A to the set B

$\circ$ — $f \circ g$ denotes the composition of maps $g : A \rightarrow B$ and $f : B \rightarrow C$, i.e ., for $p \in A$ we have $(f \circ g)(p) = f[g(p)]$

$[\,]$ — $f[A]$ denotes the image of the set A under the map f, i.e., the set $\{f(x)|x \in A\}$

C^n — the set of n-times continuously differentiable functions. Note that C^0 means simply continuous, while C^1 means that the first derivative exists and is continuous.

C^∞ — the set of infinitely continuously differentiable (i.e., smooth) functions

$\exists$ — there exists; i.e., for all $u \in \mathbb{R}$, $\exists\ v \mid v + u = 0$

$\forall$ — for all; i.e., $\forall u \in \mathbb{R}$, $\exists\ v \mid v + u = 0$

Properties of Maps

- If f is a function $f : M \to N$, M is called the **domain** of f, and N is called its **codomain**.
- The set of points in N that M gets mapped into is called the **image** of f.
- For any subset $U \subset N$, the set of elements of M that get mapped to U is called the **preimage** of U under f, or $f^{-1}(U)$.
- A map $f : M \to N$ is called **one-to-one** (or **injective**) if each element of N has at most one element of M mapped into it.
- A map $f : M \to N$ is called **onto** (or **surjective**) if each element of N has at least one element of M mapped into it.

- A map that is both one-to-one and onto is known as **invertible** (or **bijective**). In this case we can define the inverse map $f^{-1} : N \to M$ by $(f^{-1} \circ f)(x) = x$, for any $x \in M$.

Continuity

- (Weierstrass definition): For functions $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$, $f(x)$ is continuous at x_0 if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.
- (General topological definition): If open sets have been defined, then a function $f : X \to Y$ is continuous if and only if the preimage $f^{-1}(V)$, where V is an open subset of Y (which could be the whole set), is always an open subset of X.
- For the usual definition of open sets on $\mathbb{R}$, the two definitions are equivalent.

- If $f : D \to \mathbb{R}^n$, where $D \subset \mathbb{R}^m$, then the definition of continuity is a natural generalization of the $\mathbb{R} \to \mathbb{R}$ definition. f can be described as a collection of functions

$$f^i(x^1, x^2, \ldots, x^m) ,$$

where $i = 1, \ldots, n$. f is C^p if each f^i is at least C^p in each of the variables $(x^1, x^2, \ldots, x^m)$.

- Suppose that M and N are topological spaces (i.e., spaces on which open sets have been defined). Then if $f : M \to N$ is continuous, one-to-one, and onto, and its inverse is continuous, then f is called a **homeomorphism,** and the spaces M and N are said to be **homeomorphic**. As far as topology is concerned, M and N are then identical. (See Wald. Carroll never uses the word "homeomorphic".)

- Suppose that M and N are manifolds (to be defined shortly). Then if $f : M \to N$ is C^∞, one-to-one, and onto, and its inverse is C^∞, then f is called a **diffeomorphism,** and the spaces M and N are said to be **diffeomorphic**. As far as manifold properties are concerned, M and N are then identical.*

- An **open ball** is the set of all points x in $\mathbb{R}^n$ such that $|x - y| < r$ for some fixed $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$, where $|x-y|^2 = \sum_i (x^i - y^i)^2$. Note that $|x - y|$ must be less than r. The ball does not include its boundary.

- An **open set** in $\mathbb{R}^n$ is a set constructed from an arbitrary (maybe infinite) union of open balls. Equivalently, a set $V \subset \mathbb{R}^n$ is open if, for any $y \in V$, there is an open ball centered at y that is completely inside V.

*This entry has been corrected from the version shown in lecture, which mistakenly omitted the requirement that f and f^{-1} must be C^∞.

MANIFOLDS!

- If M is a set, a **chart** or **coordinate system** on M is a one-to-one map $\phi : U \to \mathbb{R}^n$ such that the image $\phi(U)$ of the map is an open subset of $\mathbb{R}^n$. We do not assume a topology on M, so U is said to be open in M if $\phi(U)$ is open in $\mathbb{R}^n$.

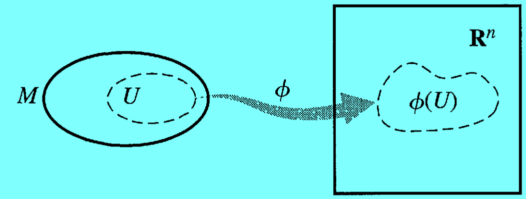

FIGURE 2.13 A coordinate chart covering an open subset U of M.

(From Sean Carroll, *Spacetime and Geometry.*)

- A C^∞ **atlas** of charts is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that satisfies two conditions:
  1) The U_α cover M, so that any point in M is contained in at least one chart U_α.
  2) The charts smoothly sew together. Whenever two charts overlap, the map from one coordinate system to the other must be C^∞. In symbols, whenever $U_\alpha \cap U_\beta \neq 0$, the map $(\phi_\alpha \circ \phi_\beta^{-1})$ takes points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ *onto* the open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$. All such maps must be C^∞ where they are defined.

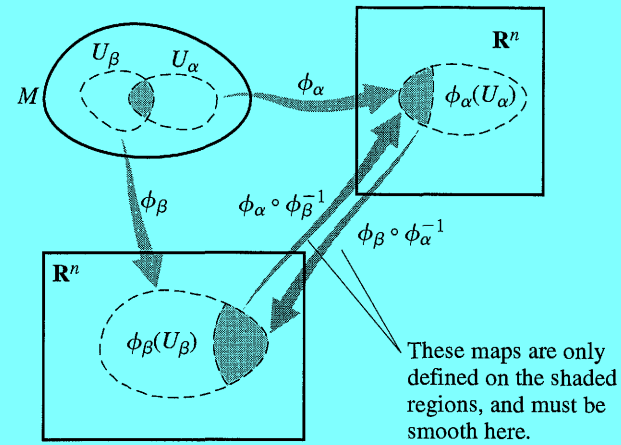

FIGURE 2.14 Overlapping coordinate charts.

(From Sean Carroll, *Spacetime and Geometry.*)

A C^∞ n-dimensional **manifold** (or n-manifold for short) is simply a set M along with a maximal atlas, one that contains every possible compatible chart.