8.962 Lecture 16
April 9, 2018

SECOND BIANCHI IDENTITY

Second Bianchi Identity

Basic idea:

$R^\rho{}_{\sigma\mu\nu}$ is a commutator.

Commutators obey the Jacobi identity:

$$[A\,,\,[B\,,\,C]] + [B\,,\,[C\,,\,A]] + [C\,,\,[A\,,\,B]] = 0\ .$$

Proof of Jacobi identity: Just write it out. For example, the term ABC occurs positively in first Jacobi term, and negatively in last Jacobi term.

So, the Jacobi identity should lead to an identity for the Riemann tensor. What is A? Ans: ∇_λ.

Application to $R^\rho{}_{\sigma\mu\nu}$

Recall: $[\nabla_\mu\,,\,\nabla_\nu]\,V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma$.

Then: $[\nabla_\mu\,,\,\nabla_\nu]\,V_\rho = R_{\rho\sigma\mu\nu}V^\sigma = -R_{\sigma\rho\mu\nu}V^\sigma = -R^\sigma{}_{\rho\mu\nu}V_\sigma$.

Now consider: $[\nabla_\mu\,,\,\nabla_\nu]\,V_\rho W_\tau$.

Claim: $[\nabla_\mu\,,\,\nabla_\nu]\,V_\rho W_\tau = -R^\sigma{}_{\rho\mu\nu}V_\sigma W_\tau - R^\sigma{}_{\tau\mu\nu}V_\rho W_\sigma$.

Explanation: When $[\nabla_\mu\,,\,\nabla_\nu]\,V_\rho W_\tau$ is expanded, using the Leibnitz rule, the terms shown above are generated when both ∇'s act on V_ρ or W_τ. When one ∇ acts on each vector, the terms cancel. For example,

$$[\nabla_\mu\,,\,\nabla_\nu]\,V_\rho W_\tau = \nabla_\mu\nabla_\nu V_\rho W_\tau - \nabla_\nu\nabla_\mu V_\rho W_\tau\ ,$$

which contains

$$(\nabla_\mu V_\rho)(\nabla_\nu W_\tau) - (\nabla_\mu V_\rho)(\nabla_\nu W_\tau) = 0\ .$$

But tensors of the form $V_\rho W_\tau$ span the space of all (0,2) tensors $T_{\rho\tau}$, so

$$[\nabla_\mu\,,\,\nabla_\nu]\,T_{\rho\tau} = -R^\sigma{}_{\rho\mu\nu}T_{\sigma\tau} - R^\sigma{}_{\tau\mu\nu}T_{\rho\sigma}\ .$$

Applying the Jacobi Identity

$$[\nabla_\lambda\,,\,[\nabla_\mu\,,\,\nabla_\nu]]\,V_\sigma + \text{cyclic}(\lambda,\mu,\nu) = 0\ ,$$

where $+\text{cyclic}(\lambda,\mu,\nu)$ means to add the two other cyclic permutations of the first term.

Expanding the outer commutator,

$$\nabla_\lambda\left([\nabla_\mu\,,\,\nabla_\nu]\,V_\sigma\right) - [\nabla_\mu\,,\,\nabla_\nu]\,\nabla_\lambda V_\sigma + \text{cyclic}(\lambda,\mu,\nu) = 0\ .$$

Using $[\nabla_\mu\,,\,\nabla_\nu]\,V_\rho = -R^\sigma{}_{\rho\mu\nu}V_\sigma$, etc.,

$$-\nabla_\lambda\left(R^\rho{}_{\sigma\mu\nu}V_\rho\right) + R^\rho{}_{\lambda\mu\nu}\nabla_\rho V_\sigma + R^\rho{}_{\sigma\mu\nu}\nabla_\lambda V_\rho + \text{cyclic}(\lambda,\mu,\nu) = 0\ .$$

Using $[\nabla_\mu \, , \, \nabla_\nu] \, V_\rho = -R^\sigma{}_{\rho\mu\nu} V_\sigma$, etc.,

$$-\nabla_\lambda \left(R^\rho{}_{\sigma\mu\nu} V_\rho \right) + R^\rho{}_{\lambda\mu\nu} \nabla_\rho V_\sigma + R^\rho{}_{\sigma\mu\nu} \nabla_\lambda V_\rho + \mathrm{cyclic}(\lambda, \mu, \nu) = 0 \ .$$

Expanding the 1st term via Leibnitz, the 3rd term is canceled, leaving

$$\nabla_\lambda \left(R^\rho{}_{\sigma\mu\nu} \right) V_\rho - R^\rho{}_{\lambda\mu\nu} \nabla_\rho V_\sigma + \mathrm{cyclic}(\lambda, \mu, \nu) = 0 \ .$$

Now notice that, by the first Bianchi identity, the 2nd term vanishes when cyclically summed. So, finally,

$$\boxed{\nabla_\lambda R^\rho{}_{\sigma\mu\nu} + \mathrm{cyclic}(\lambda, \mu, \nu) = 0 \ .}$$