8.962 Lectures 23 & 24
May 7 & 9, 2018

GRAVITATIONAL ENERGY

Carroll, p. 252

Defines "Komar energy" for an asymptotically flat, stationary spacetime, and then says

> "We could even imagine time-dependent behavior in the interior; so long as K^μ was asymptotically a timelike Killing vector, the Komar energy will be well-defined. We could, for example, consider spherically symmetric gravitational collapse from an initially static star. Evaluating the integral (6.35) [volume integral] directly over Σ would give a sensible answer for the total mass, which should not change as the star collapsed to a black hole (we are imagining spherical symmetry, so that gravitational radiation cannot carry away energy to infinity)."

Wald, p. 287

"However, despite the absence of a notion of energy density of the gravitational field, there does exist a useful and meaningful notion of the total energy of an isolated system, i.e., more precisely, the total energy-momentum 4-vector present in an asymptotically flat spacetime."

Then discusses Bondi and ADM energies, and gives references that ADM energy is conserved, that both are positive, and that the flux described by the Bondi mass agrees with an earlier calculation.

Weinberg, p. 165

"The physical significance of the Einstein equations can be clarified by writing them in an entirely equivalent form that, because not manifestly covariant, reveals their relation to the wave equations of elementary particle physics. Let us adopt a coordinate system that is quasi-Minkowskian, in the sense that the metric $g_{\mu\nu}$ approaches the Minkowski metric $\eta_{\mu\nu}$ at great distances from the finite material system under study."

Weinberg's Energy-Momentum "Tensor" of Gravity

Linearized Ricci tensor:

$$R^{(1)}{}_{\mu\kappa} = \frac{1}{2}\left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\kappa} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h^\lambda{}_\kappa}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x_\lambda}\right) .$$

Energy-Momentum tensor of gravity:

$$t_{\mu\kappa} = \frac{1}{8\pi G}\left[-\frac{1}{2}h_{\mu\kappa}R^{(1)\lambda}{}_\lambda + \frac{1}{2}\eta_{\mu\kappa}h^{\rho\sigma}R^{(1)}{}_{\rho\sigma} + R^{(2)}{}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa}\eta^{\rho\sigma}R^{(2)}{}_{\rho\sigma}\right] + \mathcal{O}(h^3) .$$

where

$$\begin{aligned} R^{(2)}{}_{\mu\kappa} = &-\frac{1}{2}h^{\lambda\nu}\left[\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] \\ &+ \frac{1}{4}\left[2\frac{\partial h^\nu{}_\sigma}{\partial x^\nu} - \frac{\partial h^\nu{}_\nu}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x^\kappa} + \frac{\partial h^\sigma{}_\kappa}{\partial x^\mu} - \frac{\partial h_{\mu\kappa}}{\partial x_\sigma}\right] \\ &- \frac{1}{4}\left[\frac{\partial h_{\sigma\kappa}}{\partial x^\lambda} + \frac{\partial h_{\sigma\lambda}}{\partial x^\kappa} - \frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x^\mu} - \frac{\partial h^\lambda{}_\mu}{\partial x_\sigma}\right] . \end{aligned}$$

Landau-Lifshitz Gravitational Pseudotensor

Define

$$h^{\mu\nu\lambda} \equiv \frac{1}{16\pi G}\frac{\partial}{\partial x^\sigma}\left[(-g)\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma}\right)\right] .$$

This is antisymmetric in ν and λ. Then define $t^{\mu\nu}$ by

$$(-g)\left(T^{\mu\nu} + t^{\mu\nu}\right) = \frac{\partial h^{\mu\nu\lambda}}{\partial x^\lambda} .$$

Then antisymmetry of $h^{\mu\nu\lambda}$ implies that

$$\partial_\nu\left[\frac{\partial h^{\mu\nu\lambda}}{x^\lambda}\right] = 0 ,$$

so

$$\partial_\nu\left[(-g)\left(T^{\mu\nu} + t^{\mu\nu}\right)\right] = 0 .$$

$h^{\mu\nu\lambda}$ was invented by using the criterion that $t^{\mu\nu} = 0$ in a locally inertial frame.

Landau & Lifshitz: Classical Theory of Fields, p. 306

The integration in (101.9) [volume integral for total energy] can be taken over any infinite hypersurface, including all of the three-dimensional space.

Misner, Thorne, & Wheeler, p. 463

"The spacetime must be asymptotically flat if there is to be any possibility of defining energy and angular momentum. Only then can linearized theory be applied; and only on the principle that linearized theory applies far away can one justify using the flux integrals (20.9) in the full nonlinear theory. Nobody can compel a physicist to move in close to define energy and angular momentum. He has no need to move in close; and he may have compelling motives not to: the internal structure of the sources may be inaccessible, incomprehensible, uninteresting, dangerous, expensively distant, or frightening. This requirement for far-away flatness is a remarkable feature of the flux integrals (20.9); it is also a decisive feature."

Poisson & Will (*Gravity*), p. 295

"Because they involve a partial-derivative operator, the differential identities of Eq. (6.8)

$$\partial_\beta \left[(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\right)\right] = 0 \ ,$$

can immediately be turned into integral identities. We consider a three-dimensional region V, a fixed (time-independent) domain of the spatial coordinates x^j, bounded by a two-dimensional surface S. We assume that V contains at least some of the matter (so that $T^{\alpha\beta}$ is non-zero somewhere within V), but that S does not intersect any of the matter (so that $T^{\alpha\beta} = 0$ everywhere on S)."

Weinberg, p. 47

"These components have no clear physical significance, and in fact can be made to vanish if we fix the origin of coordinates to coincide with the 'center of energy' at $t = 0$, that is, if at $t = 0$ the moment $\int x^i T^{00}\, d^3x$ vanishes."

Weinberg, p. 168

"By its construction, $\tau^{\nu\lambda}$ is clearly the energy-momentum 'tensor' we determine when we measure the gravitational field produced by any system. Indeed, there are many possible definitions of the energy-momentum 'tensor' of gravitation that share most of the good properties of our $t_{\mu\kappa}$ (these definitions are usually based on the action principle; see Chapter 12), but $\tau_{\mu\kappa}$ is specially picked out by its role in (7.6.3) as part of the source of $h_{\mu\nu}$."

$$R^{(1)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa} R^{(1)\lambda}{}_\lambda = -8\pi G[T_{\mu\kappa} + t_{\mu\kappa}]. \qquad (7.6.3)$$

Misner, Thorne, & Wheeler, p. 467

"Moreover, 'local gravitational energy-momentum' has no weight. It does not curve space. It does not serve as a source term on the righthand side of Einstein's field equations. It does not produce any relative geodesic deviation of two nearby world lines that pass through the region of space in question. It is not observable."

Carroll, p. 253: ADM Energy

$$E_{\rm ADM} = \frac{1}{16\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}\sigma^i\left(\partial_j h^j{}_i - \partial_i h^j{}_j\right) .$$

"The ADM energy of a nonsingular, asymptotically flat spacetime obeying Einstein's equation and the dominant energy condition is nonnegative. Furthermore, Minkowski is the only such spacetime with vanishing ADM energy."

Landau & Lifshitz: Classical Theory of Fields, p. 335

"It is interesting to note that in a closed space the total electric charge must be zero. Namely, every closed surface in a finite space encloses on each side of itself a finite region of space. Therefore the flux of the electric field through this surface is equal, on the one hand, to the total charge located in the interior of the surface, and on the other hand to the total charge outside of it, with opposite sign. Consequently, the sum of the charges on the two sides of the surface is zero.

Similarly, from the expression (101.14) for the four-momentum in the form of a surface integral there follows the vanishing of the total four-momentum P^i over all space. Thus the definition of the total four-momentum loses its meaning, since the corresponding conservation law degenerates into the empty identity $0 = 0$."

Misner, Thorne, & Wheeler, p. 457

"There is no such thing as 'the energy (or angular momentum, or charge) of a closed universe,' according to general relativity, and this for a simple reason. To weigh something one needs a platform on which to stand to do the weighing."

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