MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — February 7, 2018
Prof. Alan Guth

PROBLEM SET 1

DUE DATE: Thursday, February 15, 2018, at 5:00 pm. Put your solutions in the homework box in Bldg 8, 3rd floor, at the intersection with the 4th floors of Bldg 16 and Bldg 26.

TOPICS COVERED AND RELEVANT LECTURES: This problem set covers some basic aspects of special relativity, including material covered in Lectures 1 and 2 (2/7/18) and 2/12/18).

READING: Carroll Chapter 1, sections §1.1 – §1.8. Note: we will go into the mathematical framework describing *tangent bundles* and other ideas mentioned in §1.4, §1.5 in more detail in later weeks, for the present a loose understanding of these ideas will suffice.

PROBLEM 1: THOUGHT EXPERIMENTS FOR THE RELATIVITY OF SIMULTANEITY *(15 pts)*

In the first lecture we discussed thought experiments that can be used to derive the three kinematic consequences of special relativity: time dilation, Lorentz contraction, and the relativity of simultaneity. In this problem we will carry out two versions of a thought experiment to derive the relativity of simultaneity. You may assume that we already know about time dilation and Lorentz contraction:

- TIME DILATION: Any clock which is moving at speed v relative to an inertial reference frame will "appear" (to an observer using that reference frame) to run slower than normal by a factor

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c \ .$$

- LORENTZ-FITZGERALD CONTRACTION: Any rod which is moving at a speed v along its length relative to an inertial reference frame will "appear" (to an observer using that reference frame) to be shorter than its normal length by the same factor γ. A rod which is moving perpendicular to its length does not undergo a change in apparent length.

The quotation marks on "appear" mean that the rate at which the moving clock ticks and the length of the moving rod are not determined by what any single observer sees; instead these quantities are found by collecting data obtained by local observers who are stationary, and have synchronized their clocks, in the specified reference frame. The goal of this problem is use only knowledge of time dilation, Lorentz contraction, and the fact that light always travels at speed c in any inertial frame to derive the relativity of simultaneity:

- RELATIVITY OF SIMULTANEITY: Suppose that a rod which has rest length ℓ_0 is equipped with clocks at each end, which are synchronized ih the rest frame of the rod. If the system moves at speed v along its length relative to an inertial reference frame, then the trailing clock will "appear" (to an observer using that frame) to read a time which is later than the leading clock by an amount $v\ell_0/c^2$. If, on the other hand, the system moves perpendicular to its length, then the synchronization of the clocks is not disturbed.

To discuss the relativity of simultaneity, consider a high-speed train of rest length ℓ_0 that travels on a straight track at speed v relative to the ground (with v comparable to c), with a clock at each end.

(a) *(5 pts)* Suppose the clocks are synchronized in the rest frame of the train by using a light pulse sent from the front of the train to the rear. The pulse leaves the clock at the front when it reads $t_{\text{emit}}^{\text{clock 1}}$. When it arrives at the clock in the rear, the clock is set to

$$t_{\text{rec}}^{\text{clock 2}} = t_{\text{emit}}^{\text{clock 1}} + \frac{\ell_0}{c} \ .$$

Using time dilation, Lorentz contraction, and the fact that light always travels at speed c, calculate the time difference between the two clocks as measured in the frame of the ground. (Remember that we are imagining that we have not yet derived the formula for the Lorentz transformation, so we cannot use it here.)

(b) *(10 pts)* Now consider the same train, but consider another way that the clocks can be synchronized. Suppose that both clocks start at the front of the train, and are both set to the same time. Then suppose that one of the clocks is carried very slowly to the back of the train. In the limit as the speed at which it is carried approaches zero (and the time for which it is carried approaches infinity), the clocks remain synchronized in the frame of the train. By analyzing this process in the frame of the ground, show that the clocks do not remain synchronized in this frame, but instead the clock that is carried to the back of the train reads later than the clock at the front by the expected amount.

PROBLEM 2: PERIODIC BOUNDARY CONDITIONS AND LORENTZ INVARIANCE *(10 pts)*

Carroll Problem 1.2: Imagine that space (not spacetime) is actually a finite box, or in more sophisticated terms, a three-torus, of size L. By this we mean that there is a coordinate system $x^\mu = (t, x, y, z)$ such that every point with coordinates (t, x, y, z) is identified with every point with coordinates $(t, x + L, y, z)$, $(t, x, y + L, z)$, and $(t, x, y, z + L)$.

Note that the time coordinate is the same. Now consider two observers; observer A is at rest in this coordinate system (constant spatial coordinates), while observer B

moves in the x-direction with constant velocity v. A and B begin at the same event, and while A remains still, B moves once around the universe and comes back to intersect the worldline of A without ever having to accelerate (since the universe is periodic). What are the relative proper times experienced in this interval by A and B? Is this consistent with your understanding of Lorentz invariance?

PROBLEM 3: LORENTZ VECTORS *(8 pts)*

(a) *(2 pts)* Show that for two timelike separated events, there is an inertial frame in which $\Delta t \neq 0, \Delta\vec{x} = 0$.

(b) *(2 pts)* Show that for two spacelike separated events, there is an inertial frame in which $\Delta t = 0, \Delta\vec{x} \neq 0$.

(c) *(2 pts)* Show that the sum of any two orthogonal spacelike vectors is spacelike.

(d) *(2 pts)* Show that a timelike vector and a null vector cannot be orthogonal.

PROBLEM 4: ORDERING OF EVENTS *(10 pts)*

Carroll Problem 1.3: Three events, A, B, C, are seen by observer $\mathcal{O}$ to occur in the order ABC. Another observer, $\tilde{\mathcal{O}}$, sees the events to occur in the order CBA. Is it possible that a third observer sees the events in the order ACB? Support your conclusion by drawing a spacetime diagram.

PROBLEM 5: LORENTZ INDEX MANIPULATION *(11 pts)*

Carroll Problem 1.7: Imagine we have a tensor $X^{\mu\nu}$ and a vector V^μ, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \qquad V^\mu = (-1, 2, 0, -2)\ .$$

Find the components of

(a) *(2 pts)* $X^\mu{}_\nu$
(b) *(2 pts)* $X_\mu{}^\nu$
(c) *(2 pts)* $X^{(\mu\nu)}$
(d) *(2 pts)* $X_{[\mu\nu]}$
(e) *(1 pt)* $X^\lambda{}_\lambda$
(f) *(1 pt)* $V^\mu V_\mu$
(g) *(1 pt)* $V_\mu X^{\mu\nu}$

[Note that $X^{(\mu\nu)}$, $X^{[\mu\nu]}$ denote tensors that are *symmetrized* and *antisymmetrized* on the indices,

$$X^{(\mu\nu)} \equiv \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu})$$

$$X^{[\mu\nu]} \equiv \frac{1}{2}(X^{\mu\nu} - X^{\nu\mu})$$

as described in Carroll Eqs. (1.79) and (1.80). For parts (a)–(d), you can display your answer as a 4×4 matrix, and the answer to (g) can be displayed as a column or row vector.]

PROBLEM 6: RELATIVISTIC FORM OF E&M *(15 pts)*

(a) *(5 pts)* Using the tensor transformation law applied to $F_{\mu\nu}$, show how the electric and magnetic field 3-vectors $\vec{E}$ and $\vec{B}$ transform under a boost along the z-axis. [This is Carroll Problem 1.10(b).]

(b) *(10 pts)* Consider a uniform electric field $\vec{E} = E\hat{z}$, as would be produced by a very large stationary two-dimensional sheet of uniform (positive) charge density. Describe this field as seen by an observer moving at velocity $\vec{v} = v\hat{x}$ relative to the initial frame. Check that the resulting Lorentz-transformed field agrees with your expectation from the rules of electromagnetism in the second observer's frame. Give a qualitative description of the motion of a negatively charged particle that begins at rest in the second observer's frame. Explain this motion as seen by an observer in the first frame.