MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — February 22, 2018
Prof. Alan Guth

PROBLEM SET 3

DUE DATE: Thursday, March 1, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem set covers metrics, geodesics, and the equivalence principle, primarily following material presented in lecture. Some short segments in Carroll are closely related to some of the material covered in lectures: §2.5 describes aspects of the metric, general coordinate transformations and local inertial coordinates, and the extremal trajectory derivation of the geodesic equation is given in the last part of §3.3, beginning at Eq.(3.44). The Equivalence Principle is discussed in §2.1.

MAXIMUM GRADE: This problem set has a total of 45 points.

PROBLEM 1: ROTATIONALLY SYMMETRIC EMBEDDING SURFACES *(19 pts)*

Consider a surface described by embedding the plane $\mathbb{R}^2$ into Euclidean 3-space through the map

$$(r, \theta) \rightarrow (r, \theta, z = f(r))$$

where r, θ are standard polar coordinates and $f(r)$ describes a rotationally invariant height function for the embedding into the third dimension.

(a) [2 pts] Compute the metric $g_{ij}(r, \theta)$ for an arbitrary height function $f(r)$.

(b) [2 pts] Repeat the computation for an embedding in Euclidean coordinates through the map $(x, y) \rightarrow (x, y, z = f(x, y))$, and use a general coordinate transformation to check that your answers from the first two parts are in agreement.

(c) [2 pts] Compute the metric in the case $f(r) = \frac{1}{2}r^2$, describing a paraboloid surface embedded in three dimensions.

(d) [3 pts] Compute the Christoffel coefficients for the metric on the paraboloid surface in coordinates (r, θ).

(e) [2 pts] Compute the metric in the case $f(r) = ar$, describing a cone, where $a = \tan\phi$ is an arbitrary real parameter.

(f) [2 pts] Compute the Christoffel coefficients in the case of the cone.

(g) [3 pts] For the conical geometry with parameter a, compute a geodesic that begins at $(r, \theta) = (\cos\phi, 0)$ with initial velocity $dr(\lambda)/d\lambda = 0$ and show that it satisfies the geodesic equation. (Hint: find a coordinate system in which the metric is simpler).

(h) [3 pts] For what value of the parameter a does the geodesic from the previous part cross itself at a right angle?

PROBLEM 2: AN INERTIAL COORDINATE SYSTEM IN THE NEWTONIAN LIMIT *(11 pts)*

Consider the spacetime metric with components

$$g_{00} = -(1 + az), \quad g_{0i} = 0, \quad g_{ij} = \delta_{ij}$$

in a region near the origin $O = (0, 0, 0, 0)$ where $az \ll 1$, and a is a constant.

(a) [3 pts] Compute the nonzero Christoffel coefficients of this metric.

(b) [3 pts] Assuming that a particle is initially at the origin with velocity $v \ll 1$, compute the leading term in the geodesic equation describing the acceleration of the particle at $t = 0$.

(c) [5 pts] Find a coordinate transformation

$$x^\mu = A^\mu{}_{\nu'} x^{\nu'} + \frac{1}{2} C^\mu{}_{\nu'\sigma'} x^{\nu'} x^{\sigma'} + \mathcal{O}(x^3)$$

giving a local *inertial coordinate system* in which $g_{\mu'\nu'}(O) = \eta_{\mu'\nu'}$ and $\partial_{\lambda'} g_{\mu'\nu'}(O) = 0$. ($x^\mu$ is the old coordinate, $x^{\mu'}$ is the new, locally inertial coordinate). Explain the physical meaning of this coordinate system.

PROBLEM 3: ROTATING COORDINATES *(10 pts)*

Flat space with metric $g_{\mu\nu} = \eta_{\mu\nu}$ can be described in static cylindrical coordinates with

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2.$$

Consider a coordinate system $x^{\mu'}$ that is rotating about the z axis with angular velocity ω relative to the static cylindrical coordinate system.

(a) [4 pts] Compute the metric in the rotating coordinate frame.

(b) [6 pts] Compute the Christoffel connection and geodesic equation in the rotating coordinate frame. Indicate how the terms in the geodesic equation relate to the familiar centripetal and Coriolis forces when $\omega r \ll 1$.

PROBLEM 4: GEODESIC PARAMETERIZATION *(5 pts)*

Show directly that the quantity

$$A \equiv g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

has a constant value along any solution to the geodesic equation. You may use either the standard form of the geodesic equation,

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0\,,$$

or the "primitive" form,

$$\frac{d}{d\lambda}\left[g_{\mu\nu} \frac{dx^\nu}{d\lambda} \right] = \frac{1}{2} \frac{\partial g_{\rho\sigma}}{\partial x^\mu} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}\ .$$

Recall that when we derived the geodesic equation, we assumed that the solution was parameterized so that $A =$ constant, so this result shows that our formalism is consistent. We do not have to separately impose the property that $A =$ constant, but instead we can be assured that if we find a solution to the geodesic equation, in either form shown above, then it will automatically have this property. The result also shows that any solution to the geodesic equation that is null, timelike, or spacelike at one point in the trajectory has the same property everywhere.