MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — March 3, 2018
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PROBLEM SET 4*

DUE DATE: Thursday, March 8, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem set covers aspects of the Schwarzschild metric, primarily following material presented in lecture. The last problem is a guided derivation of the geodesic equation from the equivalence principle, with emphasis on the meaning of an affine parameter.

MAXIMUM GRADE: This problem set has a total of 60 points.

PROBLEM 1: COUNTING TENSORS *(5 pts)*

In lecture we derived the fact that if $S_{\mu_1 \cdots \mu_R}$ is fully symmetric in its indices, and each index can take on D different values, then the number of independent components is

$$N = \frac{(R + D - 1)!}{R!(D-1)!} .$$

Suppose that $A_{\mu_1 \cdots \mu_R}$ is fully antisymmetric in its indices, with each index again taking on D different values. How many independent components does $A_{\mu_1 \cdots \mu_R}$ have?

PROBLEM 2: CLOCKS ON EARTH AND IN SPACE *(10 pts)*

In this problem we consider the effects of relativity on a clock on the surface of a planet, using an idealized version of the Earth as an interesting example. Assume that the planet is perfectly spherical, and floats in empty space in a Schwarzchild geometry created by its own mass distribution. The planet has mass M, and the radius as measured by the Schwarzschild coordinates is R. The planet rotates in time T as measured by an observer at infinity who is at rest relative to the Schwarzschild coordinates. (Here we will ignore the effect of the rotation on the metric, which we will learn about later in the course.) Now consider a clock (C) that lies on the surface of the planet at a point on the equator.

(a) [2 pts] Compute the time $T_{\rm C,SR}$ measured by the clock C after a single rotation, incorporating only the effects of special relativity. (That is, assuming that the clock is moving along a circular path of radius R in Minkowski space and completes a circuit around the circle in time T as measured by a stationary observer in Minkowski space.) To lowest order in R/T, by how much does this time differ from T (i.e., find the time difference $\Delta T_{\rm C,SR} \equiv T_{\rm C,SR} - T$)? Using $R = R_\oplus = 6378.1$ km for the radius of the Earth and $T = 23$ hours, 56 minutes, and 4.09 seconds for the length of a sidereal day, what is the numerical value of this time difference?

*A preliminary version of this problem set was posted on March 2, 2018. This final version includes one more problem, Problem 5.

(b) [2 pts] Compute the time $T_{\rm C,GTD}$ measured by the clock C after a single rotation of the planet, incorporating only the effect of gravitational time dilation from the metric component g_{tt} in the Schwarzschild metric. (That is, calculate the time measured by a clock that travels relative to the surface of the planet, remaining stationary in the Schwarzschild coordinates.) To lowest order in GM/R, by how much does this time differ from T (i.e., find the time difference $\Delta T_{\rm GTD} \equiv T_{\rm C,GTD} - T$)? Using $M_\oplus = 5.9724 \times 10^{24}$ kg as the mass of the Earth, and $G = 6.6741 \times 10^{-11}\ {\rm m^3\,kg^{-1}\,s^{-2}}$, and $c = 2.998 \times 10^8$ m/s, what is the numerical value of the time difference?

(c) [2 pts] Compute the proper time $T_{\rm Sch}$ measured by the clock C along the prescribed trajectory in the full Schwarzschild metric. To lowest order in the small quantities indicated in parts (a) and (b), by how much does your result differ from the sum of the results from (a) and (b)? That is, compute

$$\Delta T_{\rm (c)} \equiv (T_{\rm Sch} - T) - \Delta T_{\rm (a)} - \Delta T_{\rm (b)}$$

to lowest nonvanishing order in the small quantities R/T and GM/R. Here $\Delta T_{\rm (a)}$ and $\Delta T_{\rm (b)}$ refer to the (correct) answers to parts (a) and (b), which means that they are each expanded only through the lowest order in their respective small quantities. Numerically, for the case of the Earth, how big is this difference.

(d) [2 pts] Compute the difference in time measured by a clock on Earth at sea level on the equator and a second clock on a mountain at 2000 m, also on the equator, over a period of one rotation of the planet. I.e., compute

$$\Delta T_{(d)} = T_{\rm C,mountain} - T_{\rm C,sea\ level}$$

to lowest nonvanishing order in small quantities, and evaluate it numerically for the Earth.

(e) [2 pts] Consider a geosynchronous satellite in a circular orbit above the equator, with an altitude of 42,164 km. Compute the time $T_{\rm C,sat}$ measured by a clock on the satellite during one rotation of the planet. Compute the time differences

$$\Delta T_1 \equiv T_{\rm C,sat} - T$$

and

$$\Delta T_2 \equiv T_{\rm C,sat} - T_{\rm C,sea\ level}\ .$$

In each case, give an expression to lowest nonvanishing order in the small quantities, and evaluate numerically for the Earth.

PROBLEM 3: GRAVITATIONAL RED SHIFT *(10 pts)*

In this problem we consider the gravitational redshift from two perspectives.

(a) [4 pts] The gravitational redshift can be seen as a direct consequence of Einstein's equivalence principle. Consider a photon that travels upward by a distance d in Earth's near-surface gravitational field. Compute its redshift $\Delta\lambda/\lambda$ by computing the time Δt needed for light to travel the distance d, the change in velocity Δv that an inertial observer would experience compared to a stationary observer in the time Δt, and the Doppler shift when you boost into the stationary observer's frame. Express your answer in terms of g, d, and c.

(b) [6 pts] Now consider the same situation in general relativity. Compute the redshift by determining the relationship between the frequencies of emitted and received light at two heights differing by d. Show that in the Newtonian limit of Earth's near-surface field you reproduce the results of part (a).

PROBLEM 4: RADIAL SCHWARZSCHILD GEODESICS *(15 pts)*

Consider a particle falling radially inward (*i.e.,* at fixed θ, ϕ) along a geodesic in a Schwarzschild geometry with Schwarzschild radius $R_* = 2GM$. The particle has total energy $E = 1$, where

$$E = \left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} .$$

Thus, the particle has the same energy as it would have if it were at rest at infinity. As $t \to -\infty$, such a geodesic has $r \to \infty$ and $dt/d\tau \to 1$.

(a) [8 pts] Use conservation of energy to compute $dt/d\tau$ and $dr/d\tau$ as functions of $r(\tau)$.

(b) [7 pts] Integrate to solve for $r(\tau)$. Compute the proper time difference experienced by the particle between the moments when it is at radius $r = 2R_*$ and at radius $r = R_*$. In class we learned that an observer at fixed radius r will see an infalling object approach the horizon but never reach it. Does this calculation indicate that an infalling observer will have a different experience?

PROBLEM 5: DERIVING THE GEODESIC EQUATION FROM THE EQUIVALENCE PRINCIPLE *(20 pts)*

In lecture some time ago, we derived the geodesic equation from the assumption that the geodesic path is a stationary point of the path length. This is perfectly well-defined for spacelike or timelike geodesics, but for lightlike geodesics the path length is zero and its first order variation is singular. So here we will derive the geodesic equation by a method that is valid for all geodesics, and which also lends more physical insight into the equation. At each point in spacetime we can find a coordinate system that is locally inertial, and a geodesic is defined by the statement that its acceleration, in the locally inertial coordinates, is zero. The treatment here roughly follows that of Steven Weinberg's *Gravitation and Cosmology* (1972), pp. 70–77, although the notation will follow that of Carroll. (You should feel free to look at Weinberg's book, but I will try to exlain the setup well enough so that it will not be necessary.)

The starting point is to imagine that we start with some global coordinate system with coordinates x^μ, and for each spacetime point X^μ we construct a locally inertial coordinate system $x_X^{\mu'}(x)$. Here the subscript X denotes that spacetime point at which the coordinate system is locally inertial, meaning that $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ and $\partial_{\lambda'} g_{\mu'\nu'} = 0$. The argument x is the spacetime point that is being described by the coordinate vector $x^{\mu'}$ in the locally inertial system. Note that x must be in the vicinity of X, but of course it need not be equal to X. Both X and x can be described concretely in the original global coordinate system.

For spacelike or timelike trajectories, there is a natural choice for the path parameter λ, which can be taken to be equal to (or at least proportional to) the proper length (spacelike case) or proper time (timelike case). The path can then be described in global coordinates as $x^\mu(\lambda)$, and the vanishing of the locally inertial acceleration can be described by

$$\left.\frac{\mathrm{d}^2 x_X^{\mu'}\big(x(\lambda)\big)}{\mathrm{d}\lambda^2}\right|_{X=x(\lambda)} = 0 \ . \tag{1}$$

Note that the point X, the center of the locally inertial coordinate system, is held fixed when the differentiation with respect to λ is carried out, since it is the acceleration in this one coordinate system that should vanish. To simplify the notation, from here we will stop writing the restriction $X = x(\lambda)$, remembering that the primed coordinates will always refer to the locally inertial coordinates of the particle at position λ along its path, and that derivatives are all taken with a fixed choice of the primed coordinate system.

For lightlike trajectories, however, the invariant distance along the path is zero, so it cannot be used as a parameter. So we begin with some completely arbitrary parameterization $x^\mu(\sigma)$. Then the statement that the particle moves without acceleration in the locally inertial coordinates becomes the statement that the particle moves along a straight line in spacetime in the locally inertial coordinates. Since the parameterization was arbitrary, however, we cannot assume that it moves along this line at a fixed rate with respect to σ. Thus, with respect to σ, the trajectory can accelerate, but to stay on the line the acceleration must be proportional to the velocity:

$$\frac{\mathrm{d}^2 x_X^{\mu'}\big(x(\sigma)\big)}{\mathrm{d}\sigma^2} = \beta(\sigma)\frac{\mathrm{d}x_X^{\mu'}\big(x(\sigma)\big)}{\mathrm{d}\sigma} \ , \tag{2}$$

for some function $\beta(\sigma)$ that depends on our choice of parameterization.

(a) [5 pts] Suppose that we have chosen an arbitrary parameterization $x^\mu(\sigma)$ and calculated $\beta(\sigma)$. Show that if we find a new parameterization $\lambda(\sigma)$ for which

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2} + \beta(\sigma)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\right)^2 = 0 \,, \tag{3}$$

then Eq. (1) will hold for the new parameterization $x^\mu(\lambda)$.

Any parameter for which Eq. (1) holds is called an *affine parameter*. For spacelike or timelike geodesics, any affine parameter is a linear function (i.e., $\lambda = c_1 + c_2\tau$) of the proper length or proper time. For lightlike trajectories, any affine parameter is a linear function of any other affine parameter.

(b) [5 pts] Show that Eq. (1) implies that the trajectory $x^\mu(\lambda)$ satisfies the equation

$$\frac{\mathrm{d}^2 x^\tau}{\mathrm{d}\lambda^2} = -\Gamma^\tau_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \,, \tag{4}$$

where

$$\Gamma^\tau_{\nu\rho} \equiv \frac{\partial x^\tau}{\partial x^{\mu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\rho} \,. \tag{5}$$

We will be showing that $\Gamma^\tau_{\nu\rho}$ is equal to the usual Christoffel connection, but for now it is defined by this equation. Since Eqs. (1) and (4) are equivalent, we can equally well say that λ is an affine parameter if and only if Eq. (4) is satisfied.

(c) [5 pts] Using the usual rules of coordinate transformations, the metric $g_{\mu\nu}(x)$ in the global coordinate system can be expressed in terms of the metric of the locally inertial coordinate system by

$$g_{\mu\nu}(x) = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu} g_{\mu'\nu'}(x), \tag{6}$$

where of course, for $x = X$, $g_{\mu'\nu'}(x) = \eta_{\mu'\nu'}$ and $\partial_{\rho'} g_{\mu'\nu'}(x) = 0$. Show that this implies, at $x = X$, that

$$\partial_\rho g_{\mu\nu}(x) = \Gamma^\tau_{\rho\mu}\, g_{\tau\nu} + \Gamma^\tau_{\rho\nu}\, g_{\mu\tau} \,, \tag{7}$$

where $\Gamma^\tau_{\rho\mu}$ is given Eq. (5).

(d) [5 pts] Use Eq. (7) to show that $\Gamma^\tau_{\mu\nu}$ is the usual Christoffel connection,

$$\Gamma^\tau_{\mu\nu} = \frac{1}{2} g^{\tau\rho}[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}] \,. \tag{8}$$

Hint: try writing out Eq. (7) with different orderings of ρ, μ, and ν. See if you can find a linear combination with only one term on the right-hand side, and then solve for Γ.