MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — March 9, 2018
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PROBLEM SET 5

DUE DATE: Thursday, March 15, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: After one problem on the Schwarszchild metric, this problem set covers some basic aspects of manifolds and tangent bundles, primarily following material presented in lecture. Under "Lecture Notes" on the course website, I have posted notes on "Geometry for General Relativity" that were written two years ago by Sam B. Johnson. These notes are a good summary of the relevant notation, definitions, and concepts. The material relevant to this problem set can be found on pp. 1-8 of Johnson's notes, and in Chapter 2 of Carroll's book.

MAXIMUM GRADE: This problem set has a total of 65 points.

PROBLEM 1: MESSAGES TO THE INFALLING OBSERVER *(10 pts)*

It is interesting to ask what happens if light signals are sent after an object that is falling into a black hole. Consider in particular an infalling object that is traveling on the the radial trajectory that was found in Problem 4 of Problem Set 4. Recall that this trajectory described the path of an infalling object that in the asymptotic past, $t \to -\infty$, was at rest at radius $r \to \infty$. As seen from the point of view of a stationary observer outside the black hole, the object approaches the black hole horizon for all time, without ever reaching it.

Imagine that such a radially infalling object passes by an observer who is stationary at a distance of one au (astronomical unit) from a solar mass black hole. The observer shines a (powerful) flashlight after the infalling object. During the period before the infalling object crosses the horizon, does it receive a finite or infinite number of photons, and are they redshifted or blueshifted? If the answer is finite, determine the total time (as measured by the external observer with the flashlight) for which light that is emitted by the flashlight will be received by the infalling object before it passes the black hole horizon. (This time is of course proportional to the number of photons that will be received.)

PROBLEM 2: A ONE-CHART ATLAS *(10 pts)*

Carroll Problem 2.1: Just because a manifold is topologically nontrivial doesn't necessarily mean it can't be covered with a single chart. In contrast to the circle S^1, show that the infinite cylinder $\mathbb{R} \times S^1$ can be covered with just one chart, by explicitly constructing the map. [Hint (not from Carroll): notice that a curve which circles the S^1 once — i.e. a circle around the cylinder — cannot be contracted to a point. Does this say something about the region in $\mathbb{R}^2$ that must be used for the chart?]

PROBLEM 3: NOWHERE-VANISHING SECTIONS OF VECTOR BUNDLES *(15 pts)*

For a vector bundle B over a manifold M defined by the projection $\pi : B \to M$ with fibers $F_x = \pi^{-1}(x)$, a section is a *continuous* map

$$\sigma : M \to B \tag{3.1}$$

where for each $x \in M$, $\sigma(x) \in F_x$. The section is nowhere-vanishing if $\sigma(x) \neq 0$.

(a) [5 pts] Define the Möbius band as an $\mathbb{R}$-bundle over S^1 via the following two coordinate charts

$$\begin{aligned} U_1 &\equiv \{(\theta, x), -\pi < \theta < \pi, x \in \mathbb{R}\} \,, \\ U_2 &\equiv \{(\theta', x'), -\pi < \theta' < \pi, x' \in \mathbb{R}\} \,, \end{aligned} \tag{3.2}$$

with the transition function (local coordinate transformation on charts)

$$\phi_2 \circ \phi_1^{-1} : (U_1 \setminus \{0, x\}) \to (U_2 \setminus \{0, x\}) \,, \tag{3.3}$$

defined by

$$(\theta', x') = \phi_2 \circ \phi_1^{-1}(\theta, x) = \begin{cases} (\pi - \theta, x), \ \theta > 0 \\ (-\pi - \theta, -x), \ \theta < 0 \,. \end{cases} \tag{3.4}$$

(Here the symbol $\equiv$ means "defined to be," and the symbol $\setminus$ means "excluding".) Prove that this bundle has no nowhere-vanishing sections.

(b) [5 pts] Consider a two-dimensional torus T^2, which can be described as $\mathbb{R}^2$ with the identifications $(x, y) \sim (x+n, y+m)$ for any $n, m \in \mathbb{Z}$. Define the *fundamental domain* as the coordinates in the range $0 \leq x, y \leq 1$, with $x = 0$ identified with $x = 1$, and $y = 0$ identified with $y = 1$. More formally, we define the fundamental domain of T^2 as

$$\begin{aligned} T^2_{\text{fd}} \equiv \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ with } x = 0 \text{ identified with } x = 1, \\ y = 0 \text{ identified with } y = 1\} \,. \end{aligned} \tag{3.5}$$

A chart is required to be an open subset of $\mathbb{R}^n$, which of course has no identifications, so T^2_{fd} cannot be taken as a chart. But we can construct charts by first introducing two ranges of x:

$$X_1 \equiv \left\{x : \tfrac{2}{10} < x < \tfrac{8}{10}\right\} \,, \tag{3.6}$$

$$X_2 \equiv \left\{x : 0 \leq x < \tfrac{3}{10} \text{ or } \tfrac{7}{10} < x \leq 1\right\} \,, \tag{3.7}$$

where $x = 0$ and $x = 1$ are identified. X_1 and X_2 are open regions whose union covers the x-range of T^2_{fd}. Similarly, we define

$$Y_1 \equiv \left\{y : \tfrac{2}{10} < y < \tfrac{8}{10}\right\} \,, \tag{3.8}$$

$$Y_2 \equiv \left\{y : 0 \leq y < \tfrac{3}{10} \text{ or } \tfrac{7}{10} < y \leq 1\right\} \,, \tag{3.9}$$

where $y = 0$ and $y = 1$ are identified. We can then define four charts U_{ij} on $T^2_{\rm fd}$, where i and j can each be either 1 or 2, indicating the ranges of x and y respectively. That is,

$$U_{ij} \equiv \left\{ (x, y) \in T^2_{\rm fd} : x \in X_i, y \in Y_j \right\} . \tag{3.10}$$

The U_{ij} then form an atlas of charts for $T^2_{\rm fd}$. The coordinate mappings $\phi_{ij} : U_{ij} \to V_{ij}$ can then be constructed by first defining

$$\begin{aligned} \psi_1(x) &= x \ , \\ \psi_2(x) &= \begin{cases} x & \text{if } 0 \le x < \frac{3}{10} \\ x - 1 & \text{if } \frac{7}{10} \le x \le 1 \ . \end{cases} \end{aligned}$$

We can then define ϕ_{ij} by

$$\phi_{ij}(x, y) = \psi_i(x)\psi_j(y) \ . \tag{3.11}$$

Finally, the question: in terms of this system of charts, construct an example of a nowhere-vanishing section of the tangent bundle of T^2.

(c) [5 pts] "Combing a hedgehog": Try to draw a nowhere-vanishing section of the tangent bundle of the 2-sphere S^2 (in charts if necessary). This would be a choice of tangent vector everywhere on the sphere with no vanishing points. Do you believe this is possible? If not, try to explain the obstruction using words and possibly pictures (you do not need to prove this mathematically). [Hint: Consider a smooth nowhere-vanishing tangent vector field on the "northern" region (say $\theta < 3\pi/4$), and a similar vector field in the "southern" region (say $\theta > \pi/4$). How do each of these vector fields behave as you go around the equator?]

PROBLEM 4: CHARTS FOR THE TORUS *(15 pts)*

Some years ago, an 8.962 professor discovered that the two-dimensional torus T^2 can be covered by a system of only three charts, and assigned the homework problem of finding such a system of charts. Some of the students did one better, finding a system of only two charts that can cover the two-dimensional torus. Here you are asked to find either a 3-chart atlas or a 2-chart atlas: your choice.

To describe your answer, begin with a definition of T^2 and its fundamental domain, as given in Problem 3(b). For each of the two or three charts, $i = 1, 2$ or maybe $i = 1, 2, 3$, describe

(1) U_i, the subset of T^2 covered by the chart, by giving the ranges of the T^2 coordinates x and y. Here x and y need not be in the fundamental domain;

(2) V_i, the subset of $\mathbb{R}^2$ of the chart, by giving the ranges of the chart coordinates x_i and y_i;

(3) the mapping $\phi_i : T^2 \to V_i$ which defines the chart, expressed explicitly as a function $(x_i, y_i) = \phi_i(x, y)$, where x and y are restricted to the fundamental domain.

In addition, choose one pair of charts, and give the form of the overlap function $\phi_j \circ \phi_i^{-1}$ as an explicit function $(x_j, y_j) = \phi_j \circ \phi_i^{-1}(x_i, y_i)$.

PROBLEM 5: OPERATIONS WITH VECTOR FIELDS *(15 pts)*

As discussed in class, a vector field V on a manifold M is a section of the tangent bundle, and can be represented as a differential operator that acts on any function f on M as

$$V f(x) \equiv V^\mu \frac{\partial}{\partial x^\mu} f(x).$$

The commutator $[U, V]$ of two vector fields U, V is another vector field that acts on a function f via

$$[U, V]f \equiv U(Vf) - V(Uf)\,.$$

(a) [5 pts] Show that the commutator of two vector fields U, V can be written as a vector field with components

$$[U, V]^\mu = U^\lambda \partial_\lambda V^\mu - V^\lambda \partial_\lambda U^\mu\,.$$

Show that this object transforms as a tensor under a general coordinate transformation $x^\mu \to x^{\mu'}$. (We will soon learn about covariant derivatives, which transform in general as tensors, but here you will show that the commutator transforms as a tensor even though the derivatives are not covariant derivatives.)

(b) [5 pts] If x^μ denotes the coordinates of a manifold, then $\partial/\partial x^\mu$ represents a vector field. The commutator of two such vector fields always vanishes. E.g.,

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$$

The converse of this statement is also true, at least locally. Compute the commutator of the two vector fields

$$U = \frac{\partial}{\partial u}, \qquad V = \frac{\partial}{\partial v} - v^2 \frac{\partial}{\partial u}\ ,$$

where u, v are coordinates, and show that it vanishes. Find a change of coordinates to a coordinate system where these two vector fields are derivatives with respect to the coordinates,

$$U = \frac{\partial}{\partial x}, \qquad V = \frac{\partial}{\partial y}.$$

(c) [5 pts] Show that any three smooth vector fields A, B, C satisfy the Jacobi identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0\,.$$