MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — March 16, 2018
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PROBLEM SET 6

DUE DATE: Thursday, March 22, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem set covers covariant derivatives and curvature, primarily following material presented in lecture. Relevant sections of Carroll are 3.1–3.4, 3.6, 3.7.

MAXIMUM GRADE: This problem set has a total of 70 points.

PROBLEM 1: WALD'S DEFINITION OF A TANGENT SPACE *(15 pts)*

Wald (Robert M. Wald, *General Relativity*, The University of Chicago Press, 1984) defines the tangent space of a point p in a manifold M by first defining $\mathscr{F}$ as the collection of C^∞ functions from M into $\mathbb{R}$. Then a tangent vector v at point $p \in M$ is a map $v : \mathscr{F} \to \mathbb{R}$ which (1) is linear and (2) obeys the Leibniz rule:

$$(1)\ \ v[af + bg] = av[f] + bv[g], \text{ for all } f, g \in \mathscr{F};\ a, b \in \mathbb{R}; \tag{1.1}$$

$$(2)\ \ v[fg] = f(p)v[g] + g(p)v[f]\quad . \tag{1.2}$$

Note that the definition allows $v[f]$ to be an arbitrary functional of f, subject only to Eqs. (1.1) and (1.2). In particular, the only indication that $v[f]$ depends on the behavior of $f(x)$ in the vicinity of p is the appearance of p in the Leibniz rule.

Our goal is to reconstruct Wald's argument that these conditions imply that, within a coordinate chart, the tangent space is spanned by the basis vectors $\partial/\partial x^\mu$, so the most general functional v can be written as

$$v[f] = v^\mu \left.\frac{\partial f(x)}{\partial x^\mu}\right|_{x=p}\ , \tag{1.3}$$

for an n-tuple of real numbers $(v^1, v^2, \ldots, v^n)$, where n is the dimension of M. This implies that Wald's definition is equivalent to the coordinate-based definition that we used in lecture, where the tangent space was defined as the space of first-order differential operators as in Eq. (1.3). You may look at Wald's argument if you wish, but we recommend that you first try the problem without looking at Wald's book.

Wald's text uses parentheses $v(f)$ for the argument of v, but we are using square brackets, which are frequently used to indicate the argument of a functional, a mapping from a function to a real number. A goal of this problem will be to simplify Wald's argument, at least from the viewpoint of most physicists, by using notation more familiar to physicists. Hence we will use a chart that contains the point p, and say that a point in the manifold can be called x, and is specified by its chart coordinates x^μ. In Wald's text, by contrast, a general point is indicated by q, and its coordinates are written as $x^\mu \circ \psi(q)$, where $\psi(q)$ maps q to the subset of $\mathbb{R}^n$ of the chart, and then x^μ maps from the point in $\mathbb{R}^n$ to the μ'th coordinate.

(a) [5 pts] Show that conditions (1) and (2) imply that if $h \in \mathscr{F}$ is a constant function (i.e., if for all x, $h(x)$ is equal to a constant c), then $v[h] = 0$. *Hint:* consider $v[h^2]$, and separately see what you can learn about it from conditions (1) and (2).

(b) [5 pts] Let the coordinates of p be x_p^μ, and consider any other point x^μ in the chart. Consider a linear path

$$\xi^\mu(t) = t(x^\mu - x_p^\mu) + x_p^\mu \ , \tag{1.4}$$

so $\xi(0) = x_p^\mu$ and $\xi(1) = x^\mu$. Starting from the identity

$$f(x^\mu) = f(x_p^\mu) + \int_0^1 \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} f\big(\xi^\mu(t)\big) \ , \tag{1.5}$$

show that $f(x^\mu)$ can be written as

$$f(x^\mu) = f(x_p^\mu) + (x^\mu - x_p^\mu)\, H_\mu(x) \ , \tag{1.6}$$

where $H_\mu(x)$ is a (C^∞) function of x with the property that

$$H_\mu(p) = \left.\frac{\partial f(x)}{\partial x^\mu}\right|_{x=p} \ . \tag{1.7}$$

Write an explicit equation for $H_\mu(x)$. You do not have to show that it is C^∞.

(For intuition, note that if $f(x^\mu)$ can be expanded in a Taylor series about x_p^μ,

$$f(x^\mu) = f(x_p^\mu) + (x^\mu - x_p^\mu) \left.\frac{\partial f(x)}{\partial x^\mu}\right|_{x=p} + \frac{1}{2}(x^\mu - x_p^\mu)(x^\nu - x_p^\nu) \left.\frac{\partial^2 f(x)}{\partial x^\mu \partial x^\nu}\right|_{x=x_p} + \ \ldots \ , \tag{1.8}$$

then Eq. (1.6) follows immediately, with

$$H_\mu(x) = \left.\frac{\partial f(x)}{\partial x^\mu}\right|_{x=p} + \frac{1}{2}(x^\nu - x_p^\nu) \left.\frac{\partial^2 f(x)}{\partial x^\mu \partial x^\nu}\right|_{x=p} + \ \ldots \ . \tag{1.9}$$

Thus Eq. (1.6) is essentially just a way of writing the Taylor series so that the first term is explicit, and the rest of the information is hidden in the behavior of $H_\mu(x)$ for $x^\mu \neq x_p^\mu$. It is important, however, that C^∞ functions do not necessarily have convergent Taylor series, but Eq. (1.6) is always valid if f is C^∞.)

(c) [5 pts] Calculate $v[f]$, where f is written as in Eq. (1.6), and show that it reduces to something of the form of Eq. (1.3). Explicitly justify each step in terms of Eqs. (1.1) and (1.2).

PROBLEM 2: COVARIANT DERIVATIVES AND LOCALLY INERTIAL FRAMES *(15 pts)*

In lecture I stated that a covariant derivative can be defined by the statement that in a locally inertial coordinate system, the covariant derivative is equal to the ordinary derivative. We implemented this definition indirectly, by noting that the definition implies that the covariant derivative of the metric must vanish (a property called metric compatibility), and we used this criterion to determine the form of the connection.

In this problem we will apply the locally-inertial-frame definition of the covariant derivative directly. We will assume that a region of the manifold is covered by coordinates x^μ, and that we have also found coordinates $\xi^{\nu'}(x)$ that are locally inertial coordinates at some point X_0. So, by the definition of locally inertial coordinates,

$$g_{\mu'\nu'} = \eta_{\mu'\nu'} \qquad \text{(at } X_0\text{)}, \tag{2.1}$$

and

$$\partial_{\rho'} g_{\mu'\nu'}(x') = 0 \qquad \text{(at } X_0\text{)}. \tag{2.2}$$

Since $g_{\mu\nu}$ is a tensor, it follows that in the original coordinates,

$$g_{\mu\nu} = \frac{\partial \xi^{\mu'}}{\partial x^\mu} \frac{\partial \xi^{\nu'}}{\partial x^\nu} \eta_{\mu'\nu'} \qquad \text{(at } X_0\text{)} \ , \tag{2.3}$$

and

$$\partial_\rho g_{\mu\nu}(x) = \left[\frac{\partial^2 \xi^{\mu'}}{\partial x^\rho \partial x^\mu} \frac{\partial \xi^{\nu'}}{\partial x^\nu} + \frac{\partial \xi^{\mu'}}{\partial x^\mu} \frac{\partial^2 \xi^{\nu'}}{\partial x^\rho \partial x^\nu} \right] \eta_{\mu'\nu'} \qquad \text{(at } X_0\text{)}. \tag{2.4}$$

(a) [5 pts] Assuming that the covariant derivative $\nabla_\mu V^\nu$ of the vector field V^ν transforms as a tensor, and assuming that in the locally inertial frame it is equal to the ordinary derivative, so

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial V^{\nu'}}{\partial \xi^{\mu'}} \ , \tag{2.5}$$

show that the covariant derivative in the original coordinate system can be written in the form

$$\nabla_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + F^\nu_{\mu\lambda} V^\lambda \ , \tag{2.6}$$

where $F^\nu_{\mu\lambda}$ is an expression in terms of partial derivatives of the coordinate transformation $x^\mu(\xi^{\nu'})$ and/or its inverse.

(b) [5 pts] Using the answer to part (a) and Eq. (2.4) for $\partial_\rho g_{\mu\nu}(x)$, find an expression for $F^\nu_{\mu\lambda}$ in terms of $g_{\mu\nu}$, its inverse, and its derivatives. Verify that $F^\nu_{\mu\lambda} = \Gamma^\nu_{\mu\lambda}$, the usual Christoffel symbol.

(c) [5 pts] Carry out the same exercise for a covariant (i.e., lower-indexed) vector field, to show that

$$\nabla_\mu V_\nu = \frac{\partial V_\nu}{\partial x^\mu} - \Gamma^\lambda_{\mu\nu} V_\lambda \ , \tag{2.7}$$

where $\Gamma^\lambda_{\mu\nu}$ is the usual Christoffel symbol, as in part (b). Again assume that the covariant derivative can be defined to be just the ordinary derivative in the locally inertial frame.1

PROBLEM 3: CURVATURE ON A TWO-SPHERE *(15 pts)*

Consider a sphere of radius A. Answer the following questions in whatever coordinate system is most convenient.

(a) [10 pts] Compute the metric, connection, Riemann curvature tensor, Ricci tensor, and Ricci scalar. Show that the Ricci scalar is constant.

(b) [5 pts] Compute the change in direction of a vector that is parallel transported around a closed counter-clockwise loop surrounding a region of area $A^2/10$ on the sphere using the Christoffel connection. The answer should be the same for any region of this area, but you are not asked to show that.

PROBLEM 4: NEGATIVE CURVATURE *(10 pts)*

Consider a 3-dimensional space given by the set of points $\{(x, y, z), x \in \mathbb{R}, y \in \mathbb{R}, z > 0\}$ with the metric

$$ds^2 = \frac{a}{z^2}(dx^2 + dy^2 + dz^2) \, . \tag{4.1}$$

Compute the metric, connection, Riemann curvature tensor, Ricci tensor, and Ricci scalar.

PROBLEM 5: CURVATURE AND THE SCHWARZSCHILD METRIC *(15 pts)*

Consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \, d\Omega^2 \ , \tag{5.1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \ . \tag{5.2}$$

(a) [10 pts] For this metric, calculate all nonvanishing components of the Riemann tensor $R^{\lambda}{}_{\sigma\mu\nu}$. You may use the affine connections

$$\begin{aligned} \Gamma^t_{tr} = \Gamma^t_{rt} = -\Gamma^r_{rr} &= \frac{GM}{r(r-2GM)}\,, \\ \Gamma^r_{tt} &= \frac{GM(r-2GM)}{r^3}\,, \\ \Gamma^r_{\theta\theta} &= -(r-2GM)\,, \\ \Gamma^r_{\phi\phi} &= -(r-2GM)\sin^2\theta\,, \\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= \frac{1}{r}\,, \\ \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta\,, \\ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \cot\theta\,. \end{aligned} \tag{5.3}$$

which are also given in Carroll as Eq. (5.52).

(b) [5 pts] Calculate the Ricci tensor, showing (hopefully) that it vanishes.