MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — March 23, 2018

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PROBLEM SET 7

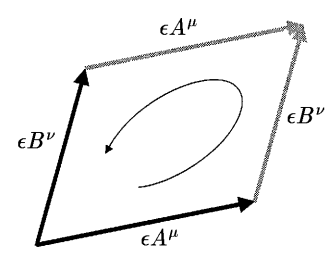
DUE DATE: Thursday, April 5, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem set focuses on two properties of curvature. Problem 1 concerns the relation between curvature and the parallel transport of a vector around an infinitesimal loop, discussed in Carroll's Sec. 3.6. The second two concern the geodesic deviation equation, which is discussed in Carroll's Sec. 3.10. Both of these topics will also be discussed in lecture on April 2, which will include a discussion of Eq. (2.4) below, which is not in Carroll's book.

MAXIMUM GRADE: This problem set has a total of 65 points.

PROBLEM 1: PARALLEL TRANSPORT AROUND AN INFINITESIMAL LOOP *(15 pts)*

Following Carroll's description on p. 121, we can imagine parallel transporting a vector V^μ around an infinitesimal loop, specified by two infinitesimal vectors. We will depart slightly from Carroll's notation by calling these two vectors ϵA^μ and ϵB^ν (rather than A^μ and B^ν), where ϵ is infinitesimal, and A^μ and B^μ are finite. We then imagine parallel transporting V^μ by first moving it along the vector ϵA^μ, then along ϵB^ν, then backward along ϵA^μ and backward along ϵB^ν, to return to the starting point, as shown in the figure below (adapted from Fig. 3.5 of Carroll):



Show that to second order in ϵ, the change in V^μ is given by

$$\delta V^\rho = -\epsilon^2 R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu \ , \tag{1.1}$$

where $R^\rho{}_{\sigma\mu\nu}$ is the Riemann curvature tensor, defined by the statement that

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \ , \tag{1.2}$$

for any vector field V^ρ, or equivalently as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \ . \tag{1.3}$$

Note that the sign shown in Eq. (1.1) is the opposite of the sign in Carroll's book, but I think the sign shown here is correct. Recall that a vector $V^\mu(\lambda)$ defined on a curve $x^\mu(\lambda)$ is said to be parallel transported along the curve if

$$\frac{DV^\mu}{\mathrm{d}\lambda} \equiv \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \nabla_\nu V^\mu = 0 \ , \tag{1.4}$$

which can be written more explicitly as

$$\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\left[\frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\sigma} V^\sigma\right] = \frac{\mathrm{d}V^\mu}{\mathrm{d}\lambda} + \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\Gamma^\mu_{\nu\sigma} V^\sigma = 0 \ . \tag{1.5}$$

PROBLEM 2: GEODESIC DEVIATION ON A TWO-SPHERE *(25 pts)*

In Problem 3 of Problem Set 6, you calculated the metric, connection, Riemann curvature tensor, Ricci tensor, and Ricci scalar on a two-sphere of radius A. In this problem we will consider a family of geodesics on the two-sphere of radius A, using them as an example to explore the meaning of the geodesic deviation equation. We will use the usual polar angles θ (angle from north pole) and ϕ (azimuthal angle) as coordinates.

Consider the family of geodesics $x^\mu(\phi_0, t)$ that start on the equator at longitude (azimuthal angle) ϕ_0, with an initial tangent vector $\mathrm{d}x^\mu/\mathrm{d}t$ pointing north. Take t to be the distance along the geodesics, starting with $t = 0$ on the equator.

(a) *(5 pts)* Consider in particular two trajecties in the family described above, one starting at $\phi_0 = 0$ and the other at $\phi_0 = \Delta\phi$. Calculate the geodesic separation h (i.e., the great circle distance) between the two trajectories as a function of t. Calculate this first exactly, and then imagine that $\Delta\phi$ is very small, giving a small angle approximation for $h(t)$ that is accurate to first order in $\Delta\phi$. Calculate the acceleration of the separation, $\mathrm{d}^2h/\mathrm{d}t^2$, using this small angle approximation.

(b) *(8 pts)* Using the notation of Carroll, Section 3.10, which we also used in lecture, consider the geodesic deviation equation,

$$A^\mu \equiv \frac{\mathrm{D}^2}{\mathrm{d}t^2} S^\mu \equiv \nabla_T(\nabla_T S^\mu) = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \ , \tag{2.1}$$

where

$$T^\mu \equiv \frac{\partial x^\mu}{\partial t} \quad \text{and} \quad S^\mu \equiv \frac{\partial x^\mu}{\partial s} \ , \tag{2.2}$$

where s parameterizes a family of geodesics $x^\mu(s,t) \in M$, so for each s, $x^\mu(s,t)$ is a geodesic with affine parameter t, and

$$\nabla_T \equiv T^\rho \nabla_\rho \quad \text{and} \quad \nabla_S \equiv S^\rho \nabla_\rho \,. \tag{2.3}$$

For the family of geodesics described in the preamble, taking $s = \phi$, calculate the two components of the right-hand side of Eq. (2.1).

(c) *(7 pts)* The left-hand side of the geodesic equation, $A^\mu \equiv \nabla_T(\nabla_T S^\mu)$ is also rather complicated, since each ∇ involves both derivative and connection terms. Expand the left-hand side, deriving expressions for A^θ and A^ϕ in terms of ordinary derivatives of S^μ with respect to t, plus other terms. Is the behavior that you found in part (a) consistent with the geodesic deviation equation?

(d) *(5 pts)* A^μ is referred to as the "relative acceleration of geodesics," but Carroll warns us that we should take this name with a grain of salt. Carroll doesn't explain the need for the salt, but the calculations in part (c) should give you an example of the complications: the calculation of A^μ involves derivatives of connections, which give nontrivial corrections which need not vanish even in a locally inertial frame. In lecture we developed a related equation which directly determines the 2nd derivative of the metric distance between infinitesimally separated geodesics:

$$\frac{d^2\ell}{dt^2} = \frac{1}{\ell} R_{\mu\nu\rho\sigma} S^\mu T^\nu T^\rho S^\sigma + \frac{1}{\ell^3}\left[S^2 \frac{\mathrm{D}S}{\mathrm{d}t} \cdot \frac{\mathrm{D}S}{\mathrm{d}t} - \left(S \cdot \frac{\mathrm{D}S}{\mathrm{d}t} \right)^2 \right] , \tag{2.4}$$

where $\ell \equiv \sqrt{g_{\mu\nu} S^\mu S^\nu}$, and for any two vectors, $A \cdot B \equiv A^\mu B_\mu$. Show that the second term on the right-hand side of Eq. (2.4) vanishes (either by direct calculation or by a more general argument), and evaluate the first term, as applied to the family of geodesics described in the preamble. Is the behavior you found in part (a) consistent with Eq. (2.4)?

PROBLEM 3: GEODESIC DEVIATION IN THE HYPERBOLIC PLANE *(25 pts)*

In Problem 4 of Problem Set 6, you calculated the connection, Riemann curvature tensor, Ricci tensor, and Ricci scalar for a space of constant negative curvature, described by the metric

$$ds^2 = \frac{a}{z^2}(dx^2 + dy^2 + dz^2) \,, \tag{3.1}$$

where $\{(x,y,z), x \in \mathbb{R}, y \in \mathbb{R}, z > 0\}$. In Problem 4 of Problem Set 2, you showed that geodsics in this space are semicircles centered around any point with $z = 0$. Here we will consider a one-parameter family of such geodesics, defined by initial conditions

$$x^\mu(s,0) = (0,0,Z+s) \tag{3.2}$$

and

$$\left.\frac{\partial x^\mu}{\partial t}(s,t)\right|_{t=0} = (1,0,0)\,. \tag{3.3}$$

(a) *(8 pts)* Using the same definitions as in Problem 1, calculate the right-hand side of the geodesic deviation equation, Eq. (2.1), for this family of geodesics, at the point $(s,t) = (0,0)$.

(b) *(7 pts)* Calculate the left-hand side of the geodesic deviation equation for this family of geodesics, again at $(s,t) = (0,0)$. Does your answer agree with what you found in part (a)?

Hint: The calculation involves taking partial derivatives of $x^\mu(s,t)$, but we only need the behavior of $x^\mu(s,t)$ near the origin, $(s,t) = (0,0)$. So we can describe $x^\mu(s,t)$ as a power series in s and t, and calculate the desired coefficients from the initial condition equations (3.2) and (3.3) and the geodesic equations of motion.

(c) *(10 pts)* Eq. (2.4) holds for any smooth family of geodesics $x^\mu(s,t)$, but it is much simpler for the cases in which the second term vanishes. The second term will not vanish for the family of geodesics described above. However, the term can be made to vanish by reparameterizing the family, replacing t by $\tilde{t}$, the distance along the geodesic curves, so that then $\tilde{T}^\mu \equiv \partial x^\mu/\partial\tilde{t}$ is a unit vector, with $\tilde{T}^2 = 1$. With this reparameterization, show that, at $(s,t) = (0,0)$, the second term in Eq. (2.4) vanishes, and evaluate the left-hand side and the first term on the right-hand side. Did you find that the equation holds?