MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity | April 13, 2018
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PROBLEM SET 9

DUE DATE: Thursday, April 19, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This is a short problem set with one problem about the Kruskal coordinate system, as discussed in lecture and in Chapter 5 of Carroll.

PROBLEM 1: KRUSKAL EXTENDED SCHWARZSCHILD SOLUTION *(35 pts)*

In the Kruskal coordinate system, the metric is given by

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM} (-\mathrm{d}T^2 + \mathrm{d}R^2) + r^2 \, \mathrm{d}\Omega^2 \,, \tag{1}$$

where the coordinates are T, R, θ, and ϕ, and r is defined implicitly by the relation

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} \,. \tag{2}$$

As usual,

$$\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \,. \tag{3}$$

Within the Kruskal description, Eq. (2) defines r as a function of T and R, but the same equation also describes how $T^2 - R^2$ is related to the Schwarzschild coordinate r. The Kruskal plane is divided into quadrants, as shown in Figure 1. Eqs. (1) and (2)

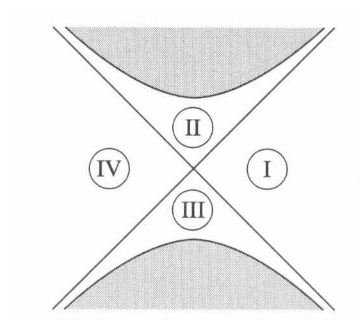

Figure 1: The quadrants of the Kruskal diagram.

define the Kruskal metric in all quadrants. We know that the metric defined in this way satisfies $R_{\mu\nu} = 0$ in quadrant I, but analyticity then implies that it satisfies $R_{\mu\nu} = 0$ everywhere, so it describes a full spacetime satisfying the vacuum Einstein equations.

The identification of the r in Eqs. (1) and (2) with the Schwarzschild coordinate r is also valid in all quadrants, since r^2 is the coefficient of $\mathrm{d}\Omega^2$ in both coordinate systems.

In the first quadrant, T and R can be related to the Schwarzschild coordinates t and r by

$$\begin{aligned} T &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \\ R &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right), \end{aligned} \tag{4}$$

which in turn implies that

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right). \tag{5}$$

in the first quadrant.

(a) [5 pts] Write the analogue of Eqs. (4) for quadrant II, the black hole interior. Your formulas should be consistent with Eqs. (1) and (2), and also with the standard form of the Schwarzschild metric for t, r, θ, and ϕ. The Schwarzschild coordinate $t(T, R)$ cannot be continuous throughout the (T, R) plane, since $t = \pm\infty$ on the 45° lines. Nonetheless, your answers to this part and the next should be consistent with the standard convention that $t(T, R)$ is defined to be as continuous as possible, in the sense that if $t(T, R)$ approaches $\pm\infty$ as a line is approached from one side, then it approaches $\pm\infty$ with the same sign as the line is approached from the other side. Hint: one way to proceed is to modify the derivation done in class (or in Carroll) to apply to the black hole interior, with $r < R_s$. Note that the equation for r_* would need to be modified.

(b) [5 pts] Write the analogue of Eqs. (4) for quadrants III and IV. In quadrant IV, does t increase in the upward or downward direction?

(c) [10 pts] Eq. (2) cannot be inverted analytically to give r as a function of T and R, but it can easily be solved numerically. To 10 significant figures, what is the value of $r/2GM$ corresponding the $T^2 - R^2 = 0.5$? Construct a plot of $r/2GM$ vs. $T^2 - R^2$ for $T^2 - R^2$ ranging from -5 up to its maximum value. (This should be a calculated plot, not a qualitative sketch. You may use any computer system that you find convenient.)

(d) [5 pts] Consider a particle that emerges from the white hole singularity at $(T, R) = (-1, 0)$, and then travels along the line $R = 0$ to $(T, R) = (1, 0)$, where it disappears into the black hole singularity. What is the total proper time τ experienced by this particle. (Hint: although the question was asked in the language of Kruskal coordinates, you may find it useful to think about Schwarzschild coordinates in answering it.)

(e) [10 pts] If one describes the Kruskal spacetime in terms of slices of constant T, it appears as two separate spacetimes that are joined temporarily by a wormhole, or Einstein–Rosen bridge, as depicted in Carroll's Figure 5.15, shown here as Figure 2.

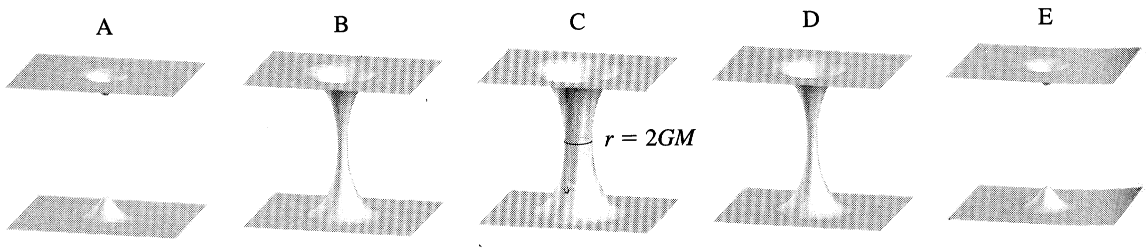

Figure 2: Spacelike slices of the Kruskal spacetime.

At what value of T does the Einstein–Rosen bridge appear? At what value of T does it disappear? Let $r(\tau)$ denote the radius (i.e., the value of r) of the thinnest part of the Einstein–Rosen bridge as a function of a time variable τ, where τ is the proper time as measured on a clock that appears and starts from $\tau = 0$ when the Einstein–Rosen bridge appears, and is always located at the smallest value of r, and at the same value of θ and ϕ. Use numerical techniques to construct a plot of $r/2GM$ vs. $\tau/2GM$. (The hint from the previous part may also be useful here.)