MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity April 21, 2018
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PROBLEM SET 10*

DUE DATE: Thursday, April 26, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem set covers the Tolman-Oppenheimer-Volkoff equation, as discussed in Carroll's Chapter 5, and as will be discussed in lecture on Monday April 23. Part (a) of the problem requires some basic quantum theory, to calculate the energy density and pressure of a degenerate gas of non-interacting neutrons. Nikhil Raghuram will discuss this subject in recitations on Friday and Monday. In addition, I will post a section from Griffith's **Introduction to Quantum Mechanics** about this subject. The second problem concerns Kruskal coordinates, as discussed in lecture and in Carroll, Section 5.7

MAXIMUM GRADE: This problem set has a total of 50 points.

PROBLEM 1: SOLVING THE TOV EQUATION FOR A DEGENERATE NEUTRON GAS *(30 pts)*

Consider a neutron star composed of a degenerate gas of non-interacting neutrons, which obey Fermi statistics. We will assume a central density which is a constant X times nuclear density, which we take as

$$\rho_N = \frac{1}{3}\frac{8\pi m_N^4}{(2\pi\hbar)^3} \; . \tag{1.1}$$

(a) [10 pts] Express the energy density and pressure in terms of the Fermi momentum k_F. This gives an equation of state $p(\rho)$ for the neutron matter, since ρ determines k_F, which in turn determines p. Remember that the neutron has two spin states, and use relativistic mechanics. (This of course is quantum mechanics, and not general relativity, but it is probably something that you know how to do. If not, review your quantum mechanics or ask at office hours.)

(b) [8 pts] For convenience, define dimensionless variables

$$\tilde{\rho} \equiv \frac{\rho}{\rho_N} \; , \quad \tilde{p} = \frac{p}{\rho_N} \; , \tag{1.2}$$

$$\tilde{k} = k_F/m_N \; , \quad \tilde{r} = r\sqrt{G\rho_N} \; , \quad \tilde{m}(\tilde{r}) = m(\tilde{r})G\sqrt{G\rho_N} \, . \tag{1.3}$$

*A preliminary version of this problem set was posted on April 19, 2018. This final version includes one more problem, Problem 2.

Use the Tolman-Oppenheimer-Volkoff equation to derive an equation for

$$\frac{\mathrm{d}\tilde{k}}{\mathrm{d}\tilde{r}} \, . \tag{1.4}$$

Recall that the TOV equation is given by

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]} \, . \tag{1.5}$$

where

$$m(r) = 4\pi \int_0^r \rho(r')\, r'^2 \, \mathrm{d}r' \, . \tag{1.6}$$

(c) [12 pts] For the special cases of $X = 0.5, 1, 2,$ and 4, integrate the TOV equations to find the mass and radius of the star. Express the mass in terms of solar masses, and express the radius in kilometers. (Hint: we recommend using $\tilde{k}(\tilde{r})$ as the function to integrate, since it is easier to express $\tilde{P}$ as a function of $\tilde{k}$ than vice versa. One complication is that the integration does not extend to some fixed maximum value of $\tilde{r}$, but rather should stop when $\tilde{k}$ reaches zero. If you use Mathematica, you should be aware that the "Method" called "EventLocator" is designed to handle exactly this sort of situation.)

PROBLEM 2: GEODESIC EQUATIONS IN KRUSKAL COORDINATES *(20 pts)*

The Kruskal metric is given by

$$\mathrm{d}s^2 = \frac{4R_S^3}{r} e^{-r/R_S}(-\,\mathrm{d}T^2 + \mathrm{d}R^2) + r^2 \, \mathrm{d}\Omega^2 \, . \tag{2.1}$$

where $R_S = 2GM$,

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \; \mathrm{d}\phi^2 \, , \tag{2.2}$$

and

$$\left(1 - \frac{r}{R_S}\right) e^{r/R_S} = T^2 - R^2 \, . \tag{2.3}$$

(a) [5 pts] Although Eq. (2.3) cannot be solved explicitly to determine $r(T, R)$, it is nonetheless possible to find analytic expressions for

$$\frac{\partial r}{\partial R} \equiv \left.\frac{\partial r}{\partial R}\right|_T \quad \text{and} \quad \frac{\partial r}{\partial T} \equiv \left.\frac{\partial r}{\partial T}\right|_R \, . \tag{2.4}$$

These expressions should depend on R, T, and r, as well as the constant R_S. Find these expressions.

(b) [10 pts] Find the geodesic equation for R for the case of radial motion; that is, find an expression for $\ddot{R}$, where overdots denote derivatives with respect to proper time τ. This expression should depend on R, T, $\dot{R}$, $\dot{T}$, r, and the constant R_S. Show that when $\dot{R} = 0$, the equation reduces to

$$\ddot{R} = \frac{R}{4R_S^2}\left(1 + \frac{R_S}{r}\right) . \tag{2.5}$$

(c) [5 pts] Suppose that a particle is instantaneously at rest in Kruskal coordinates, so $\dot{R} = \dot{\theta} = \dot{\phi} = 0$. If $R > 0$, is the acceleration (i.e., $\ddot{R}$) in the positive or negative R direction? If $\dot{R} \neq 0$, is it possible for the acceleration to be in the opposite direction?