MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — April 27, 2018
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PROBLEM SET 11

DUE DATE: Thursday, May 3, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: The first problem concerns cosmic strings, and is an exercise in solving Einstein's equations, understanding symmetries, parallel transport around closed loops, and conical geometry. You may want to review the discussion of conical geometry in the solutions to Problem 1, parts (e)-(h), of Problem Set 3. Problems 2 and 3 concern Killing vectors, and Problem 4 concerns volume integrals in curved space.

MAXIMUM GRADE: This problem set has a total of 70 points.

PROBLEM 1: COSMIC STRINGS *(30 pts)*

Cosmic strings are hypothetical massive stringlike objects that may have formed in the early universe. They are predicted by many (but not all) beyond-the-standard-model quantum field theories, where they arise as topologically stable twists in some of the fields. They can also arise in string theory, either as fundamental strings or as D-branes with one large (i.e., noncompactified) dimension. In either case, their thickness is so small that they can be treated for most purposes (including this problem) as one-dimensional objects. The metric for a single infinitely long straight cosmic string is static, invariant under rotation about the string, and also invariant under translation along its direction. These correspond to three Killing vector fields for the metric. If we ignore the coordinate that describes translations along the string, the geometry is equivalent to the metric around a massive particle in (2+1) dimensions (i.e., two space and one time dimensions). Assume that we have such a metric, which takes the form in the reduced 3D coordinate system

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \tag{1.1}$$

(with an additional term $+F(r)\,\mathrm{d}z^2$ for the full 4D cosmic string metric).

(a) [12 pts] Solve the vacuum equations and find the metric around the massive source assuming that the Ricci curvature tensor vanishes. (*Hint:* I recommend setting $A(r) \equiv e^{2\alpha(r)}$ and $B(r) \equiv e^{2\beta(r)}$, and then following the template of Carroll's derivation of the Schwarzschild metric, starting with Eq. (5.38) on p. 201.)

(b) [4 pts] Use a change of coordinates to put the metric in the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + C(r) r^2\,\mathrm{d}\theta^2\,. \tag{1.2}$$

Interpret the resulting metric and $C(r)$ physically (or geometrically).

(c) [6 pts] Consider the metric on a spatial slice (t = constant) in the reduced 3D coordinate system. Use the relationship between curvature and parallel transport to determine the total curvature $\int R^2{}_{1ij} dx^i dx^j$ that must be contained in the region containing the source if parallel transport counterclockwise around a single cosmic string rotates a vector by $2\pi/N$ counterclockwise, where we assume that the source produces nonzero Ricci curvature in a small nonsingular region in the vicinity of the origin. Write the metric (outside the source region) in this case.

(d) [8 pts] Write the (3D) metric at a large distance from a configuration of k particles, assuming that $k < N$, and that each separately gives rise to the metric described in the previous part, and assuming that the particle positions and the metric are independent of time. What is the maximum number of such particles (or, equivalently, infinite parallel cosmic strings in the 4D picture) that can inhabit space-time? (Hint: with enough particles, the space outside of the region containing the particles looks like a cylinder, while the region containing the particles caps off one end of the cylinder like the tip of a cigar.) With this maximum number of particles, what is the total curvature $\int R\sqrt{|g|}d^2x$ of a 2D spatial slice? ($|g| = |\det g_{ij}|$) [Another hint: This is an example of the Gauss-Bonnet theorem (Carroll p. 143).]

PROBLEM 2: KILLING VECTOR IDENTITIES *(15 pts)*

Carroll, Problem 3.12: Show that any Killing vector K^μ satisfies the relations mentioned in the text:

$$\begin{aligned} &\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu \qquad &[10 \text{ pts}]\ , \\ &K^\lambda \nabla_\lambda R = 0 \qquad &[5 \text{ pts}]\ . \end{aligned} \tag{2.1}$$

PROBLEM 3: KILLING VECTOR FIELDS FOR SPECIFIC SPACETIMES *(15 pts)*

Carroll, Problem 3.13: Find explicit expressions for a complete set of Killing vector fields for the following spaces:

(a) [5 pts] Minkowski space, with metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,. \tag{3.1}$$

(b) [10 pts] A spacetime with coordinates $\{u, v, x, y\}$ and metric

$$ds^2 = -(du\,dv + dv\,du) + a^2(u)dx^2 + b^2(u)dy^2\,, \tag{3.2}$$

where a and b are unspecified functions of u. This represents a gravitational wave spacetime. (Hints, which you need not show: there are five Killing vectors in all, and all of them have a vanishing u component K^u.)

Be careful, in all of these cases, about the distinction between upper and lower indices.

PROBLEM 4: THE VOLUME OF A CLOSED UNIVERSE *(10 pts)*

The metric for a Friedmann-Robertson-Walker closed universe, which we will learn more about later, can be written as

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} , \tag{4.1}$$

where $a(t)$ is the scale factor, with units of length, which for this problem can be treated as a known function. Here θ and ϕ are the usual polar angles, with $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. The radial variable r varies from 0 at one pole to 1 at the equator, and then there is a second hemisphere in which r also varies from 0 to 1. What is the total volume of the closed universe, as a function of t?