MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — May 5, 2018
Prof. Alan Guth

PROBLEM SET 12

DUE DATE: Thursday, May 10, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: Problem 1 concerns the use of differential forms, with Maxwell's equations serving as an example. These topicc were discussed in class, and Carroll discusses them in Secs. 1.8 and 2.9. Problems 2 and 3 concern Kerr black holes and the Penrose process, as were discussed in class, and are discussed by Carroll in Secs. 6.6 and 6.7.

MAXIMUM GRADE: This problem set has a total of 50 points.

PROBLEM 1: DIFFERENTIAL FORMS AND MAXWELL'S EQUATIONS *(15 pts)*

(a) [3 pts] If A is a p-form, the exterior derivative of A is defined in components by

$$(\mathrm{d}A)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]} \ . \tag{1.1}$$

Show that if the partial derivative in the definition were replaced by a covariant derivative, it would make no difference, because the Christoffel connection terms would all cancel.

(b) [3 pts] Show explicitly that for any p-form A, $\mathrm{d}\,\mathrm{d}A = 0$.

(c) [3 pts] The Hodge dual of a 2-form F (in (3+1)-dimensions) is defined by

$$(*F)_{\mu\nu} \equiv \frac{1}{2}\,\epsilon^{\lambda\sigma}{}_{\mu\nu}\,F_{\lambda\sigma} \ , \tag{1.2}$$

where $\epsilon_{\lambda\sigma\mu\nu}$ is the Levi-Civita tensor

$$\epsilon_{\lambda\sigma\mu\nu} \equiv \sqrt{g}\,\tilde{\epsilon}_{\lambda\sigma\mu\nu} \ , \tag{1.3}$$

where

$$g = -\det(g_{\mu\nu}) \tag{1.4}$$

and $\tilde{\epsilon}_{\lambda\sigma\mu\nu}$ is the fully antisymmetric quantity with $\epsilon_{0123} = 1$. Show that

$$(**F)_{\mu\nu} = -F_{\mu\nu} \ . \tag{1.5}$$

(We are interested in an arbitrary curved spacetime, so do not assume that $g = -1$.)

(d) [3 pts] The electromagnetic field strength tensor $F_{\mu\nu}$ can be described as a differential 2-form that is related to the 1-form (four-vector) potential A_μ through the relation

$$F = \mathrm{d}A \tag{1.6}$$

In Minkowski space the components of F are related to $\vec{E}$ and $\vec{B}$ by

$$F^{0i} = E^i \ , \quad F^{ij} = \tilde{\epsilon}^{ijk} B_k \ , \tag{1.7}$$

as shown in Carroll, Sec. 1.8, pp. 29–30. (In curved spacetimes one generally uses only $F^{\mu\nu}$, without reference to $\vec{E}$ or $\vec{B}$.) Write the equations $dF = 0$ in terms of these fields, and confirm that these are two of Maxwell's equations.

(e) [3 pts] The other Maxwell equations can be written as

$$\mathrm{d}(*F) = *J \ , \tag{1.8}$$

where J_μ is the electric current. Show that when this equation is expanded in components in Minkowski space, it reduces to the usual form of Maxwell's other two equations.

PROBLEM 2: MASSLESS PARTICLE ORBITS FOR A KERR BLACK HOLE *(15 pts)*

Carroll's problem 6-2 read as follows:

"Consider the orbits of massless particles, with affine parameter λ, in the equatorial plane of a Kerr black hole.

(i) Show that

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{\Sigma^2}{\rho^4}\big[E - LW_+(r)\big]\big[E - LW_-(r)\big] \ , \tag{2.1}$$

where $\Sigma^2 = (r^2 + a^2)^2 - a^2\,\Delta(r)\,\sin^2\theta$, E and L are the conserved energy and angular momentum, and you have to find expressions for $W_\pm(r)$.

(ii) Using this result, and assuming that $\Sigma^2 > 0$ everywhere, show that the orbit of a photon in the equatorial plane cannot have a turning point inside the outer event horizon r_+. This means that ingoing light rays cannot escape once they cross r_+, so it really is an event horizon."

The conserved energy and angular momentum are given by

$$E \;=\; -K_\mu p^\mu = m\left(1 - \frac{2GMr}{\rho^2}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{2mGMar}{\rho^2}\sin^2\theta\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \ , \tag{2.2}$$

$$L \;=\; R_\mu p^\mu = -\frac{2mGMar}{\rho^2}\sin^2\theta\,\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{m(r^2 + a^2)^2 - m\,\Delta\,a^2\,\sin^2\theta}{\rho^2}\sin^2\theta\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \ . \tag{2.3}$$

(a) [5 pts] Answer part (i) as Carroll asked it.

(b) [5 pts] Part (ii) is slightly flawed: the correct version of the statement is that a photon in the equatorial plane cannot have a turning point provided by that $r < r_+$ and that $r > R$, for some specific value of R. Prove the statement, and find the right value of R.

(c) [5 pts] Another flaw in the wording of part (ii) is the last sentence, "This means that ingoing light rays cannot escape once they cross r_+, so it really is an event horizon." Of course we are only talking about the equatorial plane, so any conclusion that we have an event horizon is clearly unjustified. In addition, horizons involve more than just the orbits of ingoing light rays. Show that for motion in the equatorial plane, no particle in the region $R < r < r_+$, traveling on a future-directed timelike trajectory, can escape to $r > r_+$.

PROBLEM 3: THE BLACK HOLE AREA THEOREM *(20 pts)*

It has been found that the area of the event horizon of a classical black hole can only increase and cannot decrease. This is often referred to as the "second law of black hole thermodynamics," and is related to a deep connection between the horizon area of a black hole and the entropy of a quantum black hole. In this problem we demonstrate this result for a charged Kerr black hole. [Hint: the analysis for an uncharged Kerr black hole is described in Carroll section 6.7.]

The metric for a charged Kerr black hole can be written as a slight generalization of the Kerr metric in terms of three parameters for the mass (M), angular momentum ($a = J/M$), and charge (q)

$$\begin{aligned} ds^2 &= -\left(\frac{\Delta_q - a^2\sin^2\theta}{\rho^2}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta_q)}{\rho^2}dtd\phi \\ &\quad + \frac{\rho^2}{\Delta_q}dr^2 + \rho^2 d\theta^2 + \left[\frac{(r^2+a^2)^2 - a^2\Delta_q\sin^2\theta}{\rho^2}\right]\sin^2\theta d\phi^2 \end{aligned}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta_q &= r^2 + a^2 + Gq^2 - 2GMr \end{aligned}$$

The four-vector electromagnetic potential is given by

$$A_\mu dx^\mu = \frac{-qr}{\rho^2}\left(dt - a\sin^2\theta d\phi\right).$$

(a) [5 pts] Compute the area of the outer horizon.

(b) [8 pts] For an uncharged object of energy E that enters the ergosphere and undergoes a Penrose process, compute the maximum of the ratio $L^{(2)}/|E^{(2)}|$ for the projectile that is absorbed by the black hole, which carries negative energy $E^{(2)} < 0$. Assume that the energies and angular momenta of the object and its pieces after breakup are all small compared to the corresponding quantities for the black hole.

(c) [7 pts] Show that this process cannot lead to a decrease in the horizon area.