MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity May 12, 2018
Prof. Alan Guth

# PROBLEM SET 13*

**DUE DATE:** Wednesday, May 16, 2018, at 11:00 am. That is, the problem set is due just before our last class.

**TOPICS COVERED AND RELEVANT LECTURES:** Problem 1 is concerned with the concept of total energy in general relativity, as discussed in lecture. Since the approach in lecture followed Weinberg's, I have posted a copy of Section 7.6 of Weinberg's book. Problem 2 is related to our introduction to cosmology, as discussed in lecture and in Carroll's Chapter 8. Problem 3 illustrates one method of calculating the Hawking temperature of a Schwarzschild black hole, and the analogous Gibbons–Hawking temperature of de Sitter space.

**MAXIMUM GRADE:** This problem set has a total of 55 points.

## PROBLEM 1: THE TOTAL ENERGY OF THE SCHWARZSCHILD SOLUTION *(15 pts)*

(a) [4 pts] We wish to calculate the total energy of the Schwarzschild spacetime, using the formula that expresses the total energy in a region as a surface integral over the boundary of the region,

$$P^0 = -\frac{1}{16\pi G} \int \left\{ \frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j} \right\} n_i \, \mathrm{d}s \ , \tag{1.1}$$

where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ . \tag{1.2}$$

The formalism is designed for a coordinate system that is at least approximately Minkowskian, so the polar coordinates of the standard Schwarzschild metric,

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \ , \tag{1.3}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \ , \tag{1.4}$$

are not appropriate. But we can translate it into Cartesian-like coordinates by using the usual relations between Cartesian and polar coordinates:

$$\begin{aligned} x &\equiv r \sin\theta \cos\phi \ , \\ y &\equiv r \sin\theta \sin\phi \ , \\ z &\equiv r \cos\theta \ . \end{aligned} \tag{1.5}$$

*A preliminary version of this problem set was posted on May 11, 2018. This final version includes one more problem, Problem 3, and minor comments were added to the statements of Problems 1 and 2.

(Note that in this context the above equations define $x$, $y$, and $z$, so it would be meaningless to think about deriving these equations.) Show that, in the $(t, x, y, z)$ coordinate system, the Schwarzschild metric can be written as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + dx^2 + dy^2 + dz^2 + f(r)\left(\frac{x\, dx + y\, dy + z\, dz}{r}\right)^2 , \quad (1.6)$$

where $f(r)$ is a function that you must determine.

(b) [4 pts] In an expansion in powers of $1/r$, determine $h_{ij}$ through order $1/r^2$.

(c) [7 pts] Use Eq. (1.1) to determine the total energy of the system out to radius $r$, using the expansion derived in the previous part. You should find that the total energy is $M$, plus a correction of order $1/r$. (Note that this calculation, in the limit $r \to \infty$, is identical to the calculation of the ADM (Arnowitt–Deser–Misner) energy.)

## PROBLEM 2: DE SITTER SPACE *(20 pts)*

During inflation, the universe rapidly approaches a de Sitter space, although it is never exactly de Sitter. The properties of de Sitter space are therefore very important in studies of inflationary cosmology.

Any homogeneous and isotropic spacetime can be described by the Robertson–Walker metric,

$$ds^2 = -\, dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\, d\Omega^2\right] . \quad (2.1)$$

The assumptions of homogeneity and isotropy also restrict the form of the energy-momentum tensor of the matter in the universe, which is required to have the form of a perfect fluid:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} , \quad (2.2)$$

where $\rho$ is the mass or energy density, $p$ is the pressure, and $U^\mu$ is the 4-velocity of the matter, which by isotropy must be purely in the time direction, $U^\mu = (1, 0, 0, 0)$. When Einstein's equations are applied to this system, they reduce to the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{k}{a^2} , \quad (2.3)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G(\rho + 3p) . \quad (2.4)$$

From these equations one can derive the equation describing the covariant conservation of $T_{\mu\nu}$,

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + p) . \quad (2.5)$$

Any two of Eqs. (2.3), (2.4), and (2.5) are sufficient to derive the third.

(a) [3 pts] Show that if $k > 0$, it is always possible to redefine the coordinates and $a(t)$ so that the spacetime is described by the Robertson–Walker metric with $k = 1$. Show also that if $k < 0$, it is always possible to redefine coordinates and $a(t)$ so that the spacetime is described by the Robertson–Walker metric with $k = -1$.

(b) [4 pts] De Sitter space is the unique homogeneous and isotropic spacetime in which the "matter" consists solely of vacuum energy with $\rho_v > 0$. Vacuum energy is equivalent to what Einstein called a cosmological constant $\Lambda$, with

$$\rho_v = \frac{\Lambda}{8\pi G} \ . \tag{2.6}$$

Vacuum energy does not change with time, so Eq. (2.5) implies that $p_v = -\rho_v$. For a flat ($k = 0$) metric, determine the general solution for the scale factor $a(t)$. You should find that there is no time at which $a(t)$ vanishes, but it does become arbitrarily small as $t \to -\infty$.

(c) [4 pts] Consider an open universe with matter consisting only of vacuum energy with $\rho_v > 0$, and find the general solution for the scale factor. You should find in this case that there is always a time at which $a(t) = 0$, which is conventionally taken to be $t = 0$.

(d) [4 pts] Now consider a closed universe with matter consisting only of vacuum energy with $\rho_v > 0$, and find the general solution for the scale factor. Here you should find that $a(t)$ never vanishes, and in fact has a nonzero minimum value.

A very surprising fact about de Sitter space is that the flat, open, and closed universe descriptions above actually all describe the *same* spacetime, with different choices of how to define equal-time surfaces. (Only the closed universe coordinates, however, cover the entire manifold.) Note that when $p = -\rho$, the energy-momentum tensor of Eq. (2.2) does not depend on $U^\mu$, so there is nothing that defines a rest frame. De Sitter space has 10 spacetime symmetries, the same as Minkowski space, with symmetries analogous to Lorentz boosts (3), as well as rotations (3) and spacetime translations (4). It is the presence of these additional symmetries which allows there to be multiple Robertson–Walker descriptions.

(e) [5 pts] A very simple description of de Sitter space, which makes its symmetries evident, involves embedding it into a (4+1)–dimensional Minkowski space. The metric for the embedding space can be written as

$$ds^2 = -\,dT^2 + dX^2 + dY^2 + dZ^2 + dW^2 \ , \tag{2.7}$$

where the embedded de Sitter space is the subspace satisfying the Lorentz-invariant condition

$$X^2 + Y^2 + Z^2 + W^2 - T^2 = R^2 \ , \tag{2.8}$$

for some constant $R$. The metric on the de Sitter subspace is taken to be the Minkowski metric on the full (4+1)–dimensional space, Eq. (2.7), restricted to the de Sitter subspace (i.e., it is the induced metric on this subspace). By defining the Robertson–Walker time coordinate $t$ to be a function only of $T$, show that it is possible to define coordinates on the de Sitter subspace such that the metric is exactly that of the closed universe that you found in the previous part.

## PROBLEM 3: ANALYTIC CONTINUATION AND THE TEMPERATURE OF BLACK HOLES AND DE SITTER SPACE *(20 pts)*

In this problem we will calculate the quantum-mechanical equilibrium temperature of Schwarzschild black holes and of de Sitter space, making use of an important fact from quantum statistical mechanics: the thermal expectation values of time-dependent quantities, when analytically continued to imaginary time, are periodic in the imaginary time variable, with a period equal to $1/k_B T$, where $k_B$ is the Boltzmann constant and $T$ is the temperature. The next paragraph will attempt to explain this fact. However, if you don't have enough background in quantum theory and statistical mechanics to follow this explanation, you can continue with this problem by just assuming the connection between temperature and the analytically continued time variable.

In a quantum theory with a time-independent Hamiltonian $H$, a thermal equilibrium state can be described by the Boltzmann distribution, in which each quantum state is assigned a probability proportional to $e^{-E/k_B T}$, where $T$ is the temperature and $E$ is the energy of the quantum state. The thermal expectation value of an arbitrary operator $\mathcal{O}$ is then given by

$$\langle \mathcal{O} \rangle_T = \frac{1}{Z(T)} \operatorname{Tr}\left[e^{-\beta H} \mathcal{O}\right] , \tag{3.1}$$

where $\beta \equiv 1/k_B T$.

$$Z(T) = \operatorname{Tr}\left[e^{-\beta H}\right] \tag{3.2}$$

is called the partition function, but in this equation it serves merely as a normalization factor. If $\mathcal{O}(t)$ is an operator in the Heisenberg representation, then

$$\mathcal{O}(t) = e^{iHt} \mathcal{O}(0) e^{-iHt} . \tag{3.3}$$

One can then see that if $\mathcal{O}(t)$ is analytically continued to purely imaginary values, $t = -i\tau$, where $\tau$ is real, then the thermal expectation value of $\mathcal{O}(-i\tau)$ is periodic in $\tau$, with period $\beta$. To see this, note that

$$\langle \mathcal{O}(-i\tau) \rangle_T = \frac{1}{Z(T)} \operatorname{Tr}\left[e^{-\beta H} e^{H\tau} \mathcal{O}(0) e^{-H\tau}\right] , \tag{3.4}$$

so

$$\begin{aligned}\big\langle \mathcal{O}\big(-i(\tau+\beta)\big)\big\rangle_T &= \frac{1}{Z(T)}\,\mathrm{Tr}\big[e^{-\beta H}e^{H(\tau+\beta)}\mathcal{O}(0)e^{-H(\tau+\beta)}\big] \\ &= \frac{1}{Z(T)}\,\mathrm{Tr}\big[e^{H\tau}\mathcal{O}(0)e^{-H\tau}e^{-\beta H}\big] \\ &= \frac{1}{Z(T)}\,\mathrm{Tr}\big[e^{-\beta H}e^{H\tau}\mathcal{O}(0)e^{-H\tau}\big] = \langle \mathcal{O}(-i\tau)\rangle_T\ ,\end{aligned} \tag{3.5}$$

where in the last step we used the cyclic property of the trace.

The relevance of this to black holes can be seen by analytically continuing the Schwarzschild time coordinate $t$ to purely imaginary values, $t = -i\tau$. You will find that this produces a well-defined real-valued Euclidean-signature manifold with coordinates $\tau$, $r$, $\theta$, and $\phi$. Furthermore, we will find that $\tau$ describes a periodic (i.e., angular) variable in this spacetime. We expect that the Green's functions (i.e., the expectation values of products of quantum field operators) of any quantum field theory defined on Schwarzschild space will be analytic functions of the coordinates, and will therefore remain well-defined functions in the Euclidean-signature analytic continuation. But this requires that the analytically continued Green's functions are periodic in $\tau$, and hence reflect a thermal character of the original spacetime, with a temperature equal to one over the periodicity length (in units where $k_B = c = \hbar = 1$). The conclusion is that the black hole can be in equilibrium only if it is surrounded by a bath of radiation at this temperature. If the spacetime around the black hole is empty, the black hole will continuously emit radiation at this temperature. This is of course a quantum-mechanical feature, which would be forbidden in the theory of classical general relativity.

In the above argument, it was crucial that evolution in $t$ be described by a time-independent Hamiltonian, which requires the metric to be independent of $t$. For the Schwarzschild spacetime, this means that we can apply the argument to the Schwarzschild time coordinate $t$, but not to the Kruskal time coordinate $T$. Nonetheless, it is still useful to use Kruskal coordinates to describe the manifold, since the Schwarzschild coordinate system is singular on the horizon $r = R_S = 2GM$. Recall, from lecture, your readings, or from Problem Set 9, that the Kruskal metric is given by

$$ds^2 = \frac{32G^3M^3}{r}e^{-r/2GM}(-\mathrm{d}T^2 + \mathrm{d}R^2) + r^2\,\mathrm{d}\Omega^2\,, \tag{3.6}$$

where the coordinates are $T$, $R$, $\theta$, and $\phi$, and $r$ is defined implicitly by the relation

$$T^2 - R^2 = \Big(1 - \frac{r}{2GM}\Big)e^{r/2GM}\,. \tag{3.7}$$

As usual,

$$\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,. \tag{3.8}$$

In the first quadrant, the Kruskal coordinates $T$ and $R$ can be related to the Schwarzschild coordinates $t$ and $r$ by

$$\begin{aligned} T &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \\ R &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) . \end{aligned} \tag{3.9}$$

These first-quadrant relations will be enough to completely determine the analytic continuation, since knowledge of an analytic function in one region is enough to determine the full function.

(a) [7 pts]

(i) [1 pt] When the Schwarzschild time is analytically continued to become pure imaginary, with $t = -i\tau$, does the Kruskal $T$ variable remain real, or does it become imaginary or complex?

(ii) [1 pt] Does the Kruskal $R$ variable remain real, or does it become imaginary or complex?

(iii) [1 pt] In any case, define real-valued coordinates for the Kruskal spacetime after this imaginary-$t$ analytic continuation, and give the (Euclidean-signature) metric for this manifold.

(iv) [3 pts] Is the Schwarzschild $\tau = it$ variable periodic, as advertised above? If so, determine its period, and thereby find the temperature of the black hole. This temperature is of course known as the Hawking temperature, and was first discovered by Stephen Hawking in 1974*, using very different methods from those used here.

(v) [1 pt] Use dimensional analysis to restore any factors of $k_B$, $c$, and $\hbar$, so that you can verify that the temperature goes to zero if $\hbar = 0$.

(b) [6 pts] The analytic continuation to the Euclidean-signature spacetime can be carried out directly in Schwarzschild coordinates, but one has to be careful about the singularity at $r = R_S$. Show that if $t$ is analytically continued to $t = -i\tau$, the Schwarzschild metric, for $r$ just slightly greater than $R_s$, can be written approximately as

$$ds^2 = d\tilde{r}^2 + \tilde{r}^2 \, d\chi^2 , \tag{3.10}$$

where $\tilde{r}$ is a function of $r$, with $\tilde{r}(R_S) = 0$, and $\chi$ is a function of $\tau$. The above metric has a conical singularity at $\tilde{r} = 0$, unless $\chi$ is periodic with period $2\pi$. By requiring that no such conical singularity is present, find the periodicity in $\tau$, and hopefully confirm the answer you found in part (a).

*S. W. Hawking, "Black Hole Explosions," Nature 248, 30-31 (1974).

In Problem 2, we discovered that de Sitter space can be assigned coordinates in the form of a flat, open, or closed coordinate system. In addition to these, de Sitter space can be represented by a static coordinate system:

$$ds^2 = -(1 - H^2 r^2)\,dt^2 + (1 - H^2 r^2)^{-1}\,dr^2 + r^2\,d\Omega^2 \ , \tag{3.11}$$

where $H = R^{-1}$, where $R$ is the variable appearing in Eq. (2.8), defining the scale of the de Sitter space. This coordinate system is the natural one to use when describing what a single geodesic observer can see in a perfect de Sitter space, where the hypothetical observer is at the origin. For such observers, there would be an event horizon at $r = H^{-1}$, where no light from beyond this horizon will ever reach them. For current purposes, the important point is that this metric is independent of $t$, and therefore evolution in $t$ would be described in quantum theory by a time-independent Hamiltonian, so the analytic continuation analysis described above can be applied. The coordinates shown here are singular on the horizon, like the Schwarzschild coordinates of the black hole. It is possible to find Kruskal-like coordinates for de Sitter space, which would make this part look like part (a), but the direct analysis of the static metric of Eq. (3.11) makes this part look like part (b).

(c) [7 pts] Show that if the $t$ coordinate of the static metric is analytically continued to be purely imaginary, with $t = -i\tau$, then variables can be defined so that the metric in the immediate vicinity of the singularity looks like Eq. (3.10). Again a conical singularity can be avoided only if $\chi$ is periodic with period $2\pi$. Use this fact to show that $\tau$ must be periodic, and use its period to determine the equilibrium temperature of de Sitter space. (This temperature is known as the Gibbons–Hawking temperature, because it was discovered by Gary Gibbons and Stephen Hawking[†] in 1976. The Gibbons-Hawking thermal fluctuations are at the heart of the mechanism by which inflation generates density perturbations, which are believed to be the source of the inhomogeneities of the universe, which are observed in the cosmic microwave background and are responsible for the origin of structure in the universe—clusters, galaxies, stars, planets, and us.)

[†] G. W. Gibbons and S. W. Hawking, "Cosmological event horizons, thermodynamics, and particle creation," Phys. Rev. **D15,** 2738–2751 (1976).