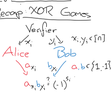
Perfect Games

Recap: XOR Games



To specify an XOR Game, specify clauses

$\{(x_1, y_1, s_1), (x_2, y_2, s_2), \ldots, (x_m, y_m, s_m)\}$

Classical Strategies are described by a variable assignment $a_x \to \{1,-1\}$ $b_y \to \{1,-1\}$.

The classical value

$$\omega(G) = \max_{strat.} \left(\frac{1}{2} + \frac{1}{2m} \sum_i a_{x_i} b_{y_i} (-1)^{s_i} \right)$$

A quantum (commuting operator) strategy is defined by state $|\psi\rangle$ and observables A_i, B_j

A and B observables

- Commute
- Have eigenvalues ± 1

The commuting operator value of a game

$$\omega^*_{co}(G) = \max_{strat} \left(\frac{1}{2} + \frac{1}{2m} \langle\psi| \sum_i A_{x_i} B_{y_i} (-1)^{s_i} |\psi\rangle \right)$$

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A game has perfect commuting operator (resp. perfect classical) strategy if $\omega^*_{co}(G) = 1$ (resp $\omega = 1$).

What makes perfect xor games interesting objects to study?

(NP) Hard to compute the classical value of 3 player XOR Game.

But!

3Xor game has $\omega(G) = 1$

$\Leftrightarrow \frac{1}{2} + \frac{1}{2m} \sum_i a_{x_i} b_{y_i} c_{z_i} (-1)^{s_i} = 1$

$\Leftrightarrow \forall i \; a_{x_i} b_{y_i} c_{z_i} = (-1)^{s_i}$

$\Leftrightarrow \forall i \; \hat{a}_{x_i} + \hat{b}_{y_i} + \hat{c}_{z_i} = s_i \pmod 2$

Easy to check if a 3Xor game has perfect classical value.

Quantumly we have an algorithm (NPA) which converges to the co. value of a game.

But that could take a long time to run. And its somewhat opaque.

Question: Is there a quantum analogue of Gaussian Elim. for Xor games?

(Partial) Ans: There is an algebraic condition characterizing XOR games w/perfect c.o. value

- For some cases this gives poly-time algorithm.
- For some cases this lets us prove stuff.

Idea: Find nec. condition, then show it's sufficient.

Fundamental objects are observables $A_{x_i}, B_{y_i} (C_{z_i} \ldots)$

- Commute
  $\Rightarrow ABA^{-1}B^{-1} = [A,B] = 1$ (group com) — relations on elements
- Order two
  $A^2 = B^2 = 1$
  (eigenvalues ± 1)
- Add a "-1" symbol σ
  $[\sigma, A] = [\sigma, B] = \sigma^2 = 1$

All words formed from these elements = Game group G.

If the game has perfect commuting operator strat

$\Rightarrow \frac{1}{2} + \frac{1}{2m} \sum_i \langle\psi| A_{x_i} B_{y_i} (-1)^{s_i} |\psi\rangle = 1$

$\Rightarrow \forall i \; \langle\psi| A_{x_i} B_{y_i} (-1)^{s_i} |\psi\rangle = 1$

$\Rightarrow \forall i \; A_{x_i} B_{y_i} (-1)^{s_i} |\psi\rangle = |\psi\rangle$

Then we also have

$A_{x_i} B_{y_i} (-1)^{s_i} A_{x_j} B_{y_j} (-1)^{s_j} |\psi\rangle = |\psi\rangle$

and so on.

Define the clause group $H < G$ by $H = \langle \{ A_{x_1} B_{y_1} \sigma^{s_1}, A_{x_2} B_{y_2} \sigma^{s_2}, \ldots, A_{x_m} B_{y_m} \sigma^{s_m} \} \rangle$

If the game has a perfect co strategy then $h|\psi\rangle = |\psi\rangle$ for all $h \in H$.

I claim a contradiction if $\sigma \in H$.

$\sigma \in H \Rightarrow (-1)|\psi\rangle = |\psi\rangle$ ✱

Thm Game has perfect co. value iff $\sigma \notin H$.

Proof

($\Rightarrow$) just showed.

($\Leftarrow$) representation theory
left action on cosets of H $|\psi\rangle = |H\rangle - |\sigma H\rangle$.

$\rho(g)|H\rangle = |gH\rangle$

Deciding if $\omega^*_{co}(G) = 1$ $\Leftrightarrow$ Solving an instance of subgroup membership problem.

Thm Game has perfect co. value iff $\sigma \notin H$.

Ex: CHSH

G generators $A_0, A_1, B_0, B_1, \sigma$

$H = \langle \{ A_0 B_0, A_1 B_0, A_0 B_1, A_1 B_1 \sigma \} \rangle$

$(A_0 B_0)(A_1 B_0)(A_1 B_1 \sigma)(A_0 B_1)$

$= A_0 A_1 A_1 A_0 B_0 B_0 B_1 B_1 \sigma = \sigma \in H$

$\Rightarrow \omega^*_{co}(CHSH) < 1$

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Cleve, Slofstra, Liu 2016

For BCS Game G deciding if $\omega_{co}(G) = 1$ $\Leftrightarrow$ Solving an instance of the word problem

General Principle: Perfect non-local games have algebraic characterization.

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Some consequences

① Undecidability

$\omega(G) = 1$ for BCS game G

$\overset{(csl)}{\Leftrightarrow}$ Solving instance of word problem

Slofstra 2016: The family of instances of the word problem corresponding to BCS games are undecidable.

$\Rightarrow C_q \neq C_{qc}$

If $C_q = C_{qc}$ then check if either

- $\exists$ a perfect dim d strategy
  $\Rightarrow \omega_{co}(G) = 1$
- At level d NPA gives $\omega_{co}(G) < 1$.

One of these must eventually happen $\Rightarrow$ Alg for deciding undecidable problem $\Rightarrow$ contradiction.

Cleve Mittal 2012

For BCS Games: $\omega_{qs} = \omega_q$

$\Rightarrow C_{qs} \neq C_{qc}$

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② Simple Strategies

For 2XOR games:

Claim: $\sigma \in H$ iff $\sigma \in H$ when all variables commute.

Pf by ex:

$H = \langle \{ A_1 B_1 \sigma, A_2 B_2, A_3 B_1, A_1 B_2, A_2 B_3, A_3 B_3 \} \rangle$

$\sigma A_1 B_1 A_1 B_2 A_2 B_2 A_2 B_3 A_3 B_3 A_3 B_1$

$= \sigma A_1 B_1 A_1 B_2 A_2 B_2 A_2 B_3 A_3 B_3 A_3 B_1$ (cancelled)

$= \sigma \in H$

$\Rightarrow$ 2XOR game G has perfect co value iff it has perfect classical value.

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General Principle: Perfect non-local games have algebraic characterization.

Why?

Fact: NPA works with observables instead of projectors.

	1	A …	B …	AA …	AB …	BB
1						
A ⋮						
B ⋮						
AA ⋮						
AB ⋮						
BB						

$(C_{A,B} = \langle\psi| AB |\psi\rangle)$

If a game is perfect
$\Rightarrow$ extra identities in matrix
$\langle\psi| \omega (A_{x_i} B_{y_i} \sigma^{s_i}) |\psi\rangle = \langle\psi| \omega |\psi\rangle$

For XOR games, NPA works with clauses

	$A_{x_1} B_{y_1} (-1)^{s_1}$, $A_{x_2} B_{y_2} (-1)^{s_2}$, …, (and products of clauses)
$A_{x_1} B_{y_1} (-1)^{s_1}$	
$A_{x_2} B_{y_2} (-1)^{s_2}$	
⋮	

$(C_{i,j} = \langle\psi| A_{x_i} B_{y_i} (-1)^{s_i} A_{x_j} B_{y_j} (-1)^{s_j} |\psi\rangle)$

All 1's matrix is PSD & has perfect value.

Valid unless $\exists i_1, i_2, \ldots i_\ell$

$A_{x_{i_1}} B_{y_{i_1}} (-1)^{s_{i_1}} \ldots A_{x_{i_\ell}} B_{y_{i_\ell}} (-1)^{s_{i_\ell}} = -1$

$\Leftrightarrow \sigma \in H$.

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Bonus comment: If all sequences $i_1 \ldots i_\ell$ satisfying $A_{x_{i_1}} B_{y_{i_1}} (-1)^{s_{i_1}} \ldots A_{x_{i_\ell}} B_{y_{i_\ell}} (-1)^{s_{i_\ell}} = -1$ have length $> L$ then all 1's matrix appears valid for NPA w/degree $< \lfloor \frac{L-1}{2} \rfloor$.
(note: degree depends on what goes go in your matrix)

$\Rightarrow$ NPA lower bound.