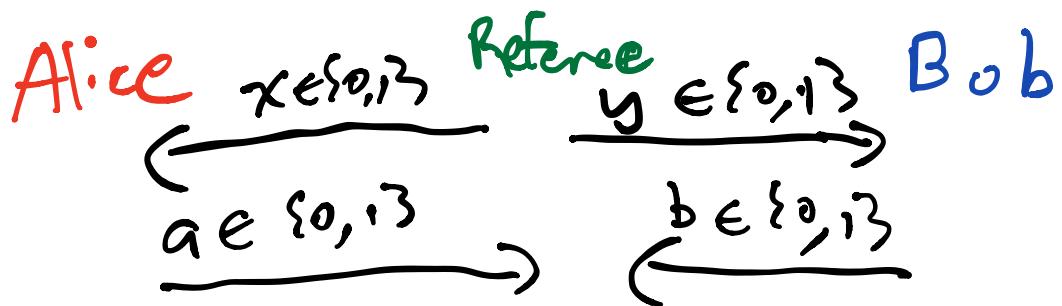


6.S979 Lecture 2: CHSH game

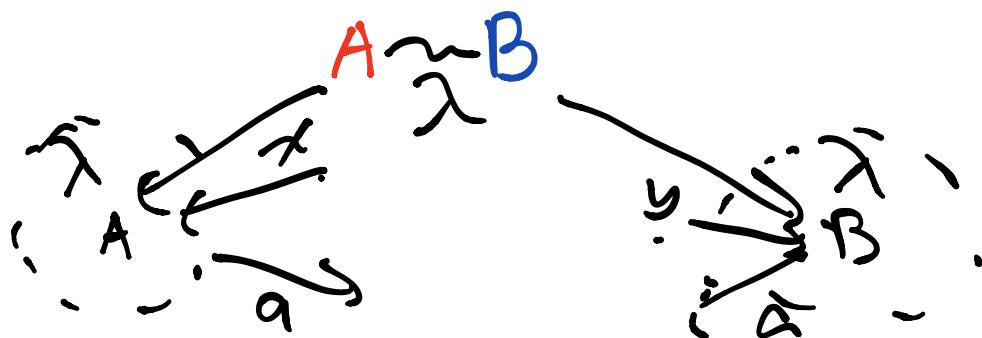
Bell '64 Clauser Horne Shimony; Holt
 '69



$$x \cdot y = ? \quad a + b \text{ mod } 2$$
$$a \oplus b$$

A Venn diagram with two overlapping circles labeled x and y . The intersection area is shaded red, while the non-overlapping parts are shaded blue. Dashed lines from the center point to the intersection and non-overlapping regions are labeled with question marks, corresponding to the question marks in the equation above.

LHVM ("Classical strategies")



$$\begin{aligned} a &= f(\underline{\lambda}, x) \\ b &= g(\overline{\lambda}, y) \end{aligned}$$

classical
value

$$\omega = \sup_{\lambda \sim \mu} \mathbb{E} \left[\mathbb{E} I[x_y = f(\underline{\lambda}, x) + g(\overline{\lambda}, y)] \right]$$

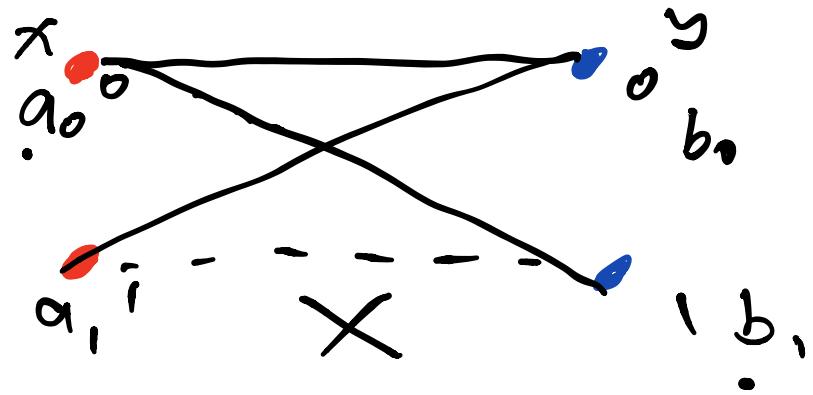
λ doesn't matter:
— Fix f, g . Then pick best λ

$$\omega = \sup_{f, g} \mathbb{E} \left[\mathbb{E} I[x_y = f(x) + g(y)] \right]$$

$$a_0 = f(0), \quad a_1 = f(1)$$

$$b_0 = g(0), \quad b_1 = g(1)$$

$$\omega = \sup_{a_0, b_0, a_1, b_1} \mathbb{E} \left[\mathbb{E} I[x_y = a_x + b_y] \right]$$



$$a_0 = b_0 = b_1 = a_1 \quad \cancel{a_1 \neq b_1}$$

You can satisfy at most $3/4$ constraint

$$\Rightarrow \omega = 3/4$$

"CHSH inequality"
Violating CHSH w/ Quantum

Observables:

$$a, b \in \{0, 1\}$$

$$A = (-1)^a$$

$$B = (-1)^b$$

$$1_{[xy = a+b]} = \frac{1}{2} + \frac{1}{2} (-1)^{xy} AB$$

$$\omega(S) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} \left(A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \right)$$

$\underbrace{\qquad\qquad\qquad}_{\text{bias}}$

Binary measurement

$$\text{s.t. } \Pi_0^2 = \Pi_0, \quad \Pi_1^2 = \Pi_1$$

$$\Pi_0 + \Pi_1 = I$$

$$\Pi_0 \cdot \Pi_1 = \Pi_1 \cdot \Pi_0 = 0 \quad \leftarrow$$

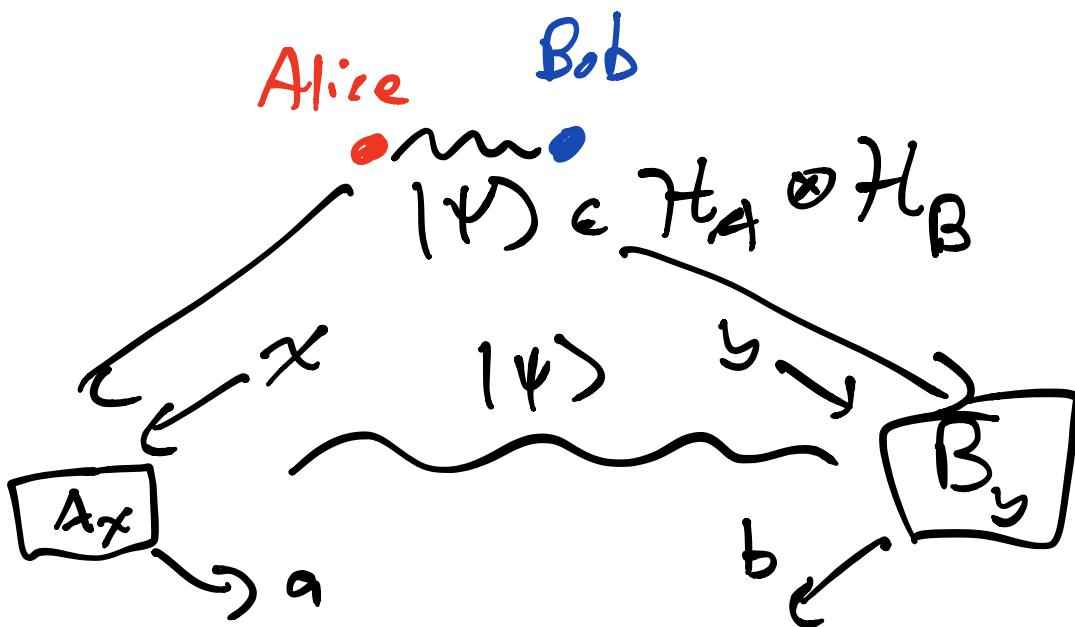
$$\Theta = \Pi_0 - \Pi_1 \quad \leftarrow$$

↑ Hermitian → eigenvalues ± 1

Proj. on +1 eigenspace].
 Proj. on -1 eigenspace

$$\langle \psi | \theta | \psi \rangle$$

= Expected value in ± 1 notation
 = $1 \cdot \Pr[\theta] - 1 \cdot \Pr[\bar{\theta}]$



$$\begin{aligned}
 C_0^* = \sup_{\substack{|\psi\rangle \\ A_x, B_y}} & \left(\frac{1}{2} + \frac{1}{8} \langle \psi | A_0 \otimes B_0 + A_0 \otimes B_1 \right. \\
 & \left. + A_1 \otimes B_0 - A_1 \otimes B_1 | \psi \rangle \right)
 \end{aligned}$$

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|00\rangle_{AB} + |11\rangle_{AB})$$

Z -basis $|0\rangle \quad |1\rangle$
 X -basis $|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$

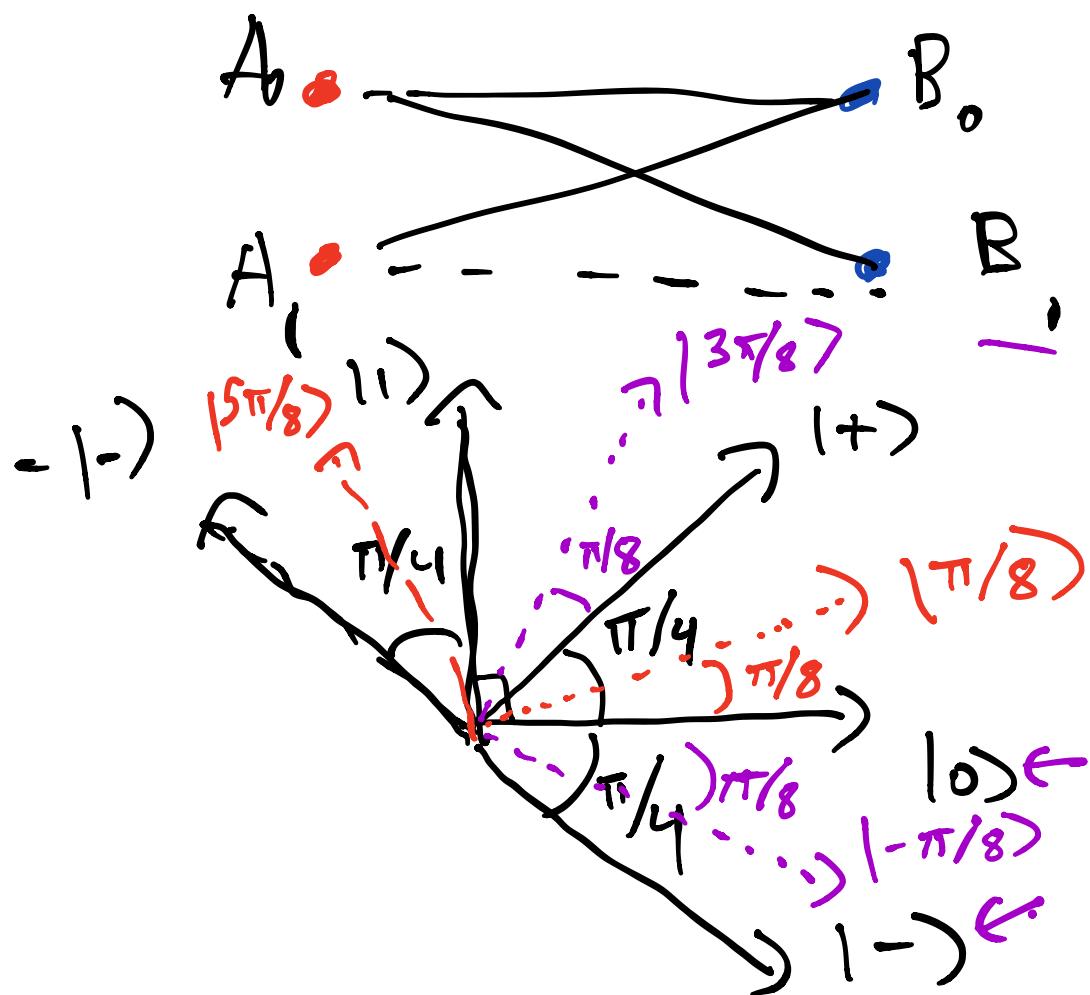
$$Z = \Pi_0 - \Pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X = \Pi_+ - \Pi_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{2} (|+\rangle\langle +|) - \frac{1}{2} (|-1\rangle\langle -1|)$$

$$A_0 = Z, A_1 = X$$



$$A_0 = Z, \quad A_1 = X$$

$$B_0 = |\pi/8\rangle(\pi/8| - |5\pi/8\rangle(5\pi/8|)$$

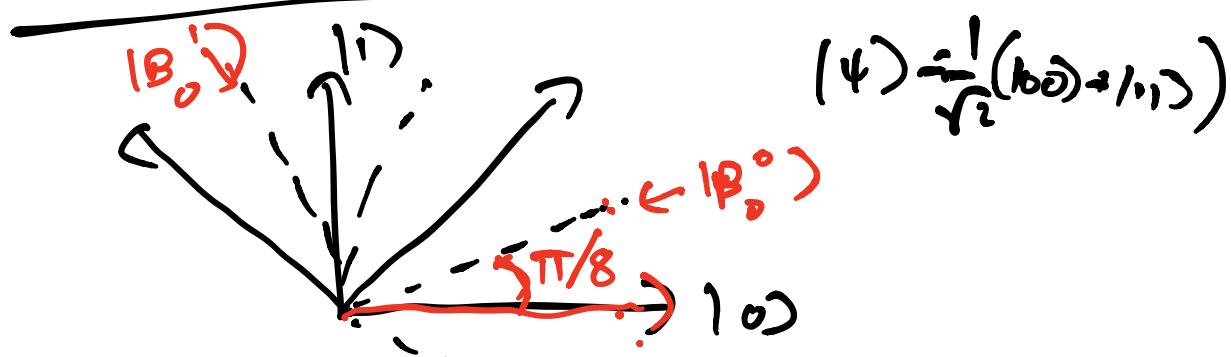
$$|\pi/8\rangle = \cos(\pi/8)|0\rangle + \sin(\frac{\pi}{8})|1\rangle$$

$$|5\pi/8\rangle = -\sin(\pi/8)|0\rangle + \cos(\frac{\pi}{8})|1\rangle$$

$$B_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\beta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Calculating min prob:



$$(4) = \frac{1}{\sqrt{2}}(10) + 11)$$

$$x=0$$

$$y=0$$

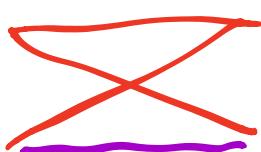
$$B_0$$

$$= 2$$

$$\alpha = \begin{cases} 0^\circ & \text{if } 10) \\ 180^\circ & \text{if } 11) \end{cases} \xrightarrow{\text{prob.}} \text{prob.}$$

$$\begin{aligned} \gamma_2 \\ b &= 1 \sim / \text{prob.} \cos^2 \frac{\pi}{8} \\ 0 &\sim / \text{prob.} \sin^2 \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} a &= 0^\circ \sim / \text{prob.} \cos^2 \left(\frac{\pi}{8}\right) \\ b &= 0^\circ \sim / \text{prob.} \sin^2 \left(\frac{\pi}{8}\right) \end{aligned}$$



$$\begin{aligned} a &= b \sim / \text{prob.} \cos^2 \frac{\pi}{8} \\ a \neq b &\sim / \text{prob.} \cos^2 \frac{\pi}{8} \end{aligned}$$

$$\Rightarrow \text{min prob. } \cos^2 \frac{\pi}{8} \\ \approx 0.854$$

Stabilizer calculation:

Observe : "stabilizers of $|\psi\rangle$ "

$$|\psi\rangle = X \otimes X |\psi\rangle \quad \text{("bit flip")}$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = Z \otimes Z |\psi\rangle \quad \text{("phase flip")}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$

$Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$

$$B_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (Z + X)$$

$$B_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (Z - X)$$

$$\omega^*(s) = \frac{1}{2} + \frac{1}{8} \langle \Psi | (A_0 \otimes B_0 - A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1) | \Psi \rangle$$

$$= \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{\sqrt{2}} \langle \Psi | (Z \otimes (Z+x) + Z \otimes (Z-x) + X \otimes (Z+x) - X \otimes (Z-x)) | \Psi \rangle$$

$$= \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{\sqrt{2}} \langle \Psi | (2Z \otimes Z + 2X \otimes X) | \Psi \rangle$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \langle \Psi | (2Z \otimes Z + 2X \otimes X) | \Psi \rangle$$

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \langle \Psi | \Psi \rangle$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos\left(\frac{\pi}{8}\right) \approx 0.854$$

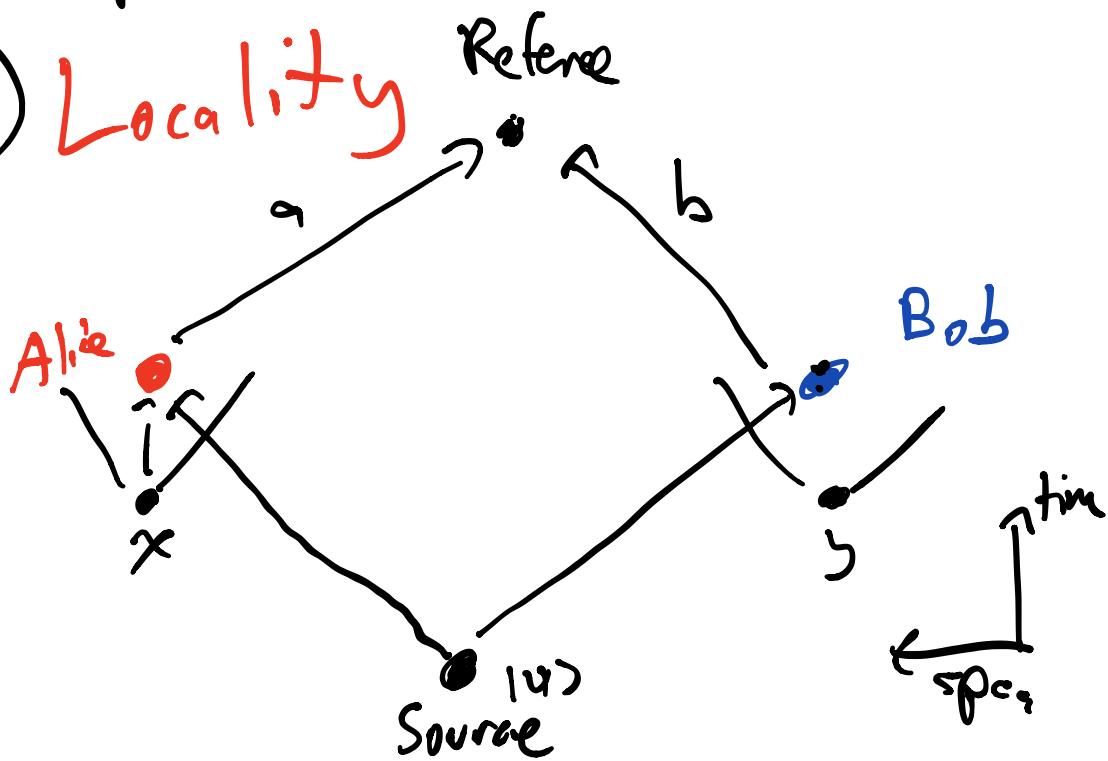
What does this mean? - - - - -

If A & B play CHSH
win w/prob. > 3/4

\Rightarrow no LHVM for their systems

Assumptions

1) Locality



2) Repeated trials / memory

3) Detection loophole
biased sample of the rounds

Aspect '80s Hensen '15
+ others

Why 0.854 ???

Is it optimal? Yes in QM (Tsirelson bound)

Could you get $\omega^+ = 1$?

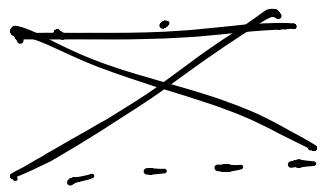
$$\langle \Psi | \left(\underbrace{A_0 \otimes B_0}_{+1} + \underbrace{\tilde{A}_0 \otimes \tilde{B}_0}_{+1}, + \underbrace{\tilde{A}_1 \otimes B_1}_{+1} - \underbrace{\tilde{A}_1 \otimes \tilde{B}_1}_{-1} \right) |\Psi \rangle \\ = 4$$

$$\underbrace{\langle \Psi | (A_0 \otimes I)}_{\langle v_0 |} \underbrace{(I \otimes B_0) | \Psi \rangle}_{| w_0 \rangle} = 1$$

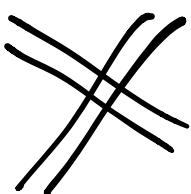
$$|v_0\rangle = A_0^\dagger \otimes I |\Psi\rangle \quad |v_0\rangle$$

$$\Rightarrow |v_0\rangle = |w_0\rangle$$

$$|v_0\rangle = \alpha |w_0\rangle \quad \langle v_0 | v_0 \rangle = \alpha^*$$



$$\langle v_1 | w_1 \rangle = -1$$



$$\begin{aligned} |v_0\rangle &= |w_0\rangle \\ |v_0\rangle &= |w_1\rangle \\ |v_1\rangle &= |w_0\rangle \\ |v_1\rangle &= -|w_1\rangle \\ \text{no } &\text{ perfect g. strat} \end{aligned}$$