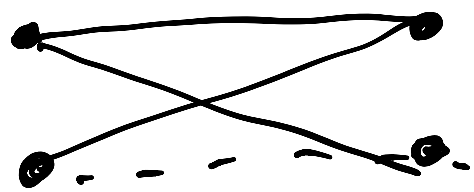
6.S979 Lecture 3

Pset 1 is out due 10/2

CHSH



$\omega_{CHSH} \leq \frac{3}{4}$ "CHSH ineq."

$\omega^*_{CHSH} \geq 0.854$ $\cos^2\left(\frac{\pi}{8}\right)$ CHSH violation

$$\omega^*_{CHSH} \leq \cos^2\left(\frac{\pi}{8}\right)$$

Tsirelson's bound

Boris Tsirelson '80

Remember:

$\nexists$ perfect q. strat for CHSH

$$\omega^*(S) = \frac{1}{2} + \frac{1}{8}\underbrace{\langle\psi|\, A_0\otimes B_0 + A_0\otimes B_1 + A_1\otimes B_0 - A_1\otimes B_1 \,|\psi\rangle}_{\langle\psi|CHSH|\psi\rangle}$$

strategy $\rightarrow S$: $|\psi\rangle, A_0, A_1, B_0, B_1$

$$\langle\psi|CHSH|\psi\rangle = 4$$

$$\Longrightarrow \quad \underbrace{\langle\psi|(A_0\otimes I)}_{\langle v_0|}\underbrace{(I\otimes B_0)|\psi\rangle}_{|w_0\rangle} = 1$$

$$\underbrace{\langle\psi|(A_0\otimes I)}_{\langle v_0|}\underbrace{(I\otimes B_1)|\psi\rangle}_{|w_1\rangle} = 1$$

$$\underbrace{\langle\psi|(A_1\otimes I)}_{\langle v_1|}\underbrace{(I\otimes B_0)|\psi\rangle}_{|w_0\rangle} = 1$$

$$\underbrace{\langle\psi|(A_1\otimes I)}_{\langle v_1|}\underbrace{(I\otimes B_1)|\psi\rangle}_{|w_1\rangle} = -1$$

$$\Longrightarrow \quad |v_0\rangle = |w_0\rangle = |w_1\rangle$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad$$

$$|v_1\rangle = -|w_1\rangle$$

$$\langle\psi|CHSH|\psi\rangle$$

$$= \langle v_0|\cdot\left(\underline{|w_0\rangle + |w_1\rangle}\right) + \langle v_1|\left(\underline{|w_0\rangle - |w_1\rangle}\right)$$

$$\leq \| |w_0\rangle + |w_1\rangle \| + \| |w_0\rangle - |w_1\rangle \|$$

$$= (1, 1) \cdot \left(\| |w_0\rangle + |w_1\rangle \|, \| |w_0\rangle - |w_1\rangle \| \right)$$

$$\leq \sqrt{2} \cdot \sqrt{\| |w_0\rangle + |w_1\rangle \|^2 + \| |w_0\rangle - |w_1\rangle \|^2}$$

$$= \sqrt{2} \cdot \sqrt{1 + 1 + \cancel{\langle w_0 | w_1 \rangle} + \cancel{\langle w_1 | w_0 \rangle} + 1 + 1 - \cancel{\langle w_0 | w_1 \rangle} - \cancel{\langle w_1 | w_0 \rangle}}$$

$$= 2\sqrt{2}$$

$$\langle \psi | CHSH | \psi \rangle \leq 2\sqrt{2}$$

$$\Rightarrow \omega^* = \frac{1}{2} + \frac{1}{8} \langle \psi | CHSH | \psi \rangle$$

$$\leq \frac{1}{2} + \frac{\sqrt{2}}{4} = \cos^2\left(\frac{\pi}{8}\right)$$

<u>Q:</u> We showed QM $\rightarrow \omega^* \leq \cos^2\left(\frac{\pi}{8}\right)$

Does $\omega^* \leq \cos^2\left(\frac{\pi}{8}\right) \rightarrow$ QM

<u>Q:</u> What about physical theories

where $\omega^* > \cos^2\left(\frac{\pi}{8}\right)$

"super quantum"

<u>Q:</u> What about the optimal strategy? Is it unique?

<u>Obs:</u> We can find $|v_0\rangle, |v_1\rangle, |w_0\rangle, |w_1\rangle$ saturating bounds

<u>But</u> need $|v_0\rangle = (A_0 \otimes I)|\psi\rangle$

$\vdots$

<u>Proof #2: Operator inequality</u>

$$\begin{aligned} CHSH = A_0 \otimes B_0 &+ A_0 \otimes B_1 \\ &+ A_1 \otimes B_0 - A_1 \otimes B_1 \\ &\leq 2\sqrt{2} \cdot I \otimes I \end{aligned}$$

$A \leq B \iff B - A \geq 0 \iff \forall\, |\psi\rangle\ \langle\psi|(B-A)|\psi\rangle \geq 0$

$\mathcal{O}^2 = I$

$$\mathrm{CHSH}^2 = \left(A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1\right)^2$$

$$= 4\, I \otimes I + I \otimes \cancel{(B_0 B_1 + B_1 B_0)}$$

$$+ I \otimes \cancel{(-B_0 B_1 - B_1 B_0)}$$

$$+ \cancel{(A_0 A_1 + A_1 A_0)} \otimes I$$

$$+ \cancel{(-A_0 A_1 - A_1 A_0)} \otimes I$$

$$- A_0 A_1 \otimes B_0 B_1 - A_1 A_0 \otimes B_1 B_0$$

$$+ A_0 A_1 \otimes B_1 B_0 + A_1 A_0 \otimes B_0 B_1$$

$$= 4 I \otimes I + \underbrace{(A_0 A_1 - A_1 A_0)}_{\leq 2I} \otimes \underbrace{(B_1 B_0 - B_0 B_1)}_{\leq 2I}$$

$$\Updownarrow$$

$$\langle \psi | (A_0 A_1 - A_1 A_0) | \psi \rangle \leq 2$$

$$\mathrm{CHSH}^2 \leq 8\, I \otimes I$$

We want

$$\text{CHSH} \le 2\sqrt{2}\, I \otimes I$$
$$= \sqrt{8}\, I \otimes I \checkmark$$

Fact: Square root is "operator monotone"

$A, B \ge 0$ $\quad A^2 \le B^2 \Rightarrow A \le B$

Warning: Not the case that
$A \le B \Rightarrow A^2 \le B^2$

Aside
k-outcome measurement
$\Pi_1, \ldots, \Pi_k$

$$\mathcal{O} = \sum_{i=1}^{k} \omega^i \Pi_i$$

ω: k-th root of unity

London '87

Proof #3: sum of squares

In spirit like Tsirelson's pf:

$$CHSH \leq 2\sqrt{2} \cdot I \otimes I$$

$$\Big(\overbrace{A_0 \otimes I - I \otimes \frac{B_0 + B_1}{\sqrt{2}}}^{P}\Big)^2$$

$$= I \otimes I + I \otimes \frac{2 \cdot I + B_0 B_1 + B_1 B_0}{2} - 2 \cdot \frac{A_0 \otimes (B_0 + B_1)}{\sqrt{2}}$$

$$\Big(\overbrace{A_1 \otimes I - I \otimes \frac{B_0 - B_1}{\sqrt{2}}}^{Q}\Big)^2$$

$$= I \otimes I + I \otimes \frac{2I - B_0 B_1 - B_1 B_0}{2} - 2\, \frac{A_1 \otimes (B_0 - B_1)}{\sqrt{2}}$$

$$p^2 + q^2 = 2 \cdot I \otimes I + 2 I \otimes I - \frac{2}{\sqrt{2}} \cdot CHSH$$

$$CHSH = \frac{\sqrt{2}}{2}\Big(4 I \otimes I - \underbrace{(p^2+q^2)}_{\geq 0}\Big)$$

$$\leq 2\sqrt{2}\, I \otimes I \checkmark$$

$$\langle \psi | CHSH | \psi \rangle = 2\sqrt{2}$$

$$\Longrightarrow \quad \langle \psi | p^2 | \psi \rangle = 0 \Rightarrow p|\psi\rangle = 0$$

$$\langle \psi | q^2 | \psi \rangle = 0 \Rightarrow q|\psi\rangle = 0$$

$$\left(A_0 \otimes I - I \otimes \frac{B_0 + B_1}{\sqrt{2}}\right)|\psi\rangle = 0$$

$$A_0 \otimes I |\psi\rangle = I \otimes \frac{B_0 + B_1}{\sqrt{2}} |\psi\rangle$$

Obs: for any obs. B_0, B_1

$$(B_0+B_1)(B_0-B_1) = -(B_0-B_1)(B_0+B_1)$$

$$I - I - B_0B_1 + B_1B_0$$

$$I - I - B_1B_0 + B_0B_1$$

$$\Rightarrow \quad C_0 = \frac{B_0+B_1}{\sqrt{2}}, \quad C_1 = \frac{B_0-B_1}{\sqrt{2}}$$

$$C_0C_1 = -C_1C_0$$

$$A_{0/1}\otimes I|\psi\rangle = I\otimes C_{0/1}|\psi\rangle$$

$$\Rightarrow \quad A_0A_1\otimes I|\psi\rangle = -A_1A_0\otimes I|\psi\rangle$$

Fact: $A_0A_1 = -A_1A_0$ dim d

$\Rightarrow \exists\, U$ s.t.

$$U^\dagger A_0 U = Z\otimes I_{d/2}$$

$$u^\dagger A_1 u = X \otimes I_{d/2}$$

Pf:

$$A_0 = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$$

$$A_1 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$A_0 A_1 = \begin{pmatrix} M_{11} & M_{12} \\ -M_{21} & -M_{22} \end{pmatrix}$$

$-\|$

$$A_1 A_0 = \begin{pmatrix} M_{11} & -M_{12} \\ M_{21} & -M_{22} \end{pmatrix}$$

$$\Rightarrow M_{11} = M_{22} = 0$$

$$A_1 = \begin{pmatrix} 0 & M_{12} \\ M_{21} & 0 \end{pmatrix} \quad M_{21} = M_{12}^\dagger$$

$$A_1^2 = \begin{pmatrix} M_{12}M_{21} & 0 \\ 0 & M_{21}M_{12} \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}$$

$$\left.\begin{aligned} \mathrm{tr}(M_{12}M_{12}^\dagger) &= m \\ \mathrm{tr}(M_{12}^\dagger M_{12}) &= n \end{aligned}\right\} \quad m = n$$

$$M_{12}M_{12}^\dagger = I_m$$

unitary C

$$\mathcal{U} = \begin{pmatrix} I_m & 0 \\ 0 & C^\dagger \end{pmatrix}$$

$$\mathcal{U}^\dagger A_0 \mathcal{U} = \begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & C^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -C \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & C^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = Z \otimes I \checkmark$$

$$\mathcal{U}^{\dagger} A_1 \mathcal{U} = \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 0 & C \\ C^{\dagger} & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & C^{\dagger} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & C \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & C^{\dagger} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = X \otimes I \quad \checkmark$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Summary:

$$\omega^{*}_{CHSH} \leq \cos^2\left(\frac{\pi}{8}\right)$$

$$\omega^{*}_{CHSH}(S) = \cos^2\left(\frac{\pi}{8}\right)$$

$$\Longrightarrow \quad A_{0/1} \otimes I \lvert \psi \rangle = I \otimes \frac{B_0 \pm B_1}{\sqrt{2}} \lvert \psi \rangle$$

$$\Rightarrow \quad A_0 A_1 \lvert \psi \rangle = - A_1 A_0 \lvert \psi \rangle$$

gap

$$A_0 A_1 = -A_1 A_0 \Rightarrow \check{A}_0 \longrightarrow Z \otimes I$$
$$A_1 \longrightarrow X \otimes I$$

Next time:

- Close the gap
- Pin down $|\psi\rangle$
- Approximate setting $\left(\omega^*(S) = \cos^2\left(\frac{\pi}{8}\right) - \varepsilon \right)$
  "self-testing"