6.S979 Lecture 6

Fill out O.H. poll on Piazza
This week Friday 1–2pm

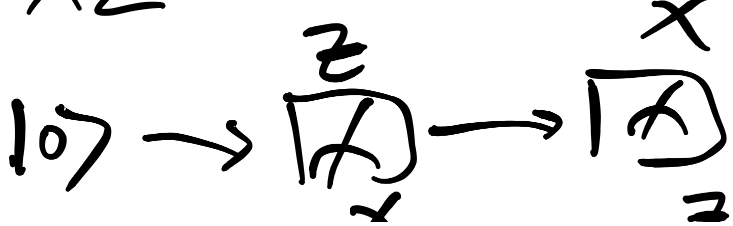
Today: "contextuality"

Bell-Kochen-Specker Thm:

Recall if observables A, B that <u>don't</u> commute
$\Rightarrow$ not compatible
not simultaneously measurable

A: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$XZ = -ZX$ anticommuting



$\boxed{\hat{\lambda}}$ $\boxed{\hat{\alpha}}$

Let's suppose outcomes determined by hidden var λ

1. Value definiteness

Outcome of measuring A

$$v(A, \lambda) \in \{\pm 1\}$$

2. Functional consistency for compatible observables

If A, B commute

$C = AB$

$$v(C, \lambda) = v(A, \lambda) \cdot v(B, \lambda)$$

This is true w/ certainty in QM.

Pf: A, B commute
$\Rightarrow$ simul. diagonalizable

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \quad B = \begin{pmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{pmatrix}$$

$$C = \begin{pmatrix} \lambda_1 \mu_1 & & & 0 \\ & \lambda_2 \mu_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \mu_n \end{pmatrix}$$

Thm [Bell-Kochen-Specker]:

If dimension ≥ 3, then $\exists$ observables s.t. no HVT obeying 1 & 2 is consistent w/ them

Historical aside:

von Neumann showed this w/ (2) assumed even for non-compatible observables

$X, Z \quad \frac{X+Z}{\sqrt{2}}$

$$v\left(\frac{X+Z}{\sqrt{2}}\right) = v(X) + v(Z)$$

Bell [early 60s]

showed thm

KS [later 60s] independently

Mermin Peres 80s

Much simpler proof

$d \geq 4$

Mermin-Peres Magic Square

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X^2 = Z^2 = I$$

$$XZ = -ZX$$

We'll work over $\mathbb{C}^2 \otimes \mathbb{C}^2$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$= I \otimes I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$= I \otimes I$
$Z \otimes X$	$X \otimes Z$	$ZX \otimes XZ$	$= I \otimes I$
	$= I \otimes I$	$= -I \otimes I = (XZZX) \otimes (XZXZ)$	

• every row col is compatible

= even
+
= even
+
= even
even

even + even + odd = odd

E.g. col 3

$X \otimes X$ ✓

$Z \otimes Z$ ✓

$ZX \otimes XZ$ ✓

$$(X \otimes X)(Z \otimes Z)$$
$$= XZ \otimes XZ$$
$$= -ZX \otimes XZ$$
$$= +ZX \otimes ZX$$
$$= (Z \otimes Z)(X \otimes X)$$

$$(X\otimes X)(ZX\otimes XZ)$$
$$= XZX \otimes XXZ$$
$$= -ZXX \otimes XXZ$$
$$= -ZXX \otimes -XZX$$
$$= +(ZX\otimes XZ)(X\otimes X)$$

$I\otimes X$	$X\otimes I$	$X\otimes X$	$= I\otimes I$
$Z\otimes I$	$I\otimes Z$	$Z\otimes Z$	$= I\otimes I$
$Z\otimes X$	$X\otimes Z$	$ZX\otimes XZ$	$= I\otimes I$
$= I\otimes I$	$= I\otimes I$	$= -I\otimes I$	

Suppose HVT $v(A,\lambda)$

(1) $\Longrightarrow$

v_0	v_1	v_2
v_3	v_4	v_5
v_6	v_7	v_8

$(2) \Rightarrow$

$A_2 = A_0 A_1$

$$
\begin{array}{c|c|c|c}
A_0 & A_1 & A_2 & \\
 & & & = I \\
\hline
 & & & = I \\
\hline
 & & & = 1 \\
= 1 & = 1 & = -1 &
\end{array}
$$

There is no possible choice of v_i's to make this work

"Contextuality"

To assign a value to A_0 need to know whether it's being measured with A_1, A_2 or w/ A_3, A_6 "contexts"

A_0	1	2
3	4	5
6	7	8

State-independent contextuality

Contextuality | Non-locality

Bell–KS

Assume some measurements are compatible $\Longrightarrow$ $\nexists$ non-contextual HVM

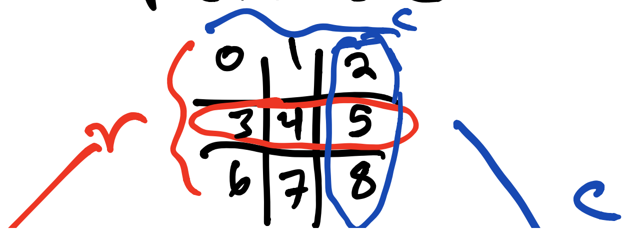
$\Uparrow$ Bell, CHSH

Use spatial nonlocality to enforce Alice measurements are compatible w/ Bob measurements

MS game [Aravind late 90s early '00s]

Referee

Alice



Bob

y_2

$\underline{x_3 x_4 x_5} \overset{?}{=} \underline{y_5 y_8}$

$\overset{?}{=} 1$; $\overset{?}{=} -1$

Win if valid row & col assignment and consistent

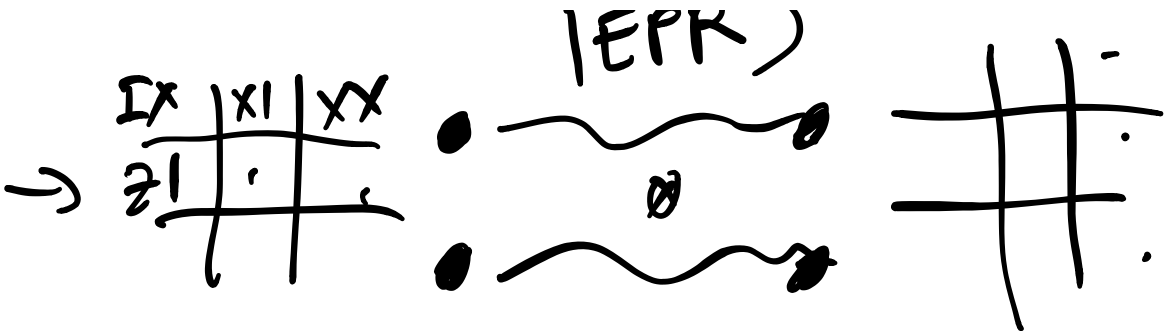
Classical value:

$\omega_{classical} < 1$
$= 8/9$

Quantum value:

$\omega^* = 1$

$)^{\otimes 2}$?



Regardless of $|\psi\rangle$, A & B satisfy row & col conditions

$|\psi\rangle = |EPR\rangle^{\otimes 2} \Longrightarrow$ A & B satisfy consistency conditions

$\Rightarrow$ MS game has "perfect completeness"

"pseudo-telepathy"

$(\omega^* = 1)$

Moreover, MS is a self-test for $|EPR\rangle^{\otimes 2}$

$\omega^*(S) \geq 1 - \varepsilon$

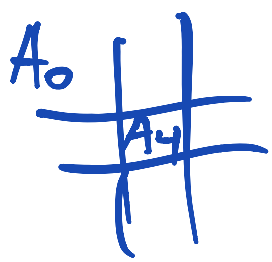
$\exists$ local isometries V_A, V_B s.t.

$$V_A \otimes V_B|\psi\rangle \approx_{\varepsilon^{1/4}} |EPR\rangle^{\otimes 2} \otimes |aux\rangle$$

$$(V_A \otimes V_B)(A_0 \otimes I)|\psi\rangle \approx_{\varepsilon^{1/4}} (IX \otimes II)|EPR\rangle^{\otimes 2} \otimes |aux\rangle$$

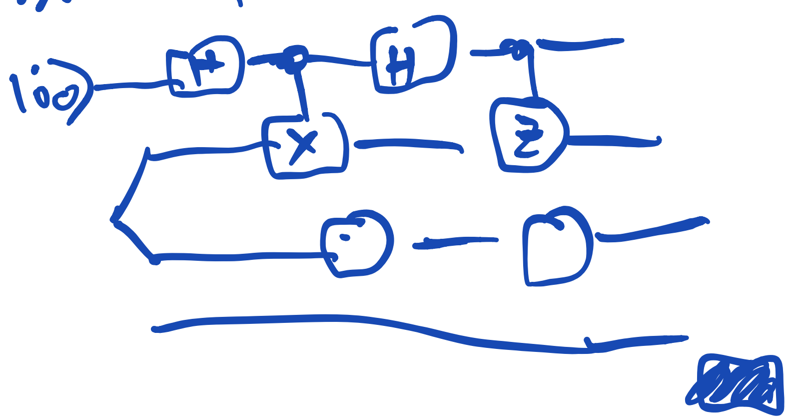
$\vdots$

Pf: $\omega^*(S) \geq 1-\varepsilon$



$$\Rightarrow A_0 A_4 \otimes I|\psi\rangle \approx -A_4 A_0 \otimes I|\psi\rangle$$

$\Rightarrow$



Generalizations:

- Robert Spekkens et al.
  operational, noise-robust versions
  "measurement contextuality"
  "preparation contextuality"

- Graph-based contextuality
  Cabello Severini Winter '10

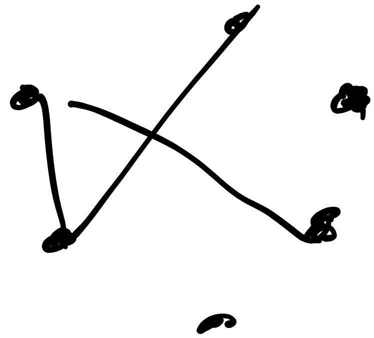
Say you have "events" represented by projectors $\{P_i\}_i$

$P_i \sim P_j$ if $P_i + P_j \leq I$

$\Rightarrow P_i P_j = 0$
$= P_j P_i = 0$

compatible, mutually exclusive

G:



$\langle \sum_i P_i \rangle$

Classical, non-contextual model

$$\langle \sum_i P_i \rangle \le \alpha(G)$$

independent set #

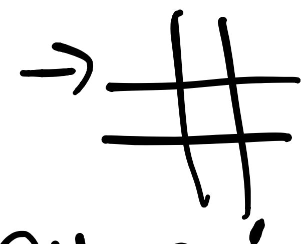
Contextuality inequality if

$\exists |\psi\rangle$ s.t.

$$\langle \psi | \sum_i P_i |\psi\rangle > \alpha(G)$$

E.g. For each row or col. P_i

for every "valid" assignment



000 011 101

110

24 P_i's in total

Connections to relaxations of α

Lovász theta function

<u>Next time:</u> Show that this doesn't work mod p, $p \neq 2$

$\exists$ non-contextual HVM for mod p versions of X and Z