6.S979: Problem Set 1 Solutions

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1. **Maximally entangled states:** In this problem, we will work with a generalization of the EPR state called the *maximally entangled state*. Consider the state space $\mathbb{C}^d \otimes \mathbb{C}^d$, and denote the standard basis of $\mathbb{C}^d$ by $\{|1\rangle, \ldots, |d\rangle\}$. The *maximally entangled state* in this space is defined to be

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle .$$

   (a) Show that for any $d \times d$ matrix A, it holds that

$$A \otimes I |\Phi\rangle = I \otimes A^T |\Phi\rangle ,$$

   where A^T is the transpose of A. (Extra food for thought: is the transpose basis-dependent?) *In bra-ket notation, $A = \sum_{ij} A_{ij} |i\rangle \langle j|$. Substituting this into the LHS of the equation we wish to show, we have*

$$\begin{aligned} A \otimes I |\Pi\rangle &= \sum_{ij} A_{ij} |i\rangle \langle j| \otimes I \left(\frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle \otimes |k\rangle \right) \\ &= \frac{1}{\sqrt{d}} \sum_{ij} A_{ij} |i\rangle \otimes |j\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{ij} (I \otimes A_{ij} |j\rangle \langle i|) |i\rangle \otimes |i\rangle \\ &= \sum_{ij} I \otimes A_{ij} |j\rangle \langle i| \left(\frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle \otimes |k\rangle \right) \\ &= I \otimes A^T |\Phi\rangle . \end{aligned}$$

   The transpose is "basis dependent" in the sense that it's not invariant under unitary changes of basis. The quantity that is invariant is the Hermitian conjugate (i.e. the conjugate transpose).

   (b) Show that for any two $d \times d$ matrices A and B, it holds that

$$\langle \Phi | A \otimes B |\Phi\rangle = \frac{1}{d} \text{tr}(AB^T).$$

Observe that by the previous part, $A \otimes B |\Phi\rangle = I \otimes BA^T |\Phi\rangle$. *So we have*

$$\begin{aligned}\langle\Phi| A \otimes B |\Phi\rangle &= \langle\Phi| I \otimes BA^T |\Phi\rangle \\ &= \frac{1}{d}\sum_{ij} \langle ii| (I \otimes BA^T) |jj\rangle \\ &= \frac{1}{d}\sum_{ij} \langle i|j\rangle \langle i| (BA^T) |j\rangle \\ &= \frac{1}{d}\sum_{i} \langle i| (BA^T) |i\rangle \\ &= \mathrm{tr}(BA^T) = \mathrm{tr}((BA^T)^T) = \mathrm{tr}(AB^T).\end{aligned}$$

(For compactness, we have written $|ii\rangle$ *instead of* $|i\rangle \otimes |i\rangle$*.)*

(c) Show that for any orthonormal basis $\{|v_1\rangle, \dots, |v_d\rangle\}$ of $\mathbb{C}^d$, the maximally entangled state can be expressed as

$$|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |v_i\rangle \otimes |v_i^*\rangle,$$

where $|v_i^*\rangle$ is the complex conjugate of the vector $|v_i\rangle$. *For any such orthonormal basis, there exists a unitary* U *such that* $U|i\rangle = |v_i\rangle$ *for all* $i \in \{1, \dots, d\}$. *The given state can be expressed in terms of* $|\Phi\rangle$ *as defined in the rest of the problem by*

$$\frac{1}{\sqrt{d}}\sum_{i} |v_i\rangle \otimes |v_i^*\rangle = U \otimes U^* |\Phi\rangle = I \otimes U^*U^T |\Phi\rangle = |\Phi\rangle,$$

where the last equality holds since U *is a unitary, and thus* $U^*U^T = (UU^\dagger)^* = I$.

2. **Stabilizers:** Recall the Pauli X and Z matrices from class

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Write an eigendecomposition for $X \otimes X$ and $Z \otimes Z$. *Both of these matrices are observables, with eigenvalues* ± 1. *For* $Z \otimes Z$, *the* $+1$ *eigenspace is spanned by* $|00\rangle$ *and* $|11\rangle$, *and the* -1 *eigenspace is spanned by* $|01\rangle$ *and* $|10\rangle$. *One can say something analogous for* $X \otimes X$ *in terms of the eigenbases* $|\pm\rangle$ *for* X; *an alternate set of eigenvectors is* $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ *and* $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ *for the* $+1$ *eigenspace, and* $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ *and* $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ *for the* -1 *eigenspace.*

(b) A state $|\psi\rangle$ is *stabilized* by an operator M if $M|\psi\rangle = |\psi\rangle$. Write down the states stabilized by

i. $X \otimes I$ and $I \otimes Z$. $|\psi\rangle = |+\rangle |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$.

ii. $X \otimes X$ and $Z \otimes Z$. $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\mathrm{EPR}\rangle$

iii. $X \otimes X$ and $-Z \otimes Z$. $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

(c) Is there a state stabilized by $X \otimes X$ and $Z \otimes I$? If not, why not? *There is no such state. One argument is to observe that* $X \otimes X$ *anticommutes with* $Z \otimes I$. *So if there were a state* $|\psi\rangle$ *stabilized by both, we would have*

$$|\psi\rangle = (X \otimes X)|\psi\rangle = (X \otimes X)(Z \otimes I)|\psi\rangle = -(Z \otimes I)(X \otimes X)|\psi\rangle = -(Z \otimes I)|\psi\rangle = -|\psi\rangle,$$

implying that $|\psi\rangle = 0$.

(d) (Optional:) Suppose that $\langle\psi|(X \otimes X + Z \otimes Z)|\psi\rangle \geq 2 - \epsilon$. Find a bound on the minimal Euclidean distance $\min_\theta \| e^{i\theta}|\psi\rangle - |EPR\rangle \|$ between a state that is a multiple of $|\psi\rangle$ and the EPR state, as a function of ϵ. (Hint: consider the eigendecomposition of the matrix $X \otimes X + Z \otimes Z$.) *Let $M = X \otimes X + Z \otimes Z$. Before we bust out Mathematica to calculate the eigendecomposition of M, let's try guessing, based on the vectors we've already constructed for the previous parts. We know already that there is a $+2$ eigenvector, namely $|EPR\rangle$, and that there's a 0 eigenvector, namely the state stabilized by $X \otimes X$ and $-Z \otimes Z$. We can guess the other two egeinvectors: one with eigenvalue 0, and one with eigenvalue -2.*

$$\lambda_1 = 2, |v_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$\lambda_2 = 0, |v_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$\lambda_3 = 0, |v_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$\lambda_4 = -2, |v_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

We can write $M = \sum_i \lambda_i |v_i\rangle\langle v_i|$. We are given

$$\langle\psi| M |\psi\rangle = \sum_i \lambda_i |\langle v_i|\psi\rangle|^2 \geq 2 - \epsilon$$
$$2|\langle EPR|\psi\rangle|^2 - 2|\langle v_4|\psi\rangle|^2 \geq 2 - \epsilon$$
$$|\langle EPR|\psi\rangle|^2 \geq 1 - \epsilon/2$$

Now observe

$$\begin{aligned} \| e^{i\theta}|\psi\rangle - |EPR\rangle \|^2 &= \langle\psi|\psi\rangle + \langle EPR|EPR\rangle - 2\Re\langle EPR| e^{i\theta}|\psi\rangle \\ &= 2 - 2\Re\langle EPR| e^{i\theta}|\psi\rangle \\ &\geq 2 - 2|\langle EPR|\psi\rangle|, \end{aligned}$$

and this inequality becomes an equality for appropriately chosen θ (so that $|\psi\rangle$ is real and has positive inner product with $|EPR\rangle$). Thus, we have

$$\begin{aligned} \min_\theta \| e^{i\theta}|\psi\rangle - |EPR\rangle \| &= \sqrt{2 - 2|\langle EPR|\psi\rangle} \\ &=\leq \sqrt{2 - 2\sqrt{1 - \epsilon/2}}. \end{aligned}$$

3. **The GHZ game:** In this problem, we will introduce *tripartite* states, corresponding to three quantum systems. Suppose Alice, Bob, and Charlie each have a single qubit. Then their joint state space is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. As usual, we denote the standard basis of $\mathbb{C}^2$ by $\{|0\rangle, |1\rangle\}$. X and Z are the Pauli matrices as in the previous problem.

(a) The *GHZ state* is the following entangled state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle).$$

(b) Write down all tensor products of X, Z, and the identity I that stabilize $|GHZ\rangle$. You should find five such matrices, including $I \otimes I \otimes I$. *They are III, XXX, ZZI, ZIZ, IZZ (where we have suppressed the tensor product symbol for compactness).*

(c) Suppose Alice and Bob have lost contact with Charlie. Show that nevertheless they can distinguish between the GHZ state and the following state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)_{AB} \otimes |1\rangle_C .$$

Do this by finding an observable $\mathcal{O}$ acting on Alice and Bob's systems such that

$$\langle\psi| \mathcal{O} \otimes I |\psi\rangle \neq \langle GHZ| \mathcal{O} \otimes I |GHZ\rangle .$$

(Hint: consider a tensor product of X or Z matrices). *Take $\mathcal{O} = X \otimes X \otimes I$. This has expectation value 0 on the GHZ state and expectation value 1 on $|\psi\rangle$.*

(d) In the *GHZ game*, Alice, Bob, and Charlie are separated so that they cannot communicate, and play together against a referee. The referee samples a triple of bits (x, y, z) from $\{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$ uniformly at random, and sends x to Alice, y to Bob, and z to Charlie. Each player responds with a single-bit answer; we denote Alice, Bob, and Charlie's answers by a, b, and c respectively. The players win if $x \vee y \vee z = a \oplus b \oplus c$.

i. What is the maximum probability of winning for Alice, Bob, and Charlie if they use a classical strategy? *First of all, we can restrict to deterministic strategies by the argument given in class. So we have four equations*

$$\begin{aligned} a_0 + b_0 + c_0 &= 0 \\ a_0 + b_1 + c_1 &= 1 \\ a_1 + b_0 + c_1 &= 1 \\ a_1 + b_1 + c_0 &= 1 \end{aligned}$$

in binary variables that we would like to satisfy. From adding the four equations together, we obtain $0 = 1$, showing that they cannot all be satisfied simultaneously. We can easily satisfy three out of the four by setting $a_i = b_i = c_i = i$ for $i \in \{0, 1\}$. So the classical value is $3/4$.

ii. Describe a quantum strategy for the players to win the game with certainty. (Hint: use the GHZ state, and the stabilizers you found in the first part of the problem.) *Just as in the previous part, the conditions for a perfect strategy can be expressed as a system of equations, this time in the observables used by the players:*

$$\begin{aligned} A_0 \otimes B_0 \otimes C_0 |\psi\rangle &= |\psi\rangle \\ A_0 \otimes B_1 \otimes C_1 |\psi\rangle &= -|\psi\rangle \\ A_1 \otimes B_0 \otimes C_1 |\psi\rangle &= -|\psi\rangle \\ A_1 \otimes B_1 \otimes C_0 |\psi\rangle &= -|\psi\rangle \end{aligned}$$

Let's guess $|\psi\rangle = |GHZ\rangle$. Then, from the first part of the problem, we know that $X \otimes X \otimes X$ is a stabilizer, so let's set $A_0 = B_0 = C_0 = X$. Now, for the remaining equations, we want three stabilizers that have an X on one tensor factor. Observe that if we multiply $X \otimes X \otimes X$ by $I \otimes Z \otimes Z$, we get $X \otimes (-iY) \otimes (-iY) = -X \otimes Y \otimes Y$. So we can choose $A_1 = B_1 = C_1 = Y$.