6.S979: Problem Set 1

Due: October 2, 2020

1. **Maximally entangled states:** In this problem, we will work with a generalization of the EPR state called the *maximally entangled state*. Consider the state space $\mathbb{C}^d \otimes \mathbb{C}^d$, and denote the standard basis of $\mathbb{C}^d$ by $\{|1\rangle, \ldots, |d\rangle\}$. The *maximally entangled state* in this space is defined to be

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle .$$

   (a) Show that for any $d \times d$ matrix A, it holds that

$$A \otimes I |\Phi\rangle = I \otimes A^T |\Phi\rangle ,$$

   where A^T is the transpose of A. (Extra food for thought: is the transpose basis-dependent?)

   (b) Show that for any two $d \times d$ matrices A and B, it holds that

$$\langle\Phi| A \otimes B |\Phi\rangle = \frac{1}{d}\mathrm{tr}(AB^T).$$

   (c) Show that for any orthonormal basis $\{|v_1\rangle, \ldots, |v_d\rangle\}$ of $\mathbb{C}^d$, the maximally entangled state can be expressed as

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |v_i\rangle \otimes |v_i^*\rangle ,$$

   where $|v_i^*\rangle$ is the complex conjugate of the vector $|v_i\rangle$.

2. **Stabilizers:** Recall the Pauli X and Z matrices from class

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Write an eigendecomposition for $X \otimes X$ and $Z \otimes Z$.

   (b) A state $|\psi\rangle$ is *stabilized* by an operator M if $M|\psi\rangle = |\psi\rangle$. Write down the states stabilized by

      i. $X \otimes I$ and $I \otimes Z$.

      ii. $X \otimes X$ and $Z \otimes Z$.

      iii. $X \otimes X$ and $-Z \otimes Z$.

   (c) Is there a state stabilized by $X \otimes X$ and $Z \otimes I$? If not, why not?

(d) (Optional:) Suppose that $\langle\psi| (X \otimes X + Z \otimes Z) |\psi\rangle \geq 2 - \epsilon$. Find a bound on the minimal Euclidean distance $\min_\theta \| e^{i\theta} |\psi\rangle - |EPR\rangle \|$ between a state that is a multiple of $|\psi\rangle$ and the EPR state, as a function of ϵ. (Hint: consider the eigendecomposition of the matrix $X \otimes X + Z \otimes Z$.)

3. **The GHZ game:** In this problem, we will introduce *tripartite* states, corresponding to three quantum systems. Suppose Alice, Bob, and Charlie each have a single qubit. Then their joint state space is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. As usual, we denote the standard basis of $\mathbb{C}^2$ by $\{|0\rangle, |1\rangle\}$. X and Z are the Pauli matrices as in the previous problem.

   (a) The *GHZ state* is the following entangled state

   $$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle).$$

   (b) Write down all tensor products of X, Z, and the identity I that stabilize $|GHZ\rangle$. You should find five such matrices, including $I \otimes I \otimes I$.

   (c) Suppose Alice and Bob have lost contact with Charlie. Show that nevertheless they can distinguish between the GHZ state and the following state

   $$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)_{AB} \otimes |1\rangle_C .$$

   Do this by finding an observable $\mathcal{O}$ acting on Alice and Bob's systems such that

   $$\langle\psi| \mathcal{O} \otimes I |\psi\rangle \neq \langle GHZ| \mathcal{O} \otimes I |GHZ\rangle .$$

   (Hint: consider a tensor product of X or Z matrices).

   (d) In the *GHZ game*, Alice, Bob, and Charlie are separated so that they cannot communicate, and play together against a referee. The referee samples a triple of bits (x, y, z) from $\{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$ uniformly at random, and sends x to Alice, y to Bob, and z to Charlie. Each player responds with a single-bit answer; we denote Alice, Bob, and Charlie's answers by a, b, and c respectively. The players win if $x \vee y \vee z = a \oplus b \oplus c$.

      i. What is the maximum probability of winning for Alice, Bob, and Charlie if they use a classical strategy?

      ii. Describe a quantum strategy for the players to win the game with certainty. (Hint: use the GHZ state, and the stabilizers you found in the first part of the problem.)