Product-state approximations to quantum ground states

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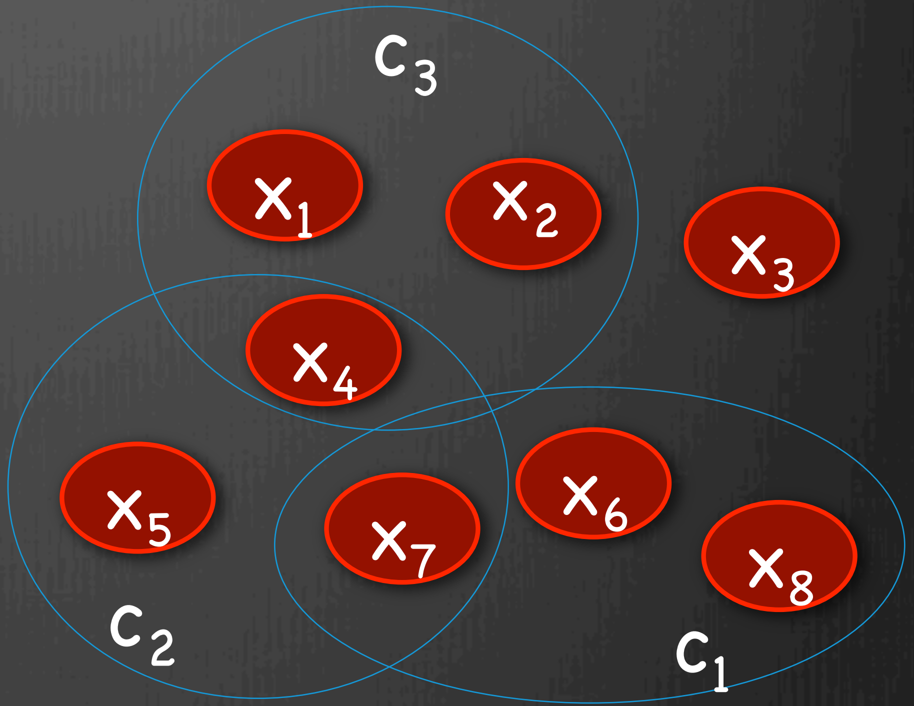
Constraint Satisfaction Problems

k-CSP:

Variables $\{x_1, \ldots, x_n\}$ in Σ^n

Alphabet Σ

Constraints $\{c_1, \ldots, c_m\}$
$c_j : \Sigma^k \rightarrow \{0,1\}$



UNSAT:= $\min_{x \in \Sigma^n} \frac{1}{m} \sum_{j=1}^{m} c_j(x_{j_1}, \ldots, x_{j_k})$

Includes 3-SAT, max-cut, vertex cover, ...
Computing UNSAT is NP-complete

CSPs $\sim$ eigenvalue problems

Hamiltonian $$H = \frac{1}{m}\sum_{j=1}^{m} C_j \in M_d^{\otimes n}$$ d = |Σ|

local terms $$C_j := \sum_{\substack{z \in \Sigma^k \\ c_j(z)=1}} |z_1, \ldots, z_k\rangle\langle z_1, \ldots, z_k|$$

UNSAT = λ_{min} (H)

e.g. Ising model, Potts model, general classical Hamiltonians

Local Hamiltonians, aka quantum k-CSPs

k-local Hamiltonian: $$H = \frac{1}{m}\sum_{i=1}^{m} H_i \in M_d^{\otimes n}$$

local terms: each H_i acts nontrivially on $\leq$ k qudits and is bounded: $\|H_i\| \leq 1$

qUNSAT = λ_{min} (H)

optimal assignment = ground state wavefunction

How hard are qCSPs?

Quantum Hamiltonian Complexity addresses this question

The local Hamiltonian problem

Problem

Given a local Hamiltonian H, decide if $\lambda_{min}(H) \leq \alpha$ or $\lambda_{min}(H) \geq \alpha + \Delta$.

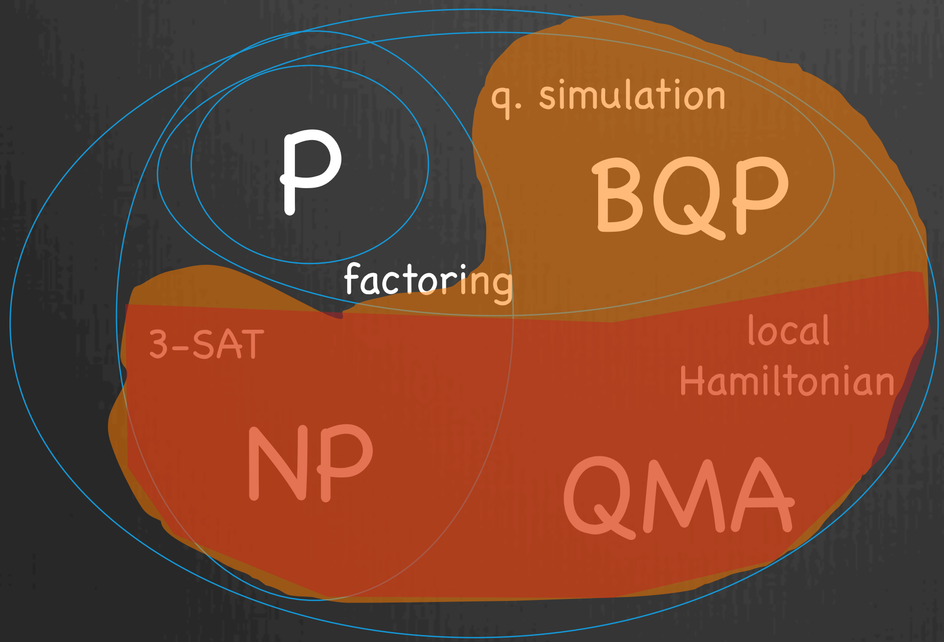
Thm [Kitaev '99] The local Hamiltonian problem is QMA-complete for $\Delta = 1/\mathrm{poly}(n)$.
(quantum analogue of the Cook-Levin theorem)

QMA := quantum analogue of NP, i.e. can verify quantum proof in poly time on quantum computer.

Even simple models are QMA-complete:
Oliveira-Terhal '05: qubits on 2-D grid
Aharanov-Gottesman-Irani-Kempe '07: qudits in 1-D
Childs-Gosset-Webb: Bose-Hubbard model in 2-D

quantum complexity theory

complexity	classical	quantum
computable in polynomial time	P	BQP
verifiable in polynomial time	NP	QMA



Conjectures

Requires exponential time to solve on classical computers.

Requires exponential time to solve even on quantum computers.

NP vs QMA

Can you give me some description I can use to get a 0.1% accurate estimate using fewer than 10^{50} steps?

NO.

I CAN, HOWEVER, GIVE YOU MANY PROTONS, WHOSE MASS YOU CAN MEASURE.

Constant accuracy?

3-SAT revisited:
NP-hard to determine if UNSAT=0 or UNSAT ≥ $1/n^3$

PCP theorem: [Babai-Fortnow-Lund '90, Arora-Lund-Motwani-Sudan-Szegedy '98]
NP-hard to determine if UNSAT(C)=0 or UNSAT(C) ≥ 0.1
Equivalent to existence of Probabilistically Checkable Proofs for NP.

Quantum PCP conjecture:
There exists a constant $\Delta>0$ such that it is QMA complete to estimate λ_{min} of a 2-local Hamiltonian H to accuracy $\Delta \cdot ||H||$.

- [Bravyi, DiVincenzo, Terhal, Loss `08] Equivalent to conjecture for O(1)-local Hamiltonians over qudits.
- ≈ equivalent to estimating the energy at constant temperature.
- Contained in QMA. At least NP-hard (by the PCP theorem).

Previous Work and Obstructions

[Aharonov, Arad, Landau, Vazirani '08]
Quantum version of 1 of 3 parts of Dinur's proof of the PCP thm (gap amplification)

But: The other two parts (alphabet and degree reductions) involve massive copying of information; not clear how to do it with a highly entangled assignment

[Bravyi, Vyalyi '03; Arad '10; Hastings '12; Freedman, Hastings '13; Aharonov, Eldar '13, ...]
No-go (NP witnesses) for large class of commuting Hamiltonians and almost-commuting Hamiltonians

But: Commuting case might really be easier

result 1: high-degree in NP

<u>Theorem</u>

If H is a 2-local Hamiltonian on a D-regular graph of n qudits, then there exists a product state $|\psi\rangle = |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$ such that

$$\lambda_{min} \leq \langle\psi|H|\psi\rangle \leq \lambda_{min} + O(d^{2/3} / D^{1/3})$$

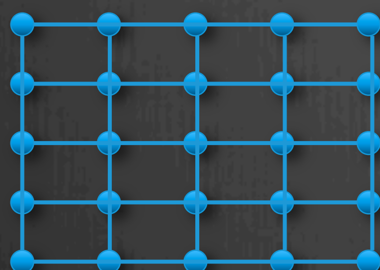
<u>Corollary</u>

The ground-state energy can be approximated to accuracy $O(d^{2/3} / D^{1/3})$ in NP.

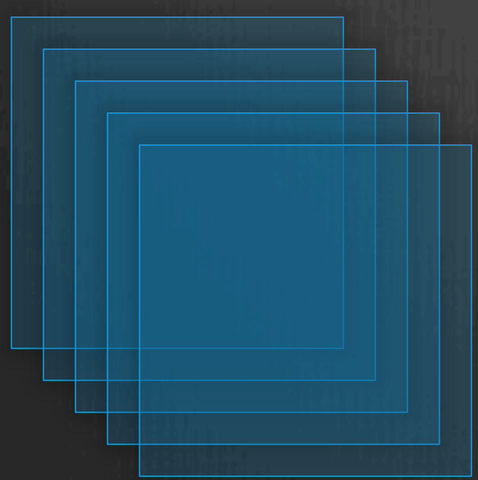
intuition: mean-field theory

1-D

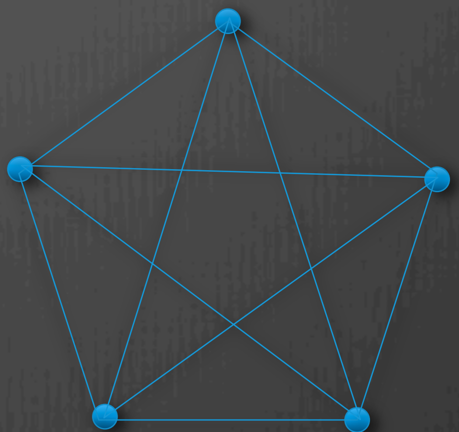
2-D



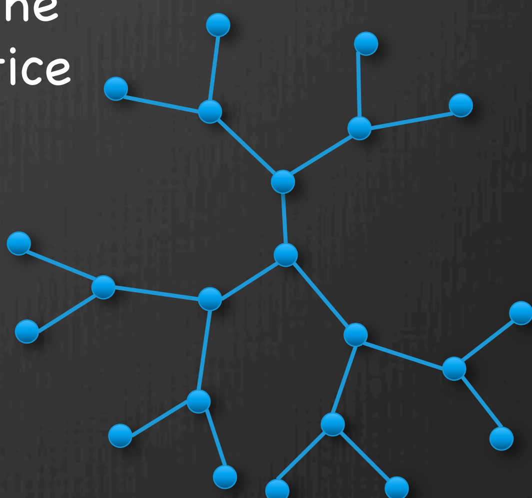
3-D



∞-D



Bethe lattice



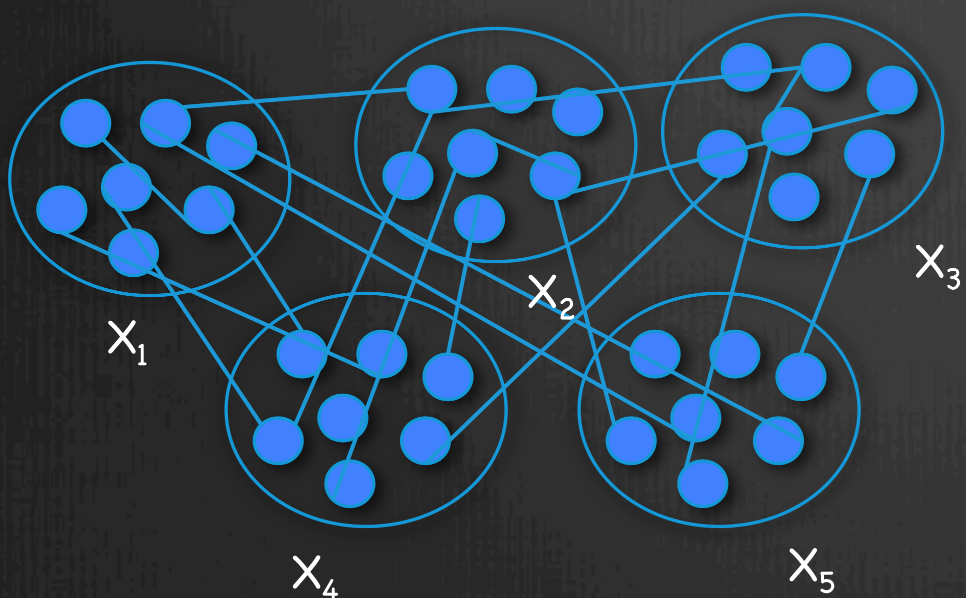
clustered approximation

Given a Hamiltonian H on a graph G with vertices partitioned into m-qudit clusters (X_1, ..., $X_{n/m}$), can approximate λ_{min} to error

$$9\left(d^2\,\mathbb{E}_i[\Phi(X_i)]\,\frac{1}{D}\,\mathbb{E}_i\,\frac{S(X_i)_{\psi_0}}{m}\right)^{1/3}$$

with a state that has no entanglement between clusters.

$$\Phi(X_i) = \Pr_{(u,v)\in E}(v \notin X_i | u \in X_i)$$



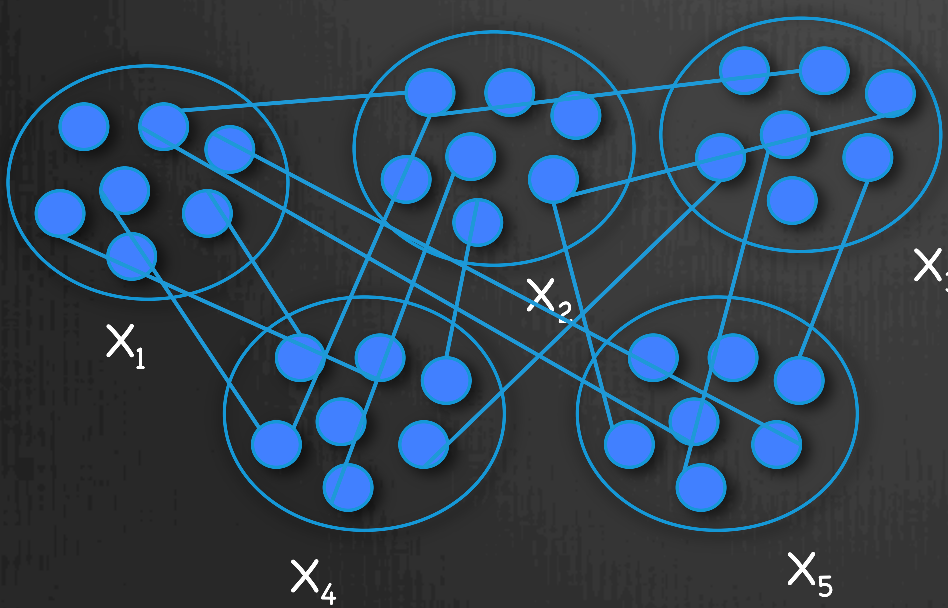
good approximation if

1. expansion is o(1)
2. degree is high
3. entanglement satisfies subvolume law

1. Approximation from low expansion

$$9\left(d^2 \mathop{\mathbb{E}}_i[\Phi(X_i)] \frac{1}{D} \mathop{\mathbb{E}}_i \frac{S(X_i)_{\psi_0}}{m}\right)^{1/3}$$

$$\Phi(X_i) = \Pr_{(u,v)\in E}(v \notin X_i | u \in X_i)$$



Hard instances must use highly expanding graphs

2. Approximation from high degree

$$9\left(d^2\,\mathbb{E}_i[\Phi(X_i)]\frac{1}{D}\,\mathbb{E}_i\frac{S(X_i)_{\psi_0}}{m}\right)^{1/3}$$

Unlike classical CSPs:

PCP + parallel repetition imply that 2-CSPs are NP-hard to approximate to error d^α/D^β for any $\alpha,\beta>0$.

Parallel repetition maps C → C' such that

1. $D' = D^k$
2. $\Sigma' = \Sigma^k$
3. UNSAT(C) = 0 → UNSAT(C')=0
   UNSAT(C) > 0 → UNSAT(C') > UNSAT(C)

Corollaries:

1. Quantum PCP and parallel repetition not both true.
2. $\Phi \leq 1/2 - \Omega(1/D)$ means highly expanding graphs in NP.

3. Approximation from low entanglement

$$9\left(d^2\,\mathbb{E}_i[\Phi(X_i)]\,\frac{1}{D}\,\mathbb{E}_i\,\frac{S(X_i)_{\psi_0}}{m}\right)^{1/3}$$

Subvolume law ($S(X_i) \ll |X_i|$) implies NP approximation

1. Previously known only if $S(X_i) \ll 1$.

2. Connects entanglement to complexity.

3. For mixed states, can use mutual information instead.

proof sketch

mostly following [Raghavendra-Tan, SODA '12]

Chain rule Lemma:

$I(X:Y_1 \ldots Y_k) = I(X:Y_1) + I(X:Y_2|Y_1) + \ldots + I(X:Y_k|Y_1 \ldots Y_{k-1})$

→ $I(X:Y_t|Y_1 \ldots Y_{t-1}) \leq \log(d)/k$ for some $t \leq k$.

Decouple most pairs by conditioning:

Choose $i, j_1, \ldots, j_k$ at random from $\{1, \ldots, n\}$

Then there exists $t<k$ such that

$$\mathop{\mathbb{E}}_{i,j,j_1,\ldots,j_t} I(X_i : X_j | X_{j_1} \ldots X_{j_t}) \leq \frac{\log(d)}{k}$$

Discarding systems $j_1,\ldots,j_t$ causes error $\leq k/n$ and leaves a distribution q for which

$$\mathop{\mathbb{E}}_{i,j} I(X_i : X_j)_q \leq \frac{\log(d)}{k} \quad \Rightarrow \quad \mathop{\mathbb{E}}_{i \sim j} I(X_i : X_j)_q \leq \frac{n}{D} \frac{\log(d)}{k}$$

Does this work quantumly?

<u>What changes?</u>
😊 Chain rule, Pinsker, etc, still work.
😧 Can't condition on quantum information.
😥 $I(A:B|C)_\rho \approx 0$ doesn't imply ρ is approximately separable [Ibinson, Linden, Winter '08]

<u>Key technique</u>: informationally complete measurement maps quantum states into probability distributions with poly(d) distortion.

$$d^{-2} \| \rho - \sigma \|_1 \leq \| M(\rho) - M(\sigma) \|_1 \leq \| \rho - \sigma \|_1$$

Proof of qPCP no-go

1. Measure ε n qudits and condition on outcomes.
   Incur error ε.

2. Most pairs of other qudits would have mutual information
   $\leq \log(d) / \varepsilon D$ if measured.

3. Thus their state is within distance $d^2(\log(d) / \varepsilon D)^{1/2}$ of product.

4. Witness is a global product state. Total error is
   $\varepsilon + d^2(\log(d) / \varepsilon D)^{1/2}$.
   Choose ε to balance these terms.

result 2: "P"TAS

PTAS for Dense k-local Hamiltonians
improves on $1/d^{k-1} + \varepsilon$ approximation from [Gharibian-Kempe '11]

PTAS for planar graphs
Builds on [Bansal, Bravyi, Terhal '07] PTAS for bounded-degree planar graphs

Algorithms for graphs with low threshold rank
Extends result of [Barak, Raghavendra, Steurer '11].
run-time for ε-approximation is
$\exp(\log(n)\ \mathrm{poly}(d/\varepsilon) \cdot \#\{\text{eigs of adj. matrix} \geq \mathrm{poly}(\varepsilon/d)\})$

The Lasserre SDP hierarchy for local Hamiltonians

	Classical	Quantum
problem	2-CSP	2-local Hamiltonian
LP hierarchy	Optimize over k-body marginals	
	E[f] for deg(f) ≤ k	$\langle\psi\|H\|\psi\rangle$ for k-local H (technically an SDP)
SDP hierarchy	Add global PSD constraint	
	$E[f^2]\geq 0$ for deg(f)≤k/2	$\langle\psi\|H^\dagger H\|\psi\rangle\geq 0$ for k/2-local H
analysis when $k = poly(d/\varepsilon)\cdot rank_{poly(\varepsilon/d)}(G)$	Barak-Raghavendra-Steurer 1104.4680	similar

Open questions

1. The Quantum PCP conjecture!
   Is quantum parallel repetition possible?
   Are commuting Hamiltonians easier?

2. Better de Finetti theorems / counterexamples
   main result says random subsets of qudits are ≈ separable
   Aharonov-Eldar have incomparable qPCP no-go.

3. Unifying various forms of Lasserre SDP hierarchy
   (a) approximating separable states via de Finetti (1210.6367)
   (b) searching for product states for local Hamiltonians (this talk)
   (c) noncommutative positivstellensatz approach to games

4. SDP approximations of lightly entangled time evolutions