operator norms & covering nets

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outline

1. separable states and their complexity
2. approximating the set of separable states
3. approximating general operator norms
4. the simple case of the simplex

entanglement and optimization

Definition: ρ is separable (i.e. not entangled) if it can be written as

$$\rho = \Sigma_i \, p_i \, |v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i|$$

probability distribution

unit vectors

$$\mathrm{Sep} = \mathrm{conv}\{|v\rangle\langle v| \otimes |w\rangle\langle w|\}$$
$$= \mathrm{conv}\{\alpha\otimes\beta\}$$
$$=$$

Weak membership problem: Given ρ and the promise that $\rho \in \mathrm{Sep}$ or ρ is far from Sep, determine which is the case.

Optimization: $h_{\mathrm{Sep}}(M) := \max \{ \mathrm{tr}[M\rho] : \rho \in \mathrm{Sep} \}$

complexity of h_{Sep}

Suppose local systems have dimension n.

$h_{Sep}(M) \pm ||M||_{op}$ / poly(n) at least as hard as

- 3-SAT[n] [Gurvits '03], [Le Gall, Nakagawa, Nishimura '12]

$h_{Sep}(M) \pm 0.99\ ||M||_{op}$ at least as hard as

- planted clique [Brubaker, Vempala '09]
- 3-SAT[$\log^2(n)$ / polyloglog(n)] [H, Montanaro '10]

$h_{Sep}(M) \pm 0.99\ h_{Sep}(M)$ at least as hard as

- small-set expansion [Barak, Brandão, H, Kelner, Steurer, Zhou '12]

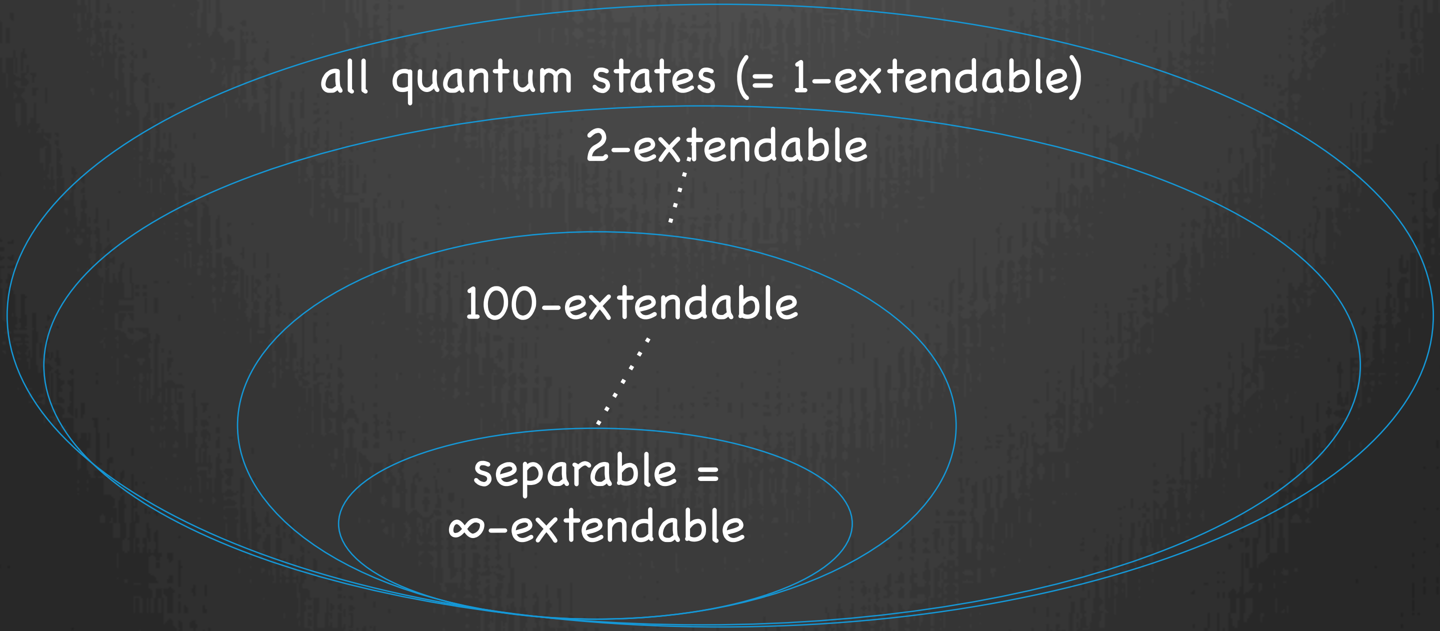
multipartite case

$h_{Sep(\sqrt{n}\text{ parties})}(M) \pm 0.99\ ||M||_{op}$ at least as hard as

- 3-SAT[n] [Chen, Drucker '10]

A hierachy of tests for entanglement

Definition: ρ^{AB} is k-extendable if there exists an extension $\rho^{AB_1\ldots B_k}$ with $\rho^{AB} = \rho^{AB_i}$ for each i.



all quantum states (= 1-extendable)

2-extendable

100-extendable

separable =
∞-extendable

Algorithms: Can search/optimize over k-extendable states in time $n^{O(k)}$.

Question: How close are k-extendable states to separable states?

SDP hierarchies for h_{Sep}

$\mathrm{Sep}(n,m) = \mathrm{conv}\{\rho_1 \otimes \dots \otimes \rho_m : \rho_i \in D_n\}$

$\mathrm{SepSym}(n,m) = \mathrm{conv}\{\rho^{\otimes m} : \rho \in D_n\}$

bipartite

doesn't match hardness

Thm: If $M = \sum_i A_i \otimes B_i$ with $\sum_i |B_i| \leq I$, each $|A_i| \leq I$, then
$h_{Sep(n,2)}(M) \leq h_{k\text{-ext}}(M) \leq h_{Sep(n,2)}(M) + c\,(\log(n)/k)^{1/2}$

[Brandão, Christandl, Yard '10], [Yang '06], [Brandão, H '12], [Li, Winter '12]

multipartite

$$M = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots,i_m} A^{(1)}_{i_1} \otimes \cdots \otimes A^{(m)}_{i_m} \qquad \sum_i |A_i^{(j)}| \leq I \qquad |c_{i_1,\ldots,i_m}| \leq 1$$

Thm:

ε-approx to $h_{SepSym(n,m)}(M)$ in time $\exp(m^2 \log^2(n)/\varepsilon^2)$.

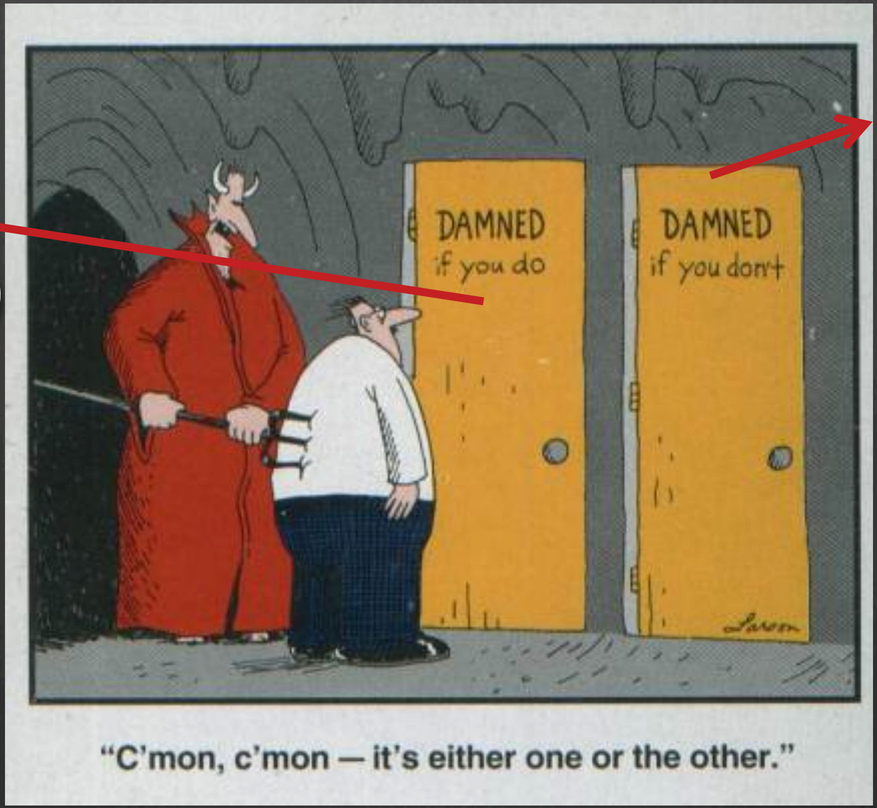
ε-approx to $h_{Sep(n,m)}(M)$ in time $\exp(m^3 \log^2(n)/\varepsilon^2)$.

[Brandão, H '12], [Li, Smith '14]

≈matches Chen-Drucker hardness

proof intuition

Measure extended state and get outcomes $p(a,b_1,\ldots,b_k)$.
Possible because of 1-LOCC form of M.



"C'mon, c'mon — it's either one or the other."

case 1

$p(a,b_1) \approx p(a) \cdot p(b_1)$

case 2

$p(a, b_2 \mid b_1)$
has less mutual
information

[Brandão, Christandl, Yard '10] (secretly in [Yang '06])
[Brandão, H '12], [Li, Winter '12], [Li, Smith '14]

suspicious coincidence!

- Run-time $\exp(c \log^2(n) / \varepsilon^2)$ appears in both
  - Algorithm for M in 1-LOCC
  - Hardness for M in SEP.

Why? Can we bridge the gap?

- Can we find multiplicative approximations, or otherwise use these approaches for small-set expansion?

h_{Sep} for 1-LOCC M

$M = \Sigma_{i\in[m]} A_i \otimes B_i$ with $\Sigma_i A_i \leq I$, each $A_i \geq 0$, $|B_i| \leq I$

$h_{Sep}(M) = \max_{\alpha,\beta} \mathrm{tr}[M(\alpha \otimes \beta)]$

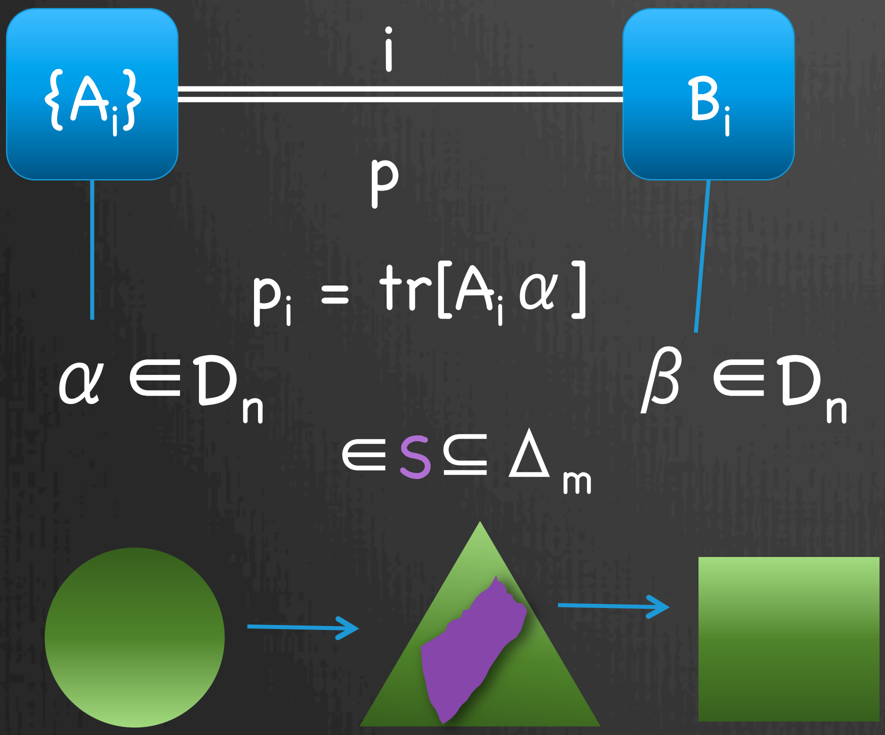
$= \max_{\alpha,\beta} \Sigma_{i\in[m]} \mathrm{tr}[A_i \alpha]\mathrm{tr}[B_i \beta]$

$= \max_{p\in S} \max_{\beta} \Sigma_{i\in[m]} p_i \mathrm{tr}[B_i \beta]$

$= \max_{p\in S} \|\Sigma_{i\in[m]} p_i B\|_{op}$

$:= \max_{p\in S} \|p\|_B$

$\|x\|_B = \|\Sigma_i x_i B_i\|_{op}$



density matrices

probabilities

measurements

nets for h_{Sep}

$$h_{Sep}(M) = \max_{p \in S} \|p\|_B$$

$$\|x\|_{\mathbf{B}} = \|\Sigma_i \, x_i \, B_i\|_{2\text{->}2} \qquad p_i = \mathrm{tr}[A_i \, \alpha]$$

$\alpha \in D_n$ $\qquad$ $p \in S \subseteq \Delta_m$ $\qquad$ $\Sigma_i \, p_i \, B_i$

net size
$O(1/ \varepsilon)^n$
☹

net size
$m^{\log(n)/ \varepsilon^2}$
☺

net size
$O(1/ \varepsilon)^n$
☹

nets from sampling

Lemma: $\forall p \in \Delta_m$ $\exists q$ k-sparse (each q_i = integer / k)

$$\|p-q\|_B \le c\sqrt{\frac{\log(n)}{k}}$$

Pf: Sample i_1, …, i_k from p. Operator Chernoff says

$$\text{Prob}\left[\left\|\sum_i p_i B_i - \frac{B_{i_1}+\ldots+B_{i_k}}{k}\right\|_{\text{op}} \ge \epsilon\right] \le ne^{-k\epsilon^2}$$

[Ahlswede-Winter '00]

Algorithm: Enumerate over k-sparse q

- check whether $\exists p \in S$, $\|p-q\|_B \le \varepsilon$
- if so, compute $\|q\|_B$

Performance

$k \simeq \log(n)/\varepsilon^2$, m=poly(n)

run-time

$O(m^k) = \exp(\log^2(n)/\varepsilon^2)$

operator norms

operator norm

$$\|X\|_{A\to B} = \sup_a \frac{\|Xa\|_B}{\|a\|_A} = \sup_{a,b} \frac{\langle \mathrm{vec}(X), a\otimes b\rangle}{\|a\|_A \|b\|_{B^*}}$$

Examples

$l_2 \rightarrow l_2$	largest singular value. $\|\|X\|\|_{2\text{->}2} = \|\|X\|\|_{op}$
$l_\infty \rightarrow l_1$	MAX-CUT = $\max\{\langle \mathrm{vec}(X), a\otimes b\rangle : \|\|a\|\|_\infty, \|\|b\|\|_\infty \leq 1\}$
$l_1 \rightarrow l_\infty$	$\max_{i,j} \|X_{i,j}\| = \max\{\langle \mathrm{vec}(X), a\otimes b\rangle : \|\|a\|\|_1, \|\|b\|\|_1 \leq 1\}$
S_1 -> S_1 of $X\otimes id$	channel distinguishability (cb norm, diamond norm)
S_1 -> S_p	max output p-norm, min output Rènyi-p entropy
$l_2 \rightarrow l_4$	hypercontractivity, small-set expansion
$S_1 \rightarrow S_\infty$	$h_{Sep} = \max\{ \langle \mathrm{Choi}(X), a\otimes b\rangle : \|\|a\|\|_{S_1}, \|\|b\|\|_{S_1} \leq 1 \}$

complexity of $l_2 \to l_4$ norm

Unique Games (UG):

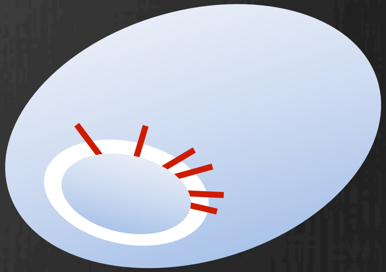
Given a system of linear equations: $x_i - x_j = a_{ij} \bmod k$.

Determine whether $\geq 1-\epsilon$ or $\leq \epsilon$ fraction are satisfiable.

Small-Set Expansion (SSE):

Is the minimum expansion of a set with $\leq \delta n$ vertices $\geq 1-\epsilon$ or $\leq \epsilon$?

$$\text{UG} \approx \text{SSE} \leq 2\text{->}4 = h_{Sep}$$



G = normalized adjacency matrix

P_λ = largest projector s.t. $G \geq \lambda P$

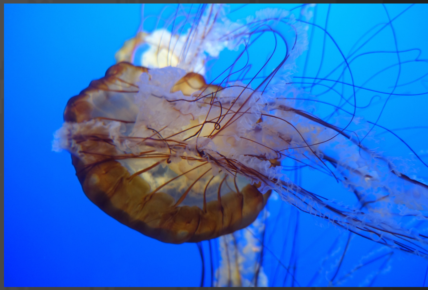
Theorem:

All sets of volume $\leq \delta$ have expansion $\geq 1 - \lambda^{O(1)}$

iff

$\|P_\lambda\|_{2\text{->}4} \leq 1/\delta^{O(1)}$

nets for Banach spaces



$X{:}A\text{->}B$

$\|X\|_{A\text{->}B} = \sup \|Xa\|_B \,/\, \|a\|_A$ **operator norm**

$\|X\|_{A\text{->}C\text{->}B} = \min \{\|Z\|_{A\text{->}C} \, \|Y\|_{C\text{->}B} : X=YZ\}$ **factorization norm**

Let A,B be arbitrary. $C = l_1^m$

Only changes are sparsification (cannot assume $m \le \mathrm{poly}(n)$)

and operator Chernoff for B.

Type-2 constant: $T_2(B)$ is smallest λ such that

$$\mathbb{E}_{\epsilon_1,\ldots,\epsilon_k \in \{\pm 1\}} \left\| \sum_{1=1}^{k} \epsilon_i Z_i \right\|_B^2 \le \lambda^2 \sum_{1=1}^{k} \|Z_i\|_B^2$$

$T_2(S_2) = 1$

$T_2(S_\infty)$
$= O(\sqrt{\log n})$

$T_2(S_1) = \sqrt{n}$

result: $\|X\|_{A\to B} \pm \epsilon \|X\|_{A\to \ell_1^m \to B}$

estimated in time $\exp(T_2(B)^2 \log(m)/ \varepsilon^2)$

applications

S_1 → S_p norms of entanglement-breaking channels
$N(\rho) = \Sigma_i \, \mathrm{tr}[A_i \rho] B_i$, where $\Sigma_i A_i = I$, $||B_i||_1 = 1$.
Can estimate $||N||_{1\to p} \pm \varepsilon$ in time $n^{f(p, \varepsilon)}$.
(uses bounds on $T_2(S_p)$ from [Ball-Carlen-Lieb ’94]

low-rank measurements:
$h_{Sep}(\Sigma_i A_i \otimes B_i) \pm \varepsilon$ for
$\Sigma_i |A_i| = 1$, $||B_i||_\infty \le 1$, rank $B_i \le r$
in time $n^{O(r/\varepsilon^2)}$

$l_2 \to l_p$ for even $p \ge 4$

$$\|X\|_{2\to p}^{p} \pm \epsilon \|X\|_{2\to 2}^{2} \|X\|_{2\to\infty}^{p-2}$$

in time $n^{O(p/\varepsilon^2)}$

Multipartite versions of 1-LOCC norm too [cf. Li-Smith ‘14]

lots of coincidences! ε -nets vs. SoS

Problem	ε -nets	SoS/info theory
$\max_{p\in\Delta} p^TAp$	BK '02, KLP '06	DF '80 BK '02, KLP '06
approx Nash	LMM '03	HNW '16
free games	AIM '14	Brandão-H '13
unique games	ABS '10	BRS '11
small-set expansion	ABS '10	BBHKSZ '12
h_{Sep}	Shi-Wu '11 BKS '13 Brandão-H '15	BCY '10 Brandão-H '12 BKS '13

simplest version: polynomial optimization over the simplex

$\Delta_n = \{p \in \mathbb{R}^n : p \geq 0, \Sigma_i\ p_i = 1\}$

Given homogenous degree-d poly $f(p_1, \ldots, p_n)$, find $\max_p f(p)$.

NP-complete: given graph G with clique number α, $\max_p p^T A p = 1 - 1/\alpha$. [Motzkin-Strauss, '65]

Approximation algorithms

- Net: Enumerate over all points in $\Delta_n(k) := \Delta_n \cap \mathbb{Z}^n/k$.
- Hierarchy: min λ s.t. $(\Sigma_i\ p_i)^k\ (\lambda (\Sigma_i\ p_i)^d - f(p))$ has all nonnegative coefficients.

Thm: Each gives error $\leq (\max_p f(p) - \min_p f(p)) \exp(d) / k$ in time $n^{O(k)}$. [de Klerk, Laurent, Parrilo, '06]

open questions

- Application to unique games, small-set expansion, etc.
- Tight hardness results, e.g. for h_{Sep}.
- What about operators with unknown factorization?
- Explain the coincidences!