Methods of optimal transportation: multidimensional mechanism design

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April 18, 2022

Outline of the talk

Based on "Beckmann's approach to multi-item multi-bidder auctions"

1. Overview: auction design problem with $I \geq 2$ items and $B \geq 2$ bidders
   - $I = 1$ item and $B \geq 1$ bidders
   - $I \geq 2$ items and $B = 1$ bidders
2. Show dual problem formulation
3. Connection with transport problems:
   - the mass transshipment problem
   - Beckmann's problem – the dynamic version of the transshipment problem
4. Numerical techniques and results

Auction design problem for the case of multiple items

Given:

- the auctioneer has I items
- a set $\mathcal{B} = \{1, 2, \ldots, B\}$ of $B \geq 1$ bidders
- each bidder b has a type $x_b = (x_{b1}, \ldots, x_{bI})$ private values

Assumptions:

- the bidders are drawn from this population independently, i.e. the value estimates x_b are sampled independently from the distribution ρ supported on $X = [0, 1]^I$
- each bidder with type $x = (x_1, \ldots, x_I)$ has an additive utility quasi-linear in money

  $$u = p \cdot x - t$$

  for receiving $p = (p_1, \ldots, p_I)$ amount of each item paying t; $(\cdot)$ is a scalar product
- the auctioneer and bidders know ρ and each bidder observes their own type.

Direct revelation mechanisms

- By the revelation principle, we can work with direct mechanisms
- Each bidder b simultaneously and confidentially announces (and may misreport) its value estimate x_b to the auctioneer.
- Using the vector $x = (x_1, \ldots, x_b)$, the auctioneer determines how much of each item each bidder receive and how much each bidder must transfer:
  - $P_b(x) = (P_{b1}(x), \ldots, P_{bI}(x))$ is the amount of good that the bidder b receive,
  - $T_b(x)$ is the price that the bidder b must pay to the auctioneer for the bundle.
- The bidders know allocation functions P_b and transfers functions T_b before the auction game. The collection of functions allocation and transfer function is called **a mechanism**.
- The revenue of the auctioneer is $R = \sum_{b=1}^{B} T_b(x_1, \ldots, x_B)$. The goal of the auctioneer is to maximize the expected revenue.

Restrictions on feasible mechanisms

- **Feasibility:** $\sum_{b=1}^{B} P_b(x_1, \dots, x_B) \le 1$ for every set of bidders $(x_1, \dots, x_B)$.
- **Reduced mechanism:**

$$\overline{P}_b(x_b) = \mathbb{E}[P_b(y) \mid y_b = x_b], \quad \overline{T}_b(x_b) = \mathbb{E}[T_b(y) \mid y_b = x_b].$$

- **Symmetry:** $\overline{P} = \overline{P}_b$, $\overline{T} = \overline{T}_b$
- **Expected utility:**

$$u(x_b) = \mathbb{E}[x_b \cdot P_b(y) - T_b(y) \mid y_b = x_b] = x_b \cdot \overline{P}(x_b) - \overline{T}(x_b).$$

- **Individual rationality:** no bidder of the type x wants to abstain from participation, i.e., nobody gets a negative expected utility, $u(x) \ge 0$.
- **Incentive compatibility:** no bidder x has an incentive to misreport their values if others report truthfully:

$$x \cdot \overline{P}(x) - \overline{T}(x) \ge x \cdot \overline{P}(\hat{x}) - \overline{T}(\hat{x}) \quad \text{for all } \hat{x} \in [0, 1].$$

Auction design problem formulation

Auctioneer's revenue: $R = B \cdot \int \overline{T}(x)\, \rho(x) dx$.

Problem (Rochet-Chone, Econometrica 1998)

Find allocation functions $(P_1, \ldots, P_B)$ *and a utility function* $u(x)$ *maximizing the auctioneer's expected revenue*

$$R = B \cdot \int \left[x \cdot \nabla u(x) - u(x)\right] \rho(x) dx$$

subject to the following constraints:

- ***feasibility:*** $\sum_{b=1}^{B} P_{b,i}(x_1, \ldots, x_B) \leq 1$ *for each item* i *and every collection of types.*
- ***individual rationality:*** $u(0) = 0$,
- ***incentive compatibility:*** $u(x)$ *is convex and* $\nabla u = \overline{P}$, *where* $\overline{P}$ *is the reduced allocation function.*

The case of $I \geq 2$ items and $B = 1$ bidder.

- $I = 1$ case: Myerson (2007 Nobel Memorial Prize in Economic Sciences)
- $I \geq 2$ and $B = 1$: solution is already complicated even in a simple case (Manelli and Vincent, Econometrica 2006)
  - $B = 1$ bidder
  - $I = 2$ independent uniformly distributed items on $[0, 1]$
- **Properties:**
  - Is selling each item separately always optimal? **No.**
  - Is bundling all items together always optimal? **No.**
- **Duality:**
  - Daskalakis, Deckelbaum, Tzamos (Econometrica 2017) established that duality is a Monge-Kantorovich problem with the stochastic dominance constraint.

What we had: feasibility

Problem (Auction design problem for $I \geq 1$ goods)

Find allocation functions $(P_1, \ldots, P_B)$ and a utility function $u(x)$ maximizing the auctioneer's expected revenue

$$R := B \cdot \int \overline{T}(x)\, \rho(x) dx = B \cdot \int \left[x \cdot \nabla u(x) - u(x)\right] \rho(x) dx$$

subject to the following constraints:

- ***feasibility:*** $\sum_{b=1}^{B} P_{b,i}(x_1, \ldots, x_B) \leq 1$ *for each item i and every collection of types.*
- ***individual rationality:*** $u(0) = 0$,
- ***incentive compatibility:*** $u(x)$ *is convex and* $\nabla u = \overline{P}$*, where* $\overline{P}$ *is the reduced allocation function.*

The feasibility condition

The expected revenue $R = B \cdot \int [x \cdot \nabla u(x) - u(x)] \, \rho(x) dx$ depends only on $u(x)$ and $\overline{P}(x) = \nabla u(x)$.

Question

Given a reduced allocation function $\overline{P} = \nabla u$, under which conditions is it possible to find the full feasible mechanism $(P_1, \ldots, P_B)$?

Stochastic dominance condition

Definition

The random variable ξ majorizes random variable η ($\xi \succeq \eta$) if $\mathbb{E}[\varphi(\xi)] \geq \mathbb{E}[\varphi(\eta)]$ for any convex increasing function φ.

Theorem (Hart and Reny)

The reduced allocation function $\overline{P}_i(x)$ is feasible if and only if $\overline{P}_i(\zeta) \preceq \xi^{B-1}$, where ζ is distributed with the density ρ and ξ is uniformly distributed on $[0,1]$.
Equivalently, for all convex increasing φ.

$$\int \varphi(\overline{P}_i(x))\,\rho(x)dx \leq \int_0^1 \varphi(z^{B-1})\,dz$$

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What we have now: feasibility → stochastic dominance

Problem (Auction design problem for $I \geq 1$ goods)

Find reduced allocation function $\overline{P}$ and a utility function $u(x)$ maximizing the auctioneer's expected revenue

$$R := B \cdot \int \overline{T}(x)\, \rho(x) dx = B \cdot \int \left[x \cdot \nabla u(x) - u(x)\right] \rho(x) dx$$

subject to the following constraints:

- ***stochastic dominance:*** $\overline{P}_i \geq 0$ *and* $\int \varphi_i(\overline{P}_i(x))\, \rho(x) dx \leq \int_0^1 \varphi_i(z^{B-1})\, dz$ *for every convex non-decreasing* φ_i,
- ***individual rationality:*** $u(0) = 0$,
- ***incentive compatibility:*** $u(x)$ *is convex and* $\nabla u = \overline{P}$, *where* $\overline{P}$ *is the reduced allocation function.*

Lagrangian: monopolist problem with production costs

Problem

Fix the convex non-decreasing cost functions $(\varphi_1, \dots, \varphi_I)$. *Find the maximum of the expected revenue over all convex non-decreasing non-negative functions* $u(x)$*:*

$$M(u;\varphi_i) = B\left(\int \Big[\underbrace{x \cdot \nabla u(x) - u(x)}_{\textit{transfer}} - \underbrace{\sum_{i=1}^{I} \varphi_i \left(\frac{\partial u}{\partial x_i} \right)}_{\textit{production cost}} \Big] \rho(x) dx + \underbrace{\sum_{i=1}^{I} \int \varphi_i(z^{B-1})\, dz}_{\textit{constant for fixed } \varphi_i} \right)$$

Interpretation

The monopolist problem:

- $B = 1$ bidder, $I \geq 2$ items
- $\varphi_i(t)$ is a cost of producing t units of the ith item

Intuition

$\varphi_i(t)$ is a nonlinear Lagrange multiplier function for the stochastic dominance constraint

Minimax principle for the monopolist problem

Theorem

*For every collection of convex non-decreasing functions $(\varphi_1, \dots, \varphi_I)$, the optimal value in the corresponding monopolist problem with the nonlinear production cost dominates the maximal revenue of the auctioneer **(weak minimax)**:*

$$R \leq \max_u M(u; \varphi_i).$$

*Moreover, there exists a collection of functions $(\varphi_1^{opt}, \dots, \varphi_I^{opt})$ such that the optimal values in the monopolist problem and in the auction design problem coincide **(strong minimax)**:*

$$R = \max_u M(u; \varphi_i^{opt}).$$

Where are we now

- Derived a minimax problem
  - the monopolist's is a nonlinear Lagrangian for the auctioneer's problem
  - but the problem is endogenous and complicated
- Next:
  - linearize the nonlinear Lagrangian:
    - using Legendre transform
  - derive a dual formulation
  - show that the dual is Beckmann's problem

Legendre transform of the production cost

Strong minimax relation between the auctioneer and the monopolist problem:

$$\frac{R}{B} = \min_{\varphi} \max_{u} \left\{ \int \left[x \cdot \nabla u(x) - u(x) - \sum_{i=1}^{I} \varphi_i \left(\frac{\partial u}{\partial x_i} \right) \right] \rho(x) dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\, dz \right\}.$$

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- Legendre transform: $\varphi_i \left(\frac{\partial u}{\partial x_i} \right) = \max_{c_i} \left\{ c_i \cdot \frac{\partial u}{\partial x_i} - \varphi_i^*(c_i) \right\}$

Legendre transform of the production cost

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- Legendre transform: $\varphi_i \left(\frac{\partial u}{\partial x_i} \right) = \max_{c_i} \left\{ c_i \cdot \frac{\partial u}{\partial x_i} - \varphi_i^*(c_i) \right\}$
- Introduce $c(x) = (c_1(x), \ldots, c_I(x))$:

$$\frac{R}{B} = \min_{\varphi} \max_{u} \min_{c} \left\{ \int \left[x \cdot \nabla u(x) - u(x) - \sum_{i=1}^{I} c_i(x) \cdot \frac{\partial u(x)}{\partial x_i} \right] \rho(x) dx \right.$$

$$\left. + \int \sum_{i=1}^{I} \varphi_i^*(c_i(x))\, \rho(x) dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\, dz \right\}.$$

Legendre transform of the production cost

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- **Minimax principle: maximize over u:**

$$\max_{u} \int \left[x \cdot \nabla u(x) - u(x) - \sum_{i=1}^{I} c_i(x) \cdot \frac{\partial u(x)}{\partial x_i} \right] \rho(x) dx = \begin{cases} 0 & \textbf{(take } u \equiv 0\textbf{)}, \\ +\infty & \textbf{(can multiply by } \lambda > 0\textbf{)} \end{cases}$$

Duality theorem for the auctioneer's problem

Theorem

In the auctioneer's problem with $B \geq 1$ bidders, $I \geq 1$ items, and bidders' types distributed on $X = [0,1]^{\mathcal{I}}$ with positive density ρ, the optimal revenue coincides with

$$R = B \cdot \inf_{(\varphi_1,\dots,\varphi_I)} \inf_{c=(c_1,\dots,c_I)} \left\{ \sum_{i=1} \int \varphi_i^*(c_i(x))\, \rho(x) dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\, dz \right\},$$

where infimum is taken over all convex non-decreasing cost functions φ_i and over all vector fields $c(x) = (c_1(x), \dots, c_I(x))$ satisfying the constraint

$$\max_u \int \left[x \cdot \nabla u(x) - u(x) - \underbrace{c(x) \cdot \nabla u(x)}_{\sum c_i \cdot \frac{\partial u}{\partial x_i}} \right] \rho(x) dx = 0.$$

Remark: McCann and Zhang (2023) discovered the related duality result in parallel for the general monopolist problem.

From inequality constraint to stochastic dominance

The constraint in the dual problem: for every convex increasing $u(x)$:

$$\int \left[x \cdot \nabla u(x) - u(x)\right] \rho(x) dx \leq \int \nabla u(x) \cdot c(x)\, \rho(x) dx.$$

- Integrate by parts the auctioneer's revenue $x \cdot \nabla u(x) - u(x)$:

$$\int [x \cdot \nabla u(x) - u(x)]\ \rho(x) dx = \int \underline{u(x)\, dm(x)};$$

- Integrate the right-hand side by parts using the divergence formula:

$$\int \nabla u(x) \cdot c(x)\, \rho(x) dx = \int \underline{u(x)\, d\pi(x)};$$

  - where $\pi + \mathrm{div}[\rho \cdot c] = 0$
  - in 1D case, $\pi + (c \cdot \rho)' = 0 +$ boundary terms
- The inequality $\int \underline{u(x)\, dm(x)} \leq \int \underline{u(x)\, d\pi(x)}$ is equivalent to the stochastic dominance constraint $m \preceq \pi$.

Dual problem formulation with stochastic dominance

We had:

$$\frac{R}{B} = \inf_{\varphi} \inf_{c} \sum_{i=1}^{I} \left\{ \sum_{i=1}^{I} \int \varphi_i^*(c_i(x))\, \rho(x) dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\, dz \right\}$$

through all convex non-decreasing $(\varphi_1, \ldots, \varphi_I)$ and vector fields c subject to

$$\max_u \int \Big[x \cdot \nabla u(x) - u(x) - c(x) \cdot \nabla u(x) \Big] \rho(x) dx = 0.$$

Dual problem formulation with stochastic dominance

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$$\max_u \int \Big[x \cdot \nabla u(x) - u(x) - c(x) \cdot \nabla u(x) \Big] \rho(x)dx = 0.$$

We proved:

$$\frac{R}{B} = \inf_{\varphi} \inf_{\pi \succeq m} \inf_{c:\,\mathrm{div}[\rho \cdot c] + \pi = 0} \left\{ \sum_{i=1} \int \varphi_i^*(c_i(x))\,\rho(x)dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\,dz \right\}$$

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Dual problem formulation with stochastic dominance

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through all convex non-decreasing $(\varphi_1, \ldots, \varphi_I)$ and vector fields c

Plan: duality between the auction problem and $\mathrm{Beck}_\rho(\pi, \Phi)$-problem

- Recall the classical transportation problems
  - Monge-Kantorovich problem
  - Kantorovich-Rubinstein problem – an alternative (less known) formulation
- Introduce $\mathrm{Beck}_\rho(\pi, \Phi)$-problem
  - Beckmann's is a **dynamic version** of Kantorovich-Rubinstein problem
- The dual to the auction problem is equivalent to $\mathrm{Beck}_\rho(\pi, \Phi)$-problem.

Reminder: Monge-Kantorovich and Kantorovich-Rubinstein problems

The classical Monge-Kantorovich problem

Given marginal distributions μ and ν and a cost function $\alpha(x, y) = ||x - y||$, find

$$\min_{\gamma} \int \alpha(x, y) \, \gamma(x, y) \, dxdy$$

subject to the constraints $\underbrace{\mathrm{pr}_1\gamma = \mu \text{ and } \mathrm{pr}_2\gamma = \nu}_{\text{fixed marginals}}$.

Reminder: Monge-Kantorovich and Kantorovich-Rubinstein problems

The classical Monge-Kantorovich problem

Given marginal distributions μ and ν and a cost function $\alpha(x,y) = ||x-y||$, find

$$\min_{\gamma} \int \alpha(x,y)\,\gamma(x,y)\,dxdy$$

subject to the constraints $\underbrace{\mathrm{pr}_1\gamma = \mu \text{ and } \mathrm{pr}_2\gamma = \nu}_{\text{fixed marginals}}$.

The mass transshipment problem (Kantorovich and Rubinstein, 1958)

Given a marginal difference $\mu - \nu$ and a cost function $\alpha(x,y) = ||x-y||$, find

$$\min_{\pi} \int \alpha(x,y)\,\gamma(x,y)\,dxdy$$

subject to the **balancing condition** $\mathrm{pr}_1\gamma - \mathrm{pr}_2\gamma = \underbrace{\mu - \nu}_{\text{fixed difference}}$.

Beckmann's problem

Idea: replace the immediate transfer $x \to y$ with the dynamical one using all the intermediate points on (x, y) as transshipment nodes.

The continuous transportation problem (Beckmann, 1952)

Given a marginal difference $\mu - \nu$, find the optimal value

$$\min_c \int |c(x)|\, dx$$

subject to the **balancing condition** $\mathrm{div}[c] + \mu - \nu = 0$.

Intuition

- $|c(x)|$ is the traffic through the point x
- the direction of $c(x)$ is the direction of the transport flow through x

Theorem

The mass transportation and Beckmann's problems are equivalent: optimal values are identical and the solution to one problem can be constructed by another one.

Generalization: $\mathrm{Beck}_\rho(\pi, \Phi)$-problem

Non-linear cost in Beckmann's problem: $\int \Phi(c(x))\, \rho(x) dx$

- the cost $\Phi(c)$ depends on both the direction and the traffic;
- $\rho(x)$ is the weight of the node x;

The balancing condition: $\mathrm{div}_\rho[c] + \mu - \nu = 0$.

- $\mathrm{div}_\rho[c] := \mathrm{div}[\rho \cdot c]$ is a weighted divergence;

$\mathrm{Beck}_\rho(\pi, \Phi)$-problem

For a given cost function $\Phi(c)$, minimize the total weighted cost over all transport flows c satisfying the balancing condition for $\pi = \mu - \nu$:

$$\mathrm{Beck}_\rho(\pi, \Phi) = \inf_{c\colon \mathrm{div}_\rho[c] + \pi = 0} \int \Phi(c)\, \rho(x) dx$$

Strong duality theorem in $\mathrm{Beck}_\rho(\pi, \Phi)$-form

Recall: $$\frac{R}{B} = \inf_{\varphi} \inf_{\pi \succeq m} \inf_{c:\, \mathrm{div}[\rho \cdot c] + \pi = 0} \left\{ \sum_{i=1}^{I} \int \varphi_i^*(c_i(x))\, \rho(x) dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\, dz \right\}$$

Definition

For given convex functions $(\varphi_1, \ldots, \varphi_I)$, define the cost $\Phi(c) = \sum_{i=1}^{I} \varphi_i^*(c_i)$

Theorem (Duality between auction design problem and $\mathrm{Beck}_\rho(\pi, \Phi)$-problem)

In the auctioneer's problem with $B \geq 1$ bidders, $I \geq 1$ items, and bidders' types distributed on $X = [0, 1]^{\mathcal{I}}$ with positive density ρ, the optimal revenue coincides with

$$B \cdot \inf_{\substack{(\varphi_i)_{i \in \mathcal{I}}, \\ \pi \succeq m}} \left[\mathrm{Beck}_\rho\left(\pi, \Phi\right) + \sum_{i \in \mathcal{I}} \int_0^1 \varphi_i \left(z^{B-1}\right) dz \right].$$

Numerical results: $I = 2$ items, multiple bidders

We solve the auction design problem with unprecedented numerical precision:

- 200×200 types, 1.6 billion incentive constraints

Outline of the methods we use:

- finite element method to approximate the continuous problem
- Oberman's approach to reduce the number of incentive constraints
- the Strassen theorem to reformulate the stochastic dominance constraint
- state of the art linear programming solvers

Example of $B = 2$ bidders and $I = 2$ independent items.

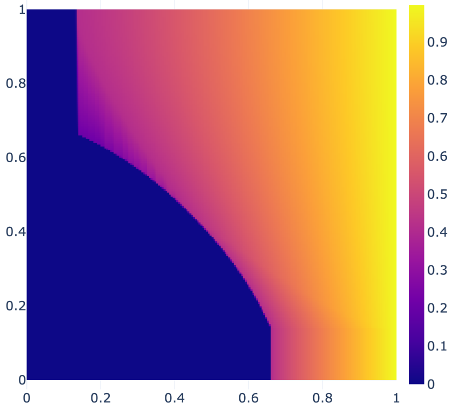


Figure: The allocation $\overline{P} = \frac{\partial u}{\partial x_1}$

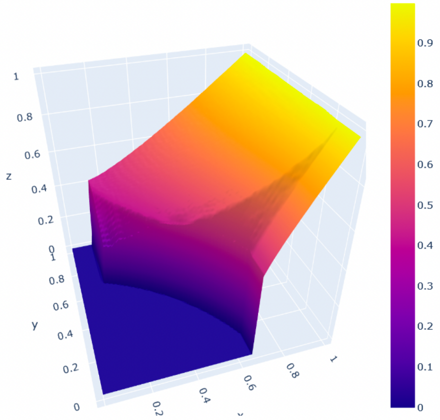


Figure: The 3D surface graph

The algorithm could be scaled to multiple bidders

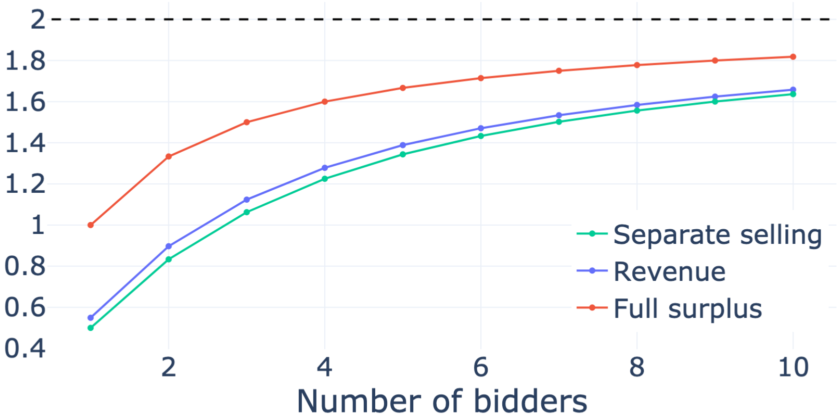


Figure: Revenue as a function of the number of bidders B for two items with i.i.d. values uniform on $[0, 1]$. Graphs from bottom to top: selling separately (light-green), selling optimally (blue), full surplus extraction (red), limit for $B \to \infty$ (the dashed line).

Bunching regions of the solution to the auction problem

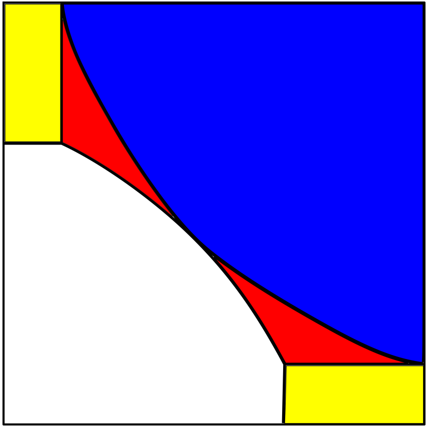


Figure: Partition of the square $[0,1]^2$ w.r.t. the rank of the hessian $\mathrm{H}(u)$.

Consider the optimal utility function u for the case of $B = 2$ bidders and $I = 2$ items with the value estimates independently uniformly distributed on $[0,1]$.

The square $[0,1]^2$ can be divided into the following regions:

- white region: $u = 0$;
- yellow regions: $\dfrac{\partial u}{\partial x_i} = 0$ for some i;
- red regions: $\det \mathrm{H}(u) = 0$;
- blue region: u is strictly convex

Conclusion

- Optimal auction design problem with multiple bidders and multiple items
  - problem at the frontier of the economics reseach
  - the methods are of the broad interest to mathematicians across fields
- $\mathrm{Beck}_\rho(\pi, \Phi)$ – generalization of the Beckmann problem
  - simple Beckmann's problem – dual to the Kantorovich-Rubinstein problem
- Main mathematical result – duality between $\mathrm{Beck}_\rho(\pi, \Phi)$ and auction problems.
- Foundation for the development of effective numerical methods

Duality result for $B = 1$ bidder

We minimize the functional $B \cdot \left(\sum_{i=1}^{I} \int \varphi_i^*(c_i)\, \rho(x) dx + \int_0^1 \varphi_i(z^{B-1})\, dz \right).$

- In the case of one bidder, $\int_0^1 \varphi(z^{B-1})\, dz = \varphi(1)$.
- Fix the vector field c_i. The minimum is reached if $\varphi_i \equiv 0$ and $\varphi_i^*(z) = z$
- The value of the functional for $\varphi_i \equiv 0$ is equal to

$$\int \sum_{i=1}^{I} c_i(x)\, \rho(x) dx = \int ||c||_{l^1}\, \rho(x) dx$$

Proposition (Duality for $B = 1$ bidder)

$$R = \min_{\pi \succeq m} \min_{\mathrm{div}_\rho[c] + \pi = 0} \int ||c||_{l^1}\, \rho(x) dx$$

Connection with the Daskalakis & Deckelbaum & Tzamos duality

We decompose π as a difference of positive and negative parts: $\pi = \pi_c - \pi_p$.

Theorem (Beckmann, 1952)

For any measure $\pi = \pi_c - \pi_p$,

$$\min_{\mathrm{div}_\rho[c]+\pi=0} \int ||c||_{l^1}\, \rho(x) dx = \min_{\gamma \in \Pi(\pi_c,\pi_p)} \int |x-y|\, \gamma(dx,dy),$$

where $\Pi(\pi_c, \pi_p)$ *is the set of transport plans with given marginals.*

Theorem (Daskalakis & Deckelbaum & Tzamos)

Consider the auction design problem with $B=1$ *bidder. Then,*

$$R = \min_{\pi_c - \pi_p \succeq m} \; \min_{\gamma \in \Pi(\pi_c,\pi_p)} \int |x-y|\, \gamma(dx,dy).$$

Nonlinear production function for the case of $I = 2$ uniform items

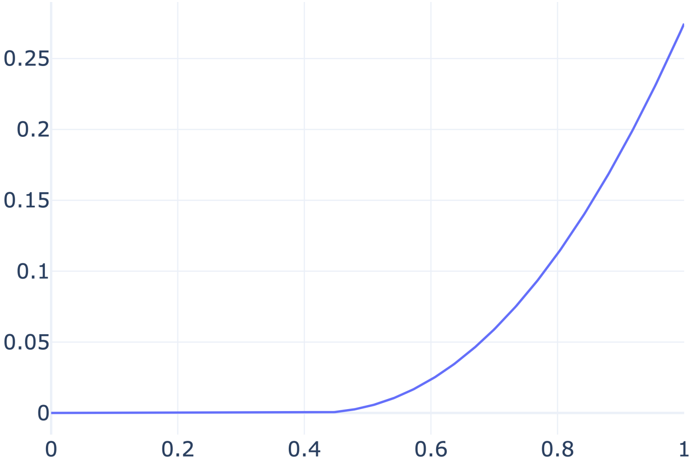


Figure: The nonlinear cost φ computed for the case of $I = 2$ independent uniformly distributed items and $B = 2$ bidders

Connection with the Daskalakis & Deckelbaum & Tzamos duality

- Recall the Daskalakis & Deckelbaum & Tzamos duality theorem:

$$\max_{u \in \mathcal{U} \cap \mathcal{L}_1} \mathcal{R}(u) = \min_{\substack{\gamma \in \mathcal{M}_+(T \times T) \\ \gamma_1 - \gamma_2 \succeq_{\mathrm{cvx}} \mu_f}} \int_{T \times T} ||x - y||_1 \, \gamma(dx, dy).$$

- Fix the projections γ_1^* and γ_2^* of the optimal γ^*.

$$\max_{u \in \mathcal{U} \cap \mathcal{L}_1} \mathcal{R}(u) = \min_{\gamma \in \Pi(\gamma_1^*, \gamma_2^*)} \int_{T \times T} ||x - y||_1 \, \gamma(dx, dy) = \mathcal{W}_1(\gamma_1^*, \gamma_2^*).$$

- **Beckmann's minimal flow problem:**

$$\mathcal{W}_1(\gamma_1^*, \gamma_2^*) = \min \left\{ \int |w(x)| \, dx \colon \, w \colon \, T \to \mathbb{R}^m, \nabla \cdot w = \gamma_1^* - \gamma_2^* \right\}.$$

- The solution w to the Beckmann's problem is an optimal vector field.

The case of $I \geq 2$ items and $B = 1$ bidder. The monopolist problem.

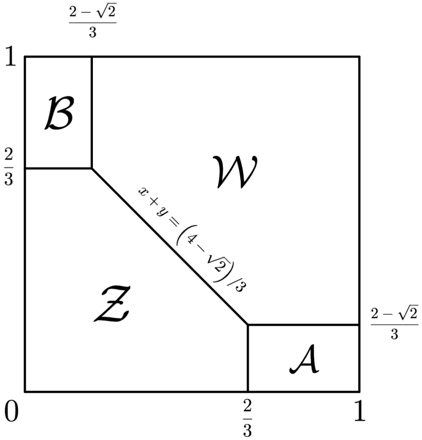


Figure: The mechanism for two i.i.d. uniform [0,1] items.

The case of $B = 1$ bidder and $I = 2$ items: the value estimates are i.i.d. uniformly distributed on $[0, 1]$.

Description of the mechanism:

- $\mathcal{Z}$: receive no goods and pay 0;
- $\mathcal{A}$: receive the 1st good and pay $\frac{2}{3}$;
- $\mathcal{B}$: receive the 2nd good and pay $\frac{2}{3}$;
- $\mathcal{W}$: receive both goods and pay $\frac{4-\sqrt{2}}{3}$.

Consequences:

- Is selling each good separately always optimal? **No.**
- Is bundling all goods together always optimal? **No.**

The Border's condition

Theorem (Border, Econometrica 1991)

The reduced allocation function $\overline{P} = (\overline{P}_1, \ldots, \overline{P}_I)$ is feasible if and only if each of its component satisfies the Border condition:

- $\overline{P}_i(x) \geq 0$ *for all* x;
- *for any set* S *of bidder types,* $$B \cdot \underbrace{\int_S \overline{P}_i(x)\, \rho(x) dx}_{\text{bidder from } S \text{ receives an item}} \leq 1 - \underbrace{\left(\int_{X \setminus S} \rho(x)\, dx\right)^B}_{\text{none of the bidders belongs to } S}.$$

Intuition:

- for simplicity, assume that the item is indivisible;
- left-hand side is probability of an intersection of 2 events:
  - at least one bidder with the type from the set S participates in the auction;
  - this bidder receives the item;
- right-hand side is probability of the event that at least one bidder with the type from the set S participates in the auction.

Second-order stochastic dominance

Definition

A random variable ξ **stochastically dominates** random variable η if

$$\mathrm{Tail}_\alpha(\xi) \geq \mathrm{Tail}_\alpha(\eta)$$

for each $0 \leq \alpha \leq 1$, where $\mathrm{Tail}_\alpha(\xi)$ is the *unconditional* expectation of the most $\alpha \times 100\%$ of the outcomes of ξ.

Equivalent definitions:

- $\mathbb{E}[\varphi(\xi)] \geq \mathbb{E}[\varphi(\eta)]$ for any convex increasing φ;
- $\mathbb{E}[(\xi - t)_+] \geq \mathbb{E}[(\eta - t)_+]$ for each t

Intuition: $\xi \succeq \eta$ if $(1 - \xi)$ is less risky than $(1 - \eta)$.

Stochastic dominance condition

- Let S be the set of $\alpha \times 100\%$ bidder types with the highest probability of receiving an item. The Border condition

$$B \cdot \underbrace{\int_S \overline{P}_i(x)\,\rho(x)dx}_{\mathrm{Tail}_\alpha} \leq 1 - \Big(\underbrace{\int_{X\setminus S} \rho(x)\,dx}_{1-\alpha}\Big)^B$$

is equivalent to the tail bound

$$\mathrm{Tail}_\alpha(\overline{P}_i(\chi)) \leq \frac{1}{B}(1-(1-\alpha)^B) = \int_\alpha^1 z^{B-1}\,dz$$

- the right-hand side is the tail size of ξ^{B-1}, where ξ is uniform on $[0,1]$:

$$\mathrm{Tail}_\alpha(\overline{P}_i(\chi)) \leq \int_\alpha^1 z^{B-1}\,dz = \mathrm{Tail}_\alpha[\xi^{B-1}].$$

Classical Lagrange multipliers

- every convex non-decreasing function φ is a positive combination of "elementary" convex functions $\varphi_t(x) = \max(x - t, 0)$: $\varphi(x) = \int \lambda(t)\varphi_t(x)\, dt$.
- stochastic dominance constraint is a union of continuum "elementary constraints":

$$\int \varphi_t(\overline{P}_i(x))\, \rho(x) dx \leq \int_0^1 \varphi_t(z^{B-1})\, dz \quad \text{for all } t \in [0,1];$$

- add these constraints to the revenue objective, using Lagrange multipliers $\lambda(t) \geq 0$:

$$M = \int \underbrace{[x \cdot \nabla u - u]}_{\text{revenue}} \rho dx - \underbrace{\sum_{i=1}^{I} \int \lambda(t) dt \left(\int \varphi_{t,i}(\overline{P}_i)\, \rho dx - \int_0^1 \varphi_{t,i}(z^{B-1})\, dz \right)}_{\text{Lagrange multiplier}}$$

- substitute $\varphi_i = \int \varphi_{t,i}\, \lambda(t) dt$ – convex and non-decreasing:

$$M = \int \left[x \cdot \nabla u(x) - u(x) - \sum_{i=1}^{I} \varphi_i \left(\frac{\partial u}{\partial x_i} \right) \right] \rho(x) dx + \sum_{i=1}^{I} \int \varphi_i(z^{B-1})\, dz$$

Plan: introduce Beckmann's problem

- Introduce the transshipment problem:
  - discrete case: the minimum-cost flow problem
  - continuous case:
    - a transshipment problem
    - an alternative (less known) version of the Monge-Kantorovich problem
- Beckmann = continuous version of the transshipment problem

Dual solution to the auction problem

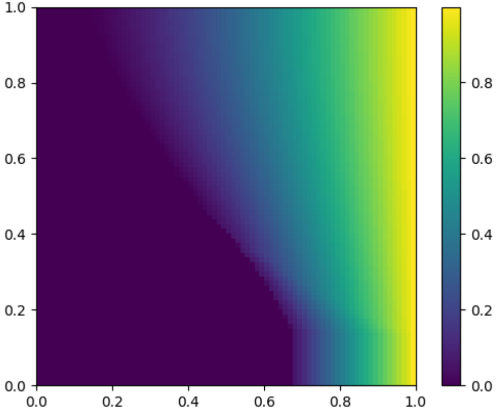


Figure: Distribution of the first component c_1 of the optimal vector field c

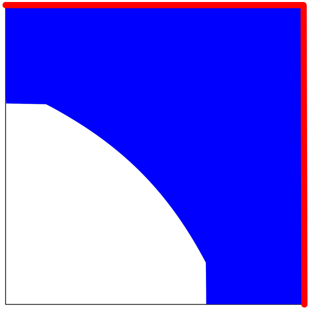


Figure: Distribution of $\nabla \cdot c$: $\int u \, d\nabla \cdot c = -\int \langle \nabla u, c \rangle dx$.

white region: $\nabla \cdot c = 0$;
blue region: $\nabla \cdot c = 3$;
red intervals: singular parts of $\nabla \cdot c$ equal to $(-1)\cdot$ uniform measures on $[0, 1]$.

Minimum-cost flow problem

We are given the set of nodes G and the set of directed edges E.

- for each node u, the supply-demand imbalance $i(u)$ is given
  - $i(u) > 0$ means that a positive amount — the supply — is added to the flow: could represent production at that node
  - $i(u) < 0$ a negative amount — the demand — is taken away from the flow: could represent consumption at that node
- for each directed edge (u, v), find the flow level $f(u, v)$:
  - non-negativity: $f(u, v) \geq 0$
  - flow conservation: $\underbrace{\sum_{(u,v)\in E} f(u, v)}_{\text{output flow}} - \underbrace{\sum_{(w,u)\in E} f(w, u)}_{\text{input flow}} = \underbrace{i(u)}_{\text{imbalance}}$
- the cost of the flow:
  - $d(u, v) \cdot |f(u, v)|$ is the cost of of pushing $f(u, v)$ units of flow through one edge
  - $\sum d(u, v) \cdot |f(u, v)|$ is the total cost.

Example of the minimum-cost flow problem

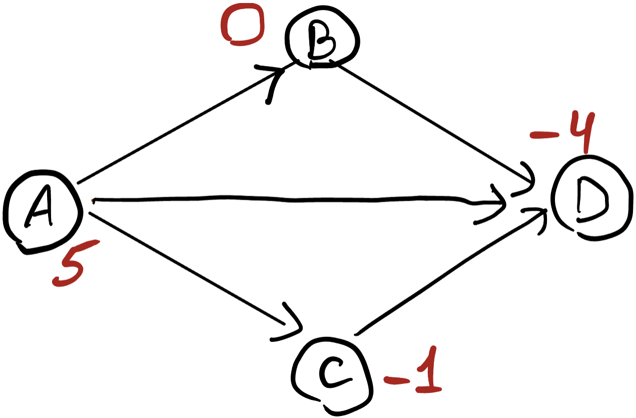


Figure: The graph consisting of 4 nodes A, B, C, D and 4 edges. The node A is a source, the nodes C and D are sinks, and B is a transshipment node

Example of the minimum-cost flow problem

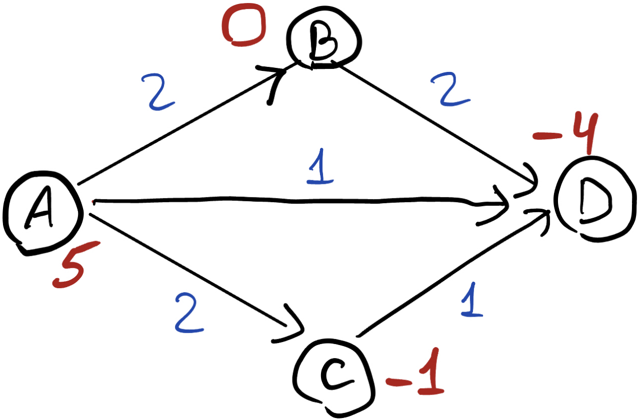


Figure: An example of the flow compensating supply-demand imbalance

Example of the minimum-cost flow problem

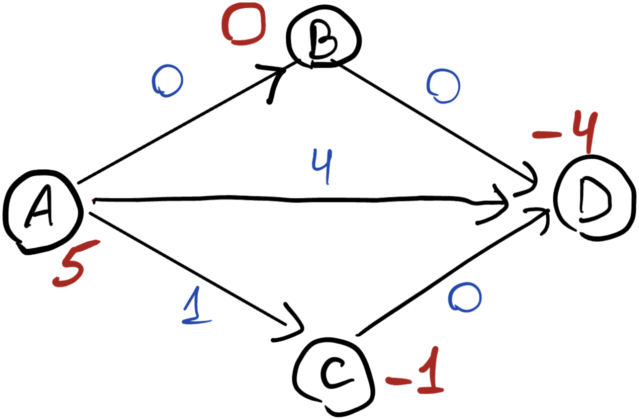


Figure: By the triangle inequality, any non-zero flow through the path $A \to B \to D$ or through the path $A \to C \to D$ could be replaced with the flow through the edge $A \to D$ reducing the total cost.

The Kantorovich–Rubinstein (mass transshipment) problem

Continuous network flow problem

- nodes are all the points x from $\mathbb{R}^n$;
- the imbalance level $i(x)$ is given by the signed measure $\mu - \nu$;
- flow is given by the transport plan $\gamma(dx, dy)$:
  - flow conservation: $\underbrace{\int \gamma(x,y)\,dy}_{\mathrm{pr}_1\gamma} - \underbrace{\int \gamma(z,x)\,dz}_{\mathrm{pr}_2\gamma} = \underbrace{\mu(x) - \nu(x)}_{\text{imbalance}}$
  - the cost of the flow: $\int d(x,y)\,\gamma(dx,dy)$

The mass transshipment problem (Kantorovich and Rubinstein, 1958)

Given a marginal difference $\mu - \nu$ and a cost function $d(x,y) = ||x-y||$, find the optimal value $\min_{\pi} \int d(x,y)\,\pi(dx,dy)$ subject to the constraint $\mathrm{pr}_1\pi - \mathrm{pr}_2\pi = \mu - \nu$.

From Kantorovich-Rubinstein to Beckmann

- Only local transfers are possible:
  - replace the immediate transfer $x \to y$ with the sequence of infinitesimal transfers $x \to (x + dc) \to (x + 2 \cdot dc) \to \cdots \to y$;
  - can be considered as a dynamic flow from x to y
- For each point x, define **the transport flow** $c(x)$:
  - the direction of $c(x)$ coincides with the local direction of the flow
  - the length of $c(x)$ is the local congestion of the flow
- the total cost of the flow is $\int \underbrace{|c(x)|}_{\text{congestion}} \underbrace{dx}_{\text{distance}}$.
- the imbalance of the flow:
  - an amount of flow entering or leaving the infinitesimal sphere around x;
  - can be described using the divergence operator $\mathrm{div}[c] = \sum \frac{\partial c_i}{\partial x_i} +$ boundary terms
  - the flow conservation condition: $\mathrm{div}[c] + \underbrace{\mu - \nu}_{\text{imbalance}} = 0$.

Beckmann's problem

The mass transshipment problem (Kantorovich and Rubinstein, 1958)

Given a marginal difference $\mu - \nu$ and a cost function $d(x, y) = ||x - y||$, find the optimal value

$$\min_{\pi} \int d(x, y)\, \pi(dx, dy)$$

subject to the constraint $\mathrm{pr}_1\pi - \mathrm{pr}_2\pi = \mu - \nu$.

The continuous transportation problem (Beckmann, 1952)

Given a marginal difference $\mu - \nu$, find the optimal value

$$\min_{c} \int |c(x)|\, dx$$

subject to the constraint $\mathrm{div}[c] + \mu - \nu = 0$.

Theorem

The mass transportation and Beckmann's problems are equivalent: the optimal values are identical and the solution to one problem can be constructed by another one.

Equivalence of dual to Kantorovich-Rubinstein and Beckmann problems

- The weak form of the constraint $\mathrm{div}[c] + \mu - \nu = 0$: for all φ,
$$\int \nabla\varphi(x) \cdot c(x)\, dx = \int \varphi(x) \cdot (\mu(dx) - \nu(dx))$$
- Introduce a Lagrangian:
$$\min_{c:\ \mathrm{div}[c]+\mu-\nu=0} \int |c|\, dx = \min_{c} \max_{\varphi} \left\{ \int |c|\, dx - \int \nabla\varphi \cdot c\, dx + \int \varphi \cdot (\mu(dx) - \nu(dx)) \right\}$$
- Apply the minimax principle:
$$\min_{c:\ \mathrm{div}[c]+\mu-\nu=0} \int |c|\, dx = \max_{\varphi} \left\{ \int \varphi \cdot (\mu(dx) - \nu(dx)) + \min_{c} \int |c|\, dx - \int \nabla\varphi \cdot c\, dx \right\}$$
- $\min_c \int |c|\, dx - \int \nabla\varphi \cdot c\, dx$ is bounded iff $|\nabla\varphi(x)| \leq 1$ for all x
- $|\nabla\varphi| \leq 1$ is 1-Lipschitz condition: $\varphi(x) - \varphi(y) \leq |x - y|$
- The problem $\max\limits_{\varphi} \int \varphi(x) \cdot (\mu(dx) - \nu(dx))$ subject to $\varphi(x) - \varphi(y) \leq |x - y|$ is dual to the transshipment problem

Beckmann's problem with nonlinear transfer cost

- the cost of pushing $f(u,v)$ units of flow depends on f non-linearly:
  - $\mathrm{cost} = \sum \Phi_{uv}(f(u,v))$
  - Φ_{uv} are edge-specific convex functions;
- in the continuous case: $\mathrm{cost} = \int \Phi(c(x))\, \rho(x) dx$
  - the cost $\Phi(c)$ depends on both the direction and the congestion of the flow;
  - $\rho(x)$ is the weight of the node x;
- the flow conservation condition: $\mathrm{div}_\rho[c] + \mu - \nu = 0$
  - $\mathrm{div}_\rho[c] := \mathrm{div}[\rho \cdot c]$ is a weighted divergence;

Problem (Beckmann's problem with non-linear transfer cost)

For a given cost function $\Phi(c)$, minimize the total weighted cost over all transport flows c compensating the supply-demand imbalance $\pi = \mu - \nu$:

$$\mathrm{Beck}_\rho(\pi, \Phi) = \inf_{c:\ \mathrm{div}[\rho\cdot \mathrm{c}]+\pi=0} \int \Phi(c)\, \rho(x) dx$$

Example of $B = 2$ bidders and $I = 2$ independent items.

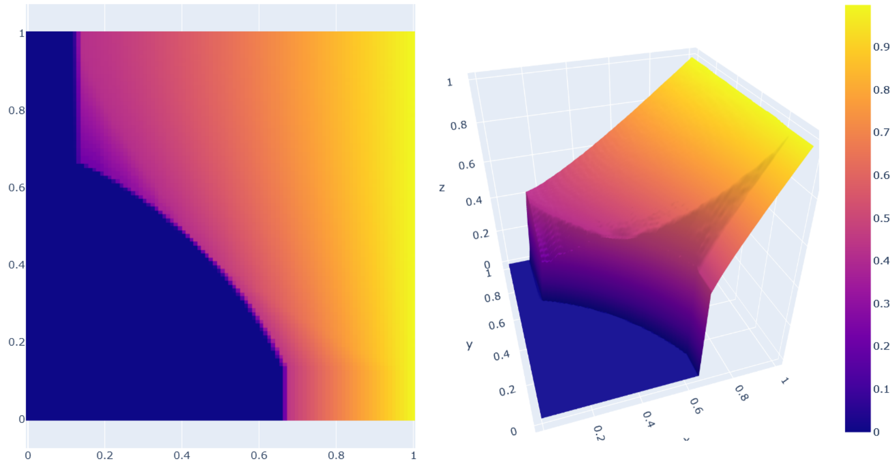


Figure: Graph of the first component of the conditional allocation function $\overline{P} = \frac{\partial u}{\partial x_1}$ for the uniformly distributed value estimate vector $x = (x_1, x_2)$.

Example of the monopolist problem with production cost

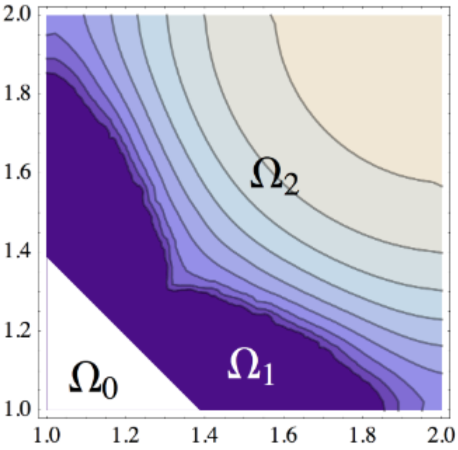


Figure: The level set of $\det \mathrm{H}(u)$ (Mirebeau 2014)

Problem example: $X = [1,2]^2$, $\rho(x)dx$ – uniform on X, $\varphi_1(x) = \varphi_2(x) = \frac{1}{2}x^2$

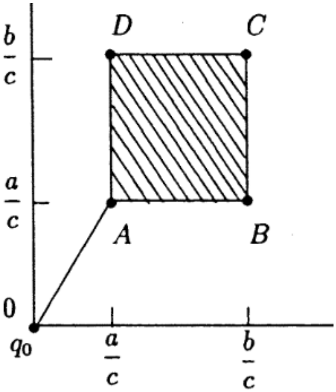
$$\int \left[\langle x, u(x) \rangle - u(x) - \frac{1}{2}||\nabla u||^2 \right] dx \to \max$$

over all $u \in \mathcal{U}$.

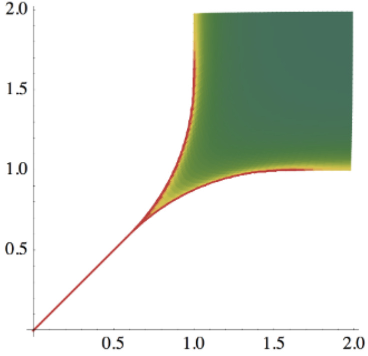
- Ω_0: $u(x) = 0$;
- Ω_1: $\det \mathrm{H}(u) = 0$;
- Ω_2: $\det \mathrm{H}(u) > 0$, the function u satisfies the Heat equation $\Delta u = 3$.

The exact solution is unknown even in this simplest case!

Solution: can solve with unprecedented numerical precision



Rochet & Chone, 1998



Mirebeau

Figure: Previously, it was expected that the image of ∇u is a union of the interval $[(0,0),(1,1)]$ and of the square $[1,2]^2$. Result of the modern computation is on the right picture.

Equivalence of dual to Kantorovich-Rubinstein and Beckmann problems

- The weak form of the constraint $\mathrm{div}[c] + \mu - \nu = 0$: for all φ, $\int \nabla\varphi(x) \cdot c(x)\, dx = \int \varphi(x) \cdot (\mu(dx) - \nu(dx))$
- Introduce a Lagrangian:
$$\min_{c:\ \mathrm{div}[c]+\mu-\nu=0} \int |c|\, dx = \min_c \max_\varphi \left\{ \int |c|\, dx - \int \nabla\varphi \cdot c\, dx + \int \varphi \cdot (\mu(dx) - \nu(dx)) \right\}$$
- Apply the minimax principle:
$$\min_{c:\ \mathrm{div}[c]+\mu-\nu=0} \int |c|\, dx = \max_\varphi \left\{ \int \varphi \cdot (\mu(dx) - \nu(dx)) + \min_c \int |c|\, dx - \int \nabla\varphi \cdot c\, dx \right\}$$
- $\min_c \int |c|\, dx - \int \nabla\varphi \cdot c\, dx$ is bounded iff $|\nabla\varphi(x)| \le 1$ for all x
- $|\nabla\varphi| \le 1$ is 1-Lipschitz condition: $\varphi(x) - \varphi(y) \le |x - y|$
- The problem $\max_\varphi \int \varphi(x) \cdot (\mu(dx) - \nu(dx))$ subject to $\varphi(x) - \varphi(y) \le |x - y|$ is dual to the transshipment problem

The algorithm could be scaled to multiple bidders

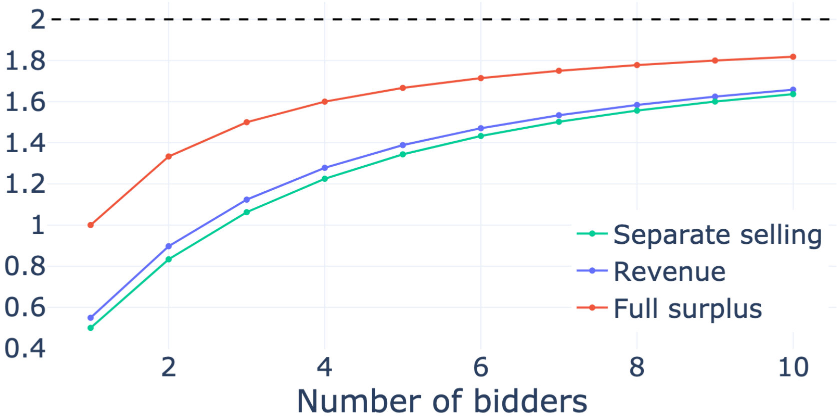


Figure: Revenue as a function of the number of bidders B for two items with i.i.d. values uniform on $[0, 1]$. Graphs from bottom to top: selling separately (light-green), selling optimally (blue), full surplus extraction (red), limit for $B \to \infty$ (the dashed line).

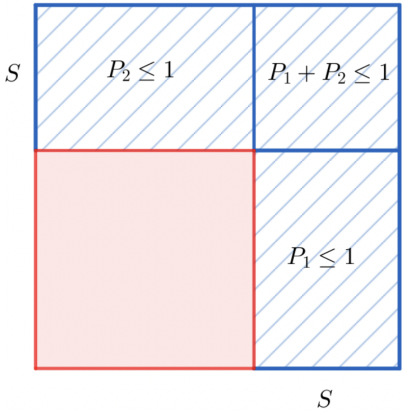
The Border's condition

Question

Given a reduced allocation function $\overline{P}$, under which conditions is it possible to find the full feasible mechanism $(P_1, \ldots, P_B)$?

Consider any set S of bidder types.

$$\sum_{b=1}^{B} \int_S \overline{P}_b(x_b)\, \rho(x_b) dx_b$$
$$= \sum_{b=1}^{B} \int_{x_b \in S} P_b(x_1, \ldots, x_B)\, \rho(x) dx$$
$$\leq \left| \cup_b \{x_b \in S\} \right| = 1 - (1 - |S|)^B.$$



skip slide

The Border's condition

Theorem (Border)

The reduced allocation function $\overline{P}(x)$ is feasible if and only if for any set S of bidder types,

$$\int_S \overline{P}(x)\,\rho(x)dx \leq \frac{1}{B}\left(1-(1-|S|)^B\right).$$

Example

Consider the case of B uniformly distributed bidders.

- $\overline{P}(x) = x^{B-1}$ for $x \geq \frac{1}{2}$;
- take $S = [t, 1]$:

$$\int_S \overline{P}(x)\,dx = \int_t^1 x^{B-1}\,dx = \frac{1}{B}(1-t^B).$$

The case of $I = 1$ item. A Vickrey auction

For $B \geq 1$ bidders, the auctioneer's revenue is equal to

$$R = \int \Big(V(x_1)P_1 + \cdots + V(x_B)P_B \Big)\, \rho(x_1, \ldots, x_B)\, dx_1 \ldots dx_B$$

subject to the constraint $P_1(x_1, \ldots, x_B) + \cdots + P_B(x_1, \ldots, x_B) \leq 1$. The maximum of the integrand is reached if $P_b = 1$ for the maximal $V(x_b)$.

Theorem (Myerson 1981)

The Vickrey auction *or* ***a second-price sealed-bid auction*** *is an optimal one: the highest bidder wins but the price paid is the second-highest bid. More precisely, denote* $x_0 = \min\{x \colon V(x) \geq 0\}$. *Then*

$$\begin{aligned} &P_b(x_1, \ldots, x_B) = 1 \textit{ and } T_b(x_1, \ldots, x_B) = \max_{d \neq b} x_d && \textit{if } x_b = \max_{0 \leq d \leq B} x_d, \\ &P_b(x_1, \ldots, x_B) = T_b(x_1, \ldots, x_B) = 0 && \textit{otherwise.} \end{aligned}$$

Time permitting: multidimensional taxation problem

- The distribution of workers $\alpha \sim \Phi$
  - $\alpha = (\alpha_c, \alpha_m)$ is a bundle of cognitive and manual skills
- Preferences: $U(c, l) = c - l_c^\rho - l_m^\rho$
- Task technology: $x_c = \alpha_c l_c$ and $x_m = \alpha_m l_m$

Problem

Maximize the total budget

$$\max_{c,x} \int \left(\frac{1}{2} x_c(\alpha)^2 + \frac{1}{2} x_c(\alpha)^2 - c(\alpha) \right) d\Phi(\alpha)$$

subject to:

- *the participation constraint:* $U(c(\alpha), x_c(\alpha)/\alpha_c, x_m(\alpha)/\alpha_m) \geq \underline{\mathcal{U}}$
- *the promise-keeping constraint:* $\int U(c(\alpha), x_c(\alpha)/\alpha_c, x_m(\alpha)/\alpha_m)\, d\Phi(\alpha) \geq \mathcal{U}$

Utility allocation

- Use

$$p_s := \alpha_s^{-\rho} \qquad\qquad x_s(p) := x_s(\alpha)^{\rho}$$

to transform preferences

$$u(c(\alpha)) - \left(\frac{x_c(\alpha)}{\alpha_c}\right)^{\rho} - \left(\frac{x_m(\alpha)}{\alpha_m}\right)^{\rho}$$

into a linear function

$$c(p) - p_c x_c(p) - p_m x_m(p)$$

Transformed planning problem

$$\min_{\{c, x_s\}} \int \left(c(p) - \frac{1}{2} x_c(p)^{2/\rho} - \frac{1}{2} x_m(p)^{2/\rho} \right) \pi(p) \mathrm{d}p$$

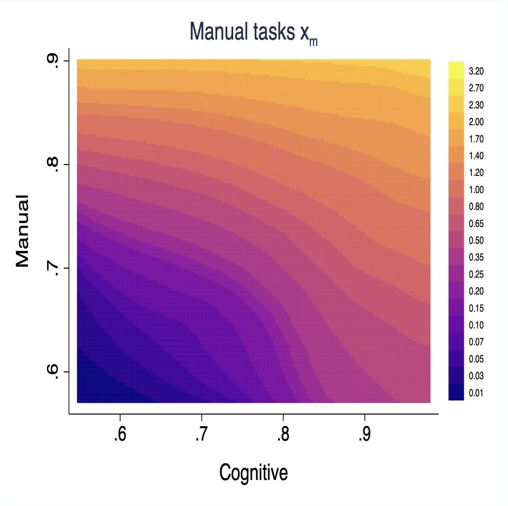
subject to:

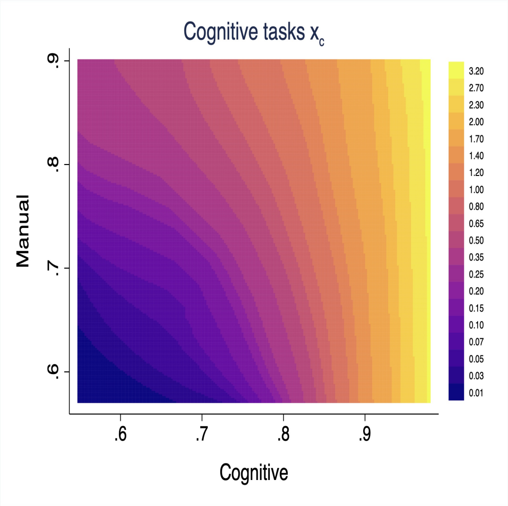
$$c(p) - p_c x_c(p) - p_m x_m(p) \geq c(q) - p_c x_c(q) - p_m x_m(q) \tag{IC}$$

$$c(p) - p_c x_c(p) - p_m x_m(p) \geq \underline{\mathcal{U}} \tag{OO}$$

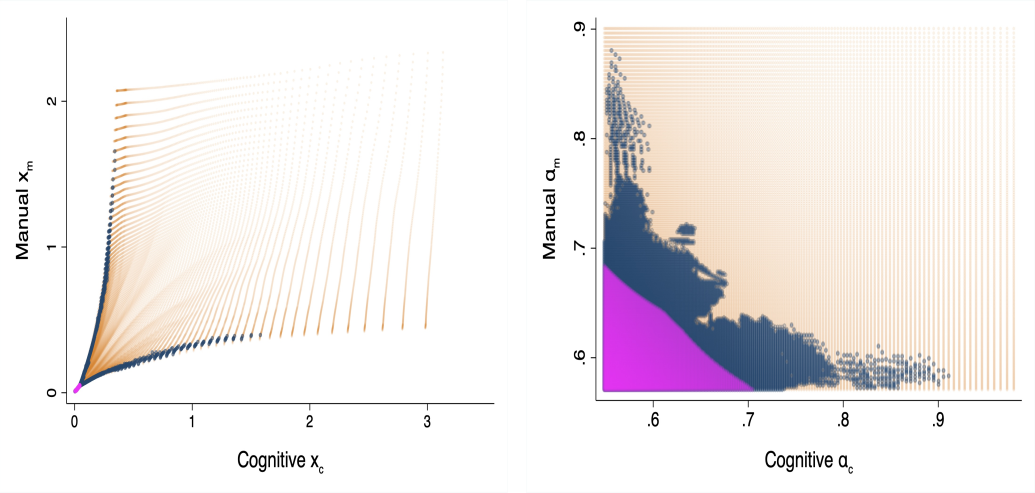
$$\int (c(p) - p_c x_c(p) - p_m x_m(p)) \pi(p) \mathrm{d}p \geq \mathcal{U} \tag{PK}$$

Task solution

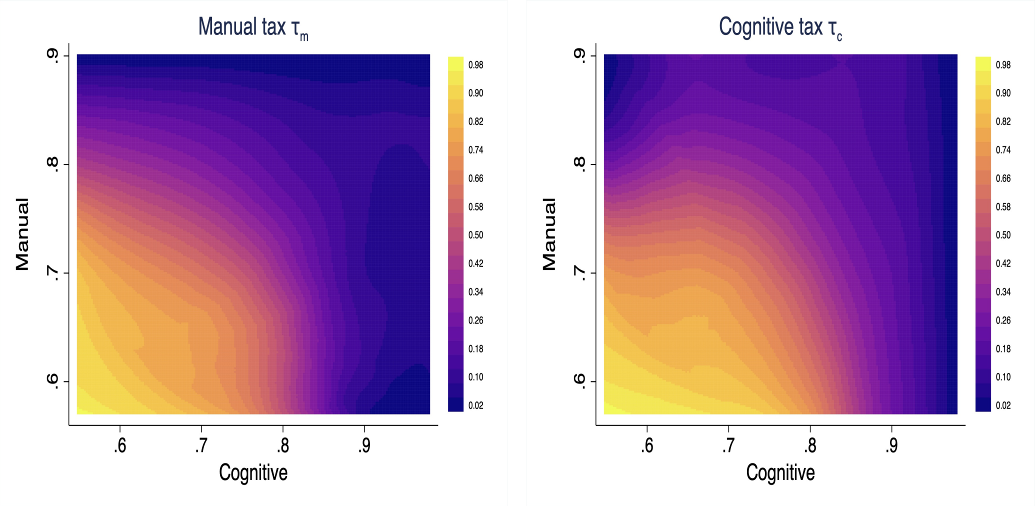




Optimal bunching

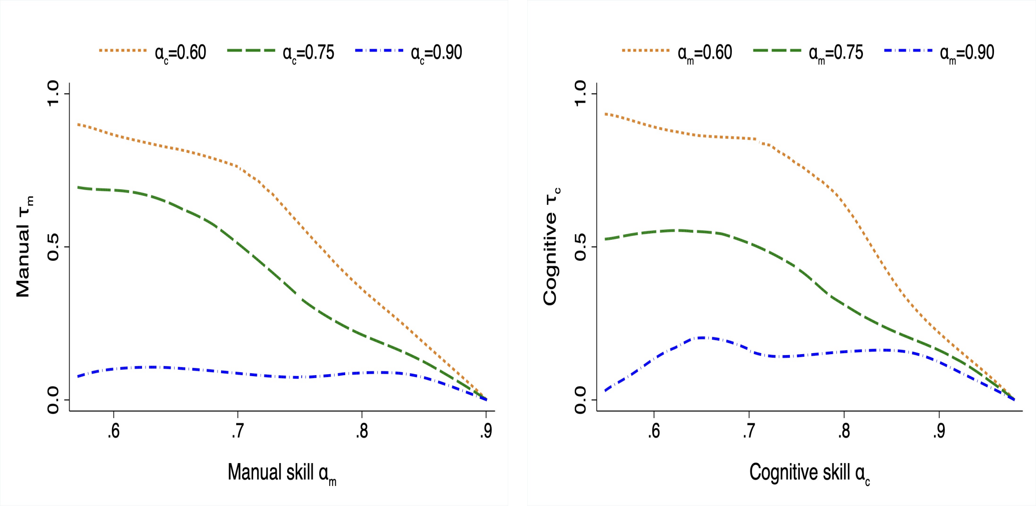


Tax wedges



Manual tax τ_m
Cognitive tax τ_c
Manual
Cognitive
.6
.7
.8
.9
0.98
0.90
0.82
0.74
0.66
0.58
0.50
0.42
0.34
0.26
0.18
0.10
0.02

Tax wedges



α_c=0.60
α_c=0.75
α_c=0.90
Manual τ_m
1.0
0.5
0.0
.6
.7
.8
.9
Manual skill α_m
α_m=0.60
α_m=0.75
α_m=0.90
Cognitive τ_c
1.0
0.5
0.0
.6
.7
.8
.9
Cognitive skill α_c

What is next

Maximize $R = B \cdot \int [x \cdot \nabla u(x) - u(x)]\, \rho(x) dx$ subject to

- **stochastic dominance:** $\overline{P}_i \geq 0$ and $\int \varphi(\overline{P}_i(x))\, \rho(x) dx \leq \int_0^1 \varphi(z^{B-1})\, dz$ for every convex non-decreasing φ
- **(IR)** and **(IC)**

Plan: writing a Lagrangian

- Put stochastic dominance constraint into the objective
- resulting problem consists of 2 steps:
  - choosing u;
  - choosing φ_i,
- problem with fixed φ_i, choosing u
- duality: then choose φ_i

Legendre transform

Definition

For a convex function f, define $f^*(y) = \sup_x\{xy - f(x)\}$.

Example

- $f(x) = \frac{1}{2}x^2$. Then $f^*(y) = \frac{1}{2}y^2$ and $\frac{1}{2}x^2 + \frac{1}{2}y^2 \geq xy$ is Cauchy's inequality
- $f(x) = \frac{x^\alpha}{\alpha}$. Then $f^*(y) = \frac{y^\beta}{\beta}$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The inequality $\frac{x^\alpha}{\alpha} + \frac{y^\beta}{\beta} \geq xy$ is Young's inequality

Theorem (Fenchel inequality)

For any convex function $f(x)$,

- $f(x) + f^*(y) \geq xy$,
- $f(x) = \sup_y\{xy - f^*(y)\}$.

Intuition: convex function f is a maximum of its tangent lines.

Use minimax principle

- **minimax principle:** $\max_u \min_c = \min_c \max_u$:

$$\frac{R}{B} = \min_{\varphi} \min_{c} \max_{u} \Bigg\{ \int \left[x \cdot \nabla u(x) - u(x) - \sum_{i=1}^{I} c_i(x) \cdot \frac{\partial u(x)}{\partial x_i} \right] \rho(x) dx + \underbrace{\sum_{i=1}^{I} \int \varphi_i^*(c_i(x))\, \rho(x) dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\, dz}_{\text{independent of } u} \Bigg\}.$$

- **maximize over u:** if the functional can take a positive value, then by replacing $u \to \lambda \cdot u$ with $\lambda > 0$ we can obtain any positive values:

$$\max_u \int \left[x \cdot \nabla u(x) - u(x) - \sum_{i=1}^{I} c_i(x) \cdot \frac{\partial u(x)}{\partial x_i} \right] \rho(x) dx = \begin{cases} 0 & \textbf{(take } u \equiv 0\textbf{)}, \\ +\infty & \textbf{(can multiply by } \lambda > 0\textbf{)} \end{cases}$$

skip slide

Can treat $\max_u$ as a constraint

- **minimax principle:** $\max_u \min_c = \min_c \max_u$:

$$\frac{R}{B} = \min_{\varphi} \min_{c} \Bigg\{ \underbrace{\max_u \int \left[x \cdot \nabla u(x) - u(x) - \sum_{i=1}^{I} c_i(x) \cdot \frac{\partial u(x)}{\partial x_i} \right] \rho(x) dx}_{\text{replace with } 0}$$

$$+ \underbrace{\sum_{i=1}^{I} \int \varphi_i^*(c_i(x))\, \rho(x) dx + \sum_{i=1}^{I} \int_0^1 \varphi_i(z^{B-1})\, dz}_{\text{independent of } u} \Bigg\}.$$

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The Kantorovich-Rubinstein problem (Dokl. Akad. Nauk SSSR, 1958)

Intuition

- In the classical problem, production and consumption nodes are separate.
- The transshipment problem: nodes can transfer and receive goods simultaneously.

The discrete mass transshipment problem

We are given:

- m points $k = 1, \dots, m$ and a vector $\psi = (\psi_1, \dots, \psi_m)$;
- ψ_k represents the volume of production (if $\psi_k \leq 0$) of consumption (if $\varphi_k > 0$)

Find a transport plan $\gamma = (\gamma_{ij})$: for each k,

- export $k \to j$ is γ_{kj}; total export: $\sum_j \gamma_{kj}$;
- import $i \to k$ is γ_{ik}; total import: $\sum_i \gamma_{ik}$;
- the **balancing condition**: $\sum \gamma_{ik} - \sum \gamma_{kj} = \psi_k$
- the total transportation cost $\sum \alpha_{ij}\gamma_{ij}$ is minimal.

Analysis of the problem. IC-constraint

Recall the incentive compatibility constraint:

$$u(x) = x \cdot \overline{P}(x) - \overline{T}(x) \geq x \cdot \overline{P}(\widehat{x}) - \overline{T}(\widehat{x})$$

- The right hand-side is equal to

$$x \cdot \overline{P}(\widehat{x}) - \overline{T}(\widehat{x}) = (x - \widehat{x}) \cdot \overline{P}(\widehat{x}) + u(\widehat{x}).$$

- The inequality

$$u(x) - u(\widehat{x}) \geq (x - \widehat{x})\overline{P}(\widehat{x})$$

holds for all $x, \widehat{x} \in [0,1]^I$ if and only if $u(x)$ is convex and $\overline{P}(x) \in \partial u(x)$ for all x.