$(3,2)-$Monge-Kantorovich problem

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Primal and Dual $(n,k)-$problem

Suppose $X_1, X_2, \dots, X_n$ are topological spaces with σ-algebras $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_n$ respectively.
Let $Pr_{X_{i_1}\times\cdots\times X_{i_k}}, Pr_I$ be the projection operator from $X = X_1 \times \cdots \times X_n$ to the coordinate $k-$dimensional subspace $X_{i_1} \times \cdots \times X_{i_k}$.

$$\mathcal{I}_k = \{(i_1, i_2, \dots, i_k) \mid 1 \le i_1 < i_2 \cdots < i_k \le n\}.$$

For any multi-index $I = (i_1, \dots i_k) \in \mathcal{I}_k$ there is given a measure μ_I on the space $X_{i_1} \ \times \ \dots \ \times \ X_{i_k}$.

$$\mathcal{P}_\mu = \{\mu \mid Pr_I \mu = \mu_I \text{ for any } I \in \mathcal{I}_k\}$$

Also, assume $c : X \to \mathbb{R} \cup \{+\infty\}$ is a cost function.

Primal and Dual $(n, k)-$problem

Definition

Primal $(n, k)-$problem is a problem of minimization of the functional

$$P(\pi) = \int_X c(x_1, \dots, x_n)\ d\pi$$

over $\pi \in \mathcal{P}_\mu$.

Definition

Dual $(n, k)-$problem is a problem of maximization of the functional

$$D(\{f_I\}) = \sum_{I \in \mathcal{I}_k} \int f_I(x_{i_1}, \dots, x_{i_k})\ d\mu_I$$

over (integrable) functions $\{f_I\}$ such that $\sum_I f_I(x_{i_1}, \dots, x_{i_k}) \leq c(x_1, \dots, x_n)$.

Duality

Theorem (Duality)

Suppose X_i are compact metric spaces, $c \geq 0$ is a continuous cost function. Then the following equality holds:

$$\min_{\mu \in \mathcal{P}_\mu} I[\mu] = \sup_{f_I \in L^1(\mu_I)} \sum_{I \in \mathcal{I}_k} \int f_I(x_{i_1}, x_{i_2}, \ldots x_{i_k}) \, d\mu_I.$$

Here one takes supremum over functions f_I such that

$$\sum_I f_I(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \leq c(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in X$.

Proof of duality

Duality theorem is proved in [G, Z, Kolesnikov, 2018] using the technique from [Villani, 2003]. The proof uses the following theorem:

Theorem (Fenchel-Rockafellar duality)

Let E be a normed vector space and E^* be the corresponding topologically dual space. Consider convex functionals Φ, Ψ on E with values in $\mathbb{R} \cup \{+\infty\}$. Let Φ^* , Ψ^* be their Legendre transforms. Assume that there exists a point $z \in E$ satisfying $\Phi(z) < +\infty$, $\Psi(z) < +\infty$ and Φ is continuous at z. Then

$$\inf(\Phi + \Psi) = \max(-\Phi^*(-z^*) - \Psi^*(z^*))$$

Prove of duality

Assume E is a space of continuous functions on X. By the Riesz-Markov-Kakutani representation theorem E^* is the space of finite signed measures on X.

$$\Phi(u) = \begin{cases} 0 \text{ if } u \geq -c, \\ +\infty \text{ else.} \end{cases} \qquad \Psi(u) = \begin{cases} \sum_{I \in \mathcal{I}_k} \int f_I \, d\mu_I & \text{if } u = \sum_I f_I, \\ +\infty & \text{else.} \end{cases}$$

$$\inf(\Phi(u) + \Psi(u)) = - \sup_{\sum_I f_I \leq c} \sum_I \int f_I \, d\mu_I$$

After Legendre transformation one obtains:

$$\Phi^*(-\pi) = \begin{cases} \int c \, d\pi, \text{ if } \pi \geq 0, \\ +\infty, \text{ else.} \end{cases} \quad \Psi^*(\pi) = \begin{cases} 0, \text{ if } \pi \in \mathcal{P}_\mu, \\ +\infty, \text{ else.} \end{cases}$$

Therefore $\max(-\Phi^*(-z^*) - \Psi^*(z^*)) = -\min_{\pi \in \mathcal{P}_\mu} \int c \, d\pi$.

The following plan

- Under which assumptions there exists at least one measure with given projections?
- Under which assumptions there exist dual solutions?
- Is dual solution bounded? Is it continuous?

Existence of a uniting measure

Definition

Uniting measures – measures in $\mathcal{P}_\mu$.

Unlike classical Monge-Kantorovich problem the set of measures with needed projections can be empty.

Proposition (Weak sufficient condition)

The set $\mathcal{P}_\mu$ is non-empty (there exist uniting measures) if $\mu_I = \mu_{i_1} \times \cdots \times \mu_{i_k}, I = (i_1, \ldots, i_k) \in \mathcal{I}_k$ for some probability measures $\mu_1, \ldots \mu_n$ on respective spaces $X_1, \ldots, X_n$.

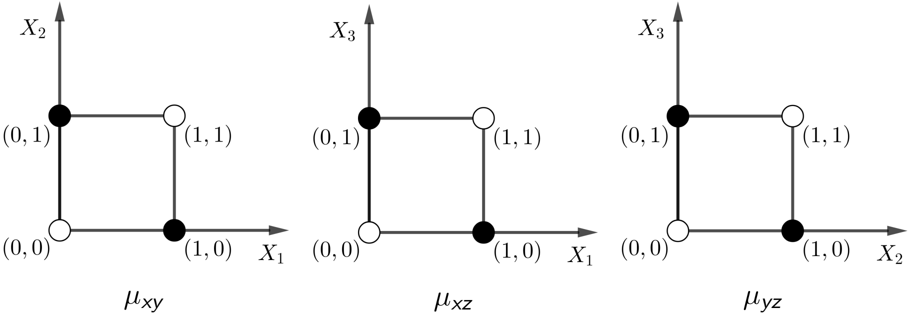
Proposition (Weak necessary condition)

Suppose $\mathcal{P}_\mu$ is non-empty; then for any $I, J \in \mathcal{I}_k$ there holds

$$Pr_{I \cap J}\ \mu_I = Pr_{I \cap J}\ \mu_J$$

Existence of a uniting measure

Weak necessary condition is not sufficient. Suppose $X = Y = Z = \{0, 1\}$. Define measures on $X \times Y$, $X \times Z$ and $Y \times Z$



There is no uniting measure for μ_{xy}, μ_{xz} and μ_{yz} but there exists uniting signed measure.

Notation for $(3, 2)-$problem

Suppose X, Y, Z are some measurable spaces; assume $\mu_{xy}, \mu_{xz}, \mu_{yz}$ are finite measures on $X \times Y$, $Y \times Z$, $X \times Z$. For the existence of a measure μ on $X \times Y \times Z$ with projections μ_{xy}, μ_{xz}, μ_{yz} the following equalities must hold:

$$Pr_X \mu_{xy} = Pr_X \mu_{xz} = \mu_x,$$
$$Pr_Y \mu_{xy} = Pr_Y \mu_{yz} = \mu_y,$$
$$Pr_Z \mu_{xz} = Pr_Z \mu_{yz} = \mu_z.$$

Also let ν_x, ν_y, ν_z be arbitrary finite measures on X, Y, Z.

Existence of a uniting measure in $(3, 2)$

We recall the existence of the uniting measure in case $\mu_{xy} = \mu_x \times \mu_y$, $\mu_{xz} = \mu_x \times \mu_z$, $\mu_{yz} = \mu_y \times \mu_z$. For example, there fits the measure $\mu_x \times \mu_y \times \mu_z$. The following theorem gives a generalization of this construction:

Theorem (Density condition)

Suppose X, Y, Z are spaces equipped with finite measures ν_x, ν_y, ν_z. Suppose that $\mu_{xy}, \mu_{xz}, \mu_{yz}$ are absolutely continuous with respect to $\nu_x \times \nu_y, \nu_x \times \nu_z, \nu_y \times \nu_z$ respectively. Assume p_{xy}, p_{xz}, p_{yz} are the respective densities. If for $\lambda \leq \frac{3}{2}$ there holds

$$1 \leq p_{xy}, p_{xz}, p_{yz} \leq \lambda,$$

then there exists a uniting measure for μ_{xy}, μ_{xz} and μ_{yz}.

Existence of a uniting measure in $(3, 2)$

It's sufficient to prove the density condition theorem only for $\lambda = \frac{3}{2}$. Without loss of generality ν_x, ν_y, ν_z are probability measures.

$$M = \mu_{xy}(X \times Y) = \mu_{xz}(X \times Z) = \mu_{yz}(Y \times Z),$$

Assume p_x, p_y, p_z are the densities of μ_x, μ_y, μ_z with respect to ν_x, ν_y, ν_z. There holds $1 \le p_x, p_y, p_z, M \le \lambda$.
For example, if $M = \lambda$, the following equalities hold: $\mu_{xy} = \lambda(\nu_x \times \nu_y), \mu_{xz} = \lambda(\nu_x \times \nu_z), \mu_{yz} = \lambda(\nu_y \times \nu_z)$. The measure $\mu = \lambda(\nu_x \times \nu_y \times \nu_z)$ has projections μ_{xy}, μ_{xz} and μ_{yz}. The same argument works for $M = 1$.

Existence of a uniting measure in $(3, 2)$

The following signed measure is uniting

$$\mu = \frac{4}{M^2}\mu_x \times \mu_y \times \mu_z - \\ - \frac{2}{M}\left(\nu_x \times \mu_y \times \mu_z + \mu_x \times \nu_y \times \mu_z + \mu_x \times \mu_y \times \nu_z\right) + \\ + 2\left(\mu_{xy} \times \nu_z + \mu_{xz} \times \nu_y + \mu_{yz} \times \nu_x\right) - \\ - \frac{1}{M}\left(\mu_{xy} \times \mu_z + \mu_{xz} \times \mu_y + \mu_{yz} \times \mu_x\right)$$

since

$$Pr_{XY}\mu = \frac{4}{M}\mu_x \times \mu_y - 2\nu_x \times \mu_y - 2\mu_x \times \nu_y - \frac{2}{M}\mu_x \times \mu_y + \\ + 2\mu_{xy} + 2\mu_x\nu_y + 2\nu_x\mu_y - \mu_{xy} - \frac{2}{M}\mu_x \times \mu_y = \mu_{xy}.$$

Existence of a uniting measure in $(3, 2)$

Check the non-negativity of this measure. To this end, check

$$\frac{4}{M^2}a_1b_1c_1 - \frac{2}{M}(a_1b_1 + a_1c_1 + b_1c_1) + 2(a_2 + b_2 + c_2) - \\ - \frac{1}{M}(a_1a_2 + b_1b_2 + c_1c_2) \geq 0$$

for $1 \leq a_1, b_1, c_1, a_2, b_2, c_2, M \leq \frac{3}{2}$. This expression is greater than $\varepsilon(M) > 0$ for all $a_1, b_1, c_1, a_2, b_2, c_2$, and $M \in \left(1, \frac{3}{2}\right)$.

Proposition

In the assumptions of the density condition there exists a uniting measure μ absolutely continuous with respect to $\nu_x \times \nu_y \times \nu_z$; the density of this measure is bounded and separated from zero.

Existence of a uniting measure

Theorem

For $\lambda \leq 2$ there exists a (not necessary absolutely continuous) uniting measure μ.

For $\lambda > 2$ this theorem fails.
Most of the results can be generalized to $(n, k)-$problem.

Theorem

Suppose $\{\mu_I \mid I \in \mathcal{I}_k\}$ satisfy the weak necessary conditions. Then there exists a signed measure μ such that

$$Pr_I \mu = \mu_I, I \in \mathcal{I}_k.$$

There exists an analogue of density condition in $(n, k)-$problem for some $\lambda_{n,k}$.

$(3,2)-$function

Definition

A function $F : X \times Y \times Z \to \mathbb{R}$ is called a $(3,2)-$function if there exist functions f_{xy}, f_{xz}, f_{yz} such that

$$F(x,y,z) = f_{xy}(x,y) + f_{xz}(x,z) + f_{yz}(y,z)$$

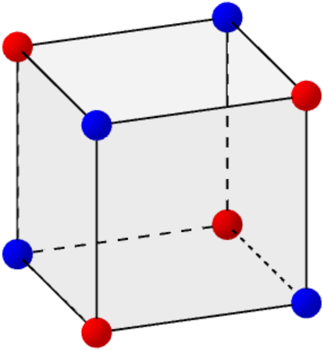
for any $(x,y,z) \in X \times Y \times Z$.

Proposition

F is a $(3,2)-$function iff for any $x_0, x_1 \in X$, $y_0, y_1 \in Y$, $z_0, z_1 \in Z$ there holds

$$F(x_0,y_0,z_0) + F(x_1,y_1,z_0) + F(x_1,y_0,z_1) + F(x_0,y_1,z_1) = \\ = F(x_1,y_1,z_1) + F(x_1,y_0,z_0) + F(x_0,y_1,z_0) + F(x_0,y_0,z_1)$$

$(3, 2)-$function



For any $(3, 2)-$function F the sum of the values in the red points equals the sum of the values in the blue points.

$(3,2)-$function

One can construct f_{xy}, f_{xz}, f_{yz} using $(3,2)-$function F.

Proposition

Suppose F is a $(3,2)-$function, (x_0, y_0, z_0) is an arbitrary point of $X \times Y \times Z$.

$$f_{xy}(x,y) = F(x,y,z_0) - \frac{1}{2}F(x,y_0,z_0) - \frac{1}{2}F(x_0,y,z_0) + \frac{1}{3}F(x_0,y_0,z_0),$$
$$f_{xz}(x,z) = F(x,y_0,z) - \frac{1}{2}F(x,y_0,z_0) - \frac{1}{2}F(x_0,y_0,z) + \frac{1}{3}F(x_0,y_0,z_0),$$
$$f_{yz}(y,z) = F(x_0,y,z) - \frac{1}{2}F(x_0,y_0,z) - \frac{1}{2}F(x_0,y,z_0) + \frac{1}{3}F(x_0,y_0,z_0).$$

Then $F(x,y,z) = f_{xy}(x,y) + f_{xz}(x,z) + f_{yz}(y,z)$.

Definition

The functions f_{xy}, f_{xz}, f_{yz} from the proposition above are called **frame functions** of $(3,2)-$function F **at the point** (x_0, y_0, z_0).

$(3,2)-$function

Remark

Suppose $F = f_{xy} + f_{xz} + f_{yz}$ *and* $f_{xy} \in L^1(\mu_{xy})$, $f_{xz} \in L^1(\mu_{xz})$, $f_{yz} \in L_1(\mu_{yz})$. *Then* $F \in L^1(\mu)$ *for any uniting measure* μ.
But it's not true, that if F *is a* $(3,2)-$*function and* $F \in L^1(\mu)$ *for a uniting measure* μ, *then there exist* $f_{xy} \in L^1(\mu_{xy})$, $f_{xz} \in L^1(\mu_{xz})$, $f_{yz} \in L_1(\mu_{yz})$ *such that* $F = f_{xy} + f_{xz} + f_{yz}$.

Definition

Measures μ and ν are called **uniformly equivalent** if $L^1(\mu) = L^1(\nu) \Leftrightarrow d\mu = r\ d\nu$ for some bounded and separated from zero density function r.
A measure μ on the space $X \times Y \times Z$ is called **almost product** if there exist measures ν_x, ν_y, ν_z such that μ is uniformly equivalent to $\nu_x \times \nu_y \times \nu_z$.
It's easy to prove that it's sufficient to take $Pr_X\mu, Pr_Y\mu, Pr_Z\mu$ as ν_x, ν_y, ν_z.

$(3,2)-$function

Theorem

Denote by F a $(3,2)-$function on the space $X \times Y \times Z$, μ is a finite uniting measure on $X \times Y \times Z$; suppose that μ is almost product. Suppose $F \in L^1(\mu)$; then for almost all $(x_0, y_0, z_0) \in X \times Y \times Z$ the frame functions f_{xy}, f_{xz}, f_{yz} are integrable with respect to $\mu_{xy}, \mu_{xz}, \mu_{yz}$.

Corollary

Assume μ and ν are measures on the space $X \times Y \times Z$ such that there holds

$$Pr_{XY}\mu = Pr_{XY}\nu, \; Pr_{XZ}\mu = Pr_{XZ}\nu, \; Pr_{YZ}\mu = Pr_{YZ}\nu.$$

Suppose μ is almost product; $F \in L^1(\mu)$ is a $(3,2)-$function. Then $F \in L^1(\nu)$ and

$$\int F \, d\mu = \int F \, d\nu$$

Existence of a solution of the dual $(3,2)-$problem

Definition (Dual $(3,2)-$problem)

Suppose $\mu_{xy}, \mu_{xz}, \mu_{yz}$ are the measures on $X \times Y$, $X \times Z$, $Y \times Z$; c is a cost function on $X \times Y \times Z$. Dual $(3,2)-$problem is a problem of maximization

$$\int f_{xy}\; d\mu_{xy} + \int f_{xz}\; d\mu_{xz} + \int f_{yz}\; d\mu_{yz}$$

over (integrable) functions f_{xy}, f_{xz}, f_{yz} such that $f_{xy}(x,y) + f_{xz}(x,z) + f_{yz}(y,z) \leq c(x,y,z)$.

Definition (Dual $(3,2)-$problem)

Suppose μ is a measure on $X \times Y \times Z$ with the projections $\mu_{xy}, \mu_{xz}, \mu_{yz}$; c is a cost function. Dual $(3,2)-$problem is a problem of maximization $\int F\; d\mu$ over (integrable) $(3,2)-$functions F such that $F \leq c$.

This definitions are equivalent if μ is almost product.

Existence of a solution of the dual $(3,2)-$problem

Theorem

Assume μ is a probability measure on $X \times Y \times Z$ and c is a cost function such that $c(x,y,z) \leq c_{xy}(x,y) + c_{xz}(x,z) + c_{yz}(y,z)$ for some (integrable) $c_{xy}, c_{xz}, c_{yz} < +\infty$. Suppose μ is almost product. Assume that c is greater than some $(3,2)-$function. Then there exist integrable with respect to μ_{xy}, μ_{xz} and μ_{yz} functions $-\infty \leq f_{xy}, f_{xz}, f_{yz} < +\infty$ such that $F_0 = f_{xy} + f_{xz} + f_{yz} \leq c$, and $\sup_{F \leq c} \int F \; d\mu = \int F_0 \; d\mu$.

Remark

The same conditions for $(n,1)-$problem were used in [Kellerer 1984]

Existence of a solution of the dual $(3, 2)-$problem

Without loss of generality we can assume that $c \leq 0$.

Theorem (Komlosh)

Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose $\{f_n\} \subset L^1(\mu)$ and $\sup_n ||f_n||_{L^1(\mu)} < \infty$. Then there exists a subsequence $\{g_n\} \subset \{f_n\}$ and a function $g \in L^1(\mu)$ such that for any subsequence $\{h_n\} \subset \{g_n\}$ arithmetic means of the first n partial sums $(h_1 + \cdots + h_n)/n$ tend to g almost everywhere.

By this theorem, there exists a sequence of $(3, 2)-$functions $\{F_n\} \subset L^1(\mu)$ and $F \in L^1(\mu)$ such that $F_n \leq c$, $\lim_{n\to\infty} \int F_n \, d\mu = \sup_{F \leq c} \int F \, d\mu$ and F_n tend to F almost everywhere.

Existence of a solution of the dual $(3,2)-$problem

All the functions F_n are bounded from above. Therefore, it follows from reverse Fatou's lemma, that $\int F\ d\mu \geq \lim_{n\to\infty} \int F_n\ d\mu = \sup_{F\leq c} \int F\ d\mu$.

Definition

A point $(x,y,z) \in X \times Y \times Z$ is called **regular** if $\lim_{n\to+\infty} F_n(x,y,z) = F(x,y,z) \neq \infty$.

For F and $(x_0,y_0,z_0) \in X \times Y \times Z$ we define f_{xy}, f_{xz}, f_{yz} as follows:

$$f_{xy}(x,y) = \begin{cases} F(x,y,z_0) - \frac{1}{2}F(x,y_0,z_0) - \frac{1}{2}F(x_0,y,z_0) + \frac{1}{3}F(x_0,y_0,z_0) \\ \text{if } (x,y,z_0),(x,y_0,z_0),(x_0,y,z_0),(x_0,y_0,z_0) \text{ are regular,} \\ -\infty \text{ otherwise.} \end{cases}$$

f_{yz}, f_{xz} are constructed in the same way.

Existence of a solution of the dual $(3, 2)-$problem

Lemma

For almost all points $(x_0, y_0, z_0) \in X \times Y \times Z$ *the functions* f_{xy}, f_{xz}, f_{yz} *from the previous slide are so that*

- f_{xy}, f_{xz}, f_{yz} *are integrable with respect to* μ_{xy}, μ_{xz}, μ_{yz},
- $f_{xy} + f_{xz} + f_{yz} \leq F$,
- $f_{xy} + f_{xz} + f_{yz} = F$ *almost everywhere.*

This lemma is proved by Fubini's theorem.
Then the functions f_{xy}, f_{xz}, f_{yz} are the solution of the dual problem. Q.E.D.

Remark

The same technique works for (n, k).

Nonexistence of a dual solution

Aim: construct a measure μ and a bounded cost function c such that there exists no «maximal» $(3,2)-$function for the related dual problem. So, measure μ will not be almost product.

Suppose $X = Y = Z = \mathbb{N}$ are discrete measurable spaces. Assume $p_n = \frac{1}{n^2}$. Denote by A_n the set $\{(n+1,n,n),(n,n+1,n),(n,n,n+1),(n,n+1,n+1),(n+1,n,n+1),(n+1,n+1,n)\}$ for any $n \in \mathbb{N}$. Denote by μ the following measure on $X \times Y \times Z$:

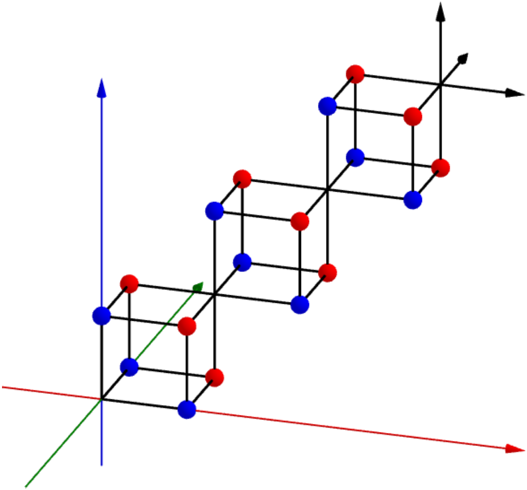
$$\mu(x,y,z) = \begin{cases} p_n & \text{if } (x,y,z) \in A_n, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mu_{xy}, \mu_{xz}, \mu_{yz}$ the projections of μ. Suppose

$$c(x,y,z) = \begin{cases} 1 & \text{if } (x,y,z) \in \{(n+1,n,n),(n,n+1,n),(n,n,n+1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

is a cost function.

Nonexistence of a dual solution



Support of μ is the set of colored points. The cost function equals 1 on the blue points and 0 elsewhere.

Nonexistence of a dual solution

Lemma

There exists a unique uniting measure μ for $\mu_{xy}, \mu_{xz}, \mu_{yz}$.

Let ν be a measure with projections $\mu_{xy}, \mu_{xz}, \mu_{yz}$. Then ν is supported on $\{(x, y, z) \mid \max(|x-y|, |x-z|, |y-z|) \leq 1\}$. Assume $a_n = \nu(n, n, n)$. It's easy to prove that

$$\nu(n+1, n, n) = \nu(n, n+1, n) = \nu(n, n, n+1) = p_n - \sum_{i=1}^{n} a_n$$

$$\nu(n, n+1, n+1) = \nu(n+1, n, n+1) = \nu(n+1, n+1, n) = p_n + \sum_{i=1}^{n} a_n$$

p_n tend to 0. Therefore, if $a_k > 0$ for some k, then there exists $n \in \mathbb{N}$ such that

$$\nu(n+1, n, n) < 0.$$

Nonexistence of a dual solution

In particular μ is the primal solution of the related $(3,2)-$problem. Suppose F is the dual problem solution. It follows from the complementary slackness conditions that:

$$F(n+1,n,n) = F(n,n+1,n) = F(n,n,n+1) = 1,$$
$$F(n,n+1,n+1) = F(n+1,n,n+1) = F(n+1,n+1,n) = 0.$$

It follows from the property of $(3,2)-$function that $F(n+1,n+1,n+1) - F(n,n,n) = -3$. Since $F(1,1,1) \leq 0$, we obtain $F(n,n,n) \leq 3-3n$.
If $F = f_{xy} + f_{xz} + f_{yz}$ then there folds

$$\int |f_{xy}|\, d\mu_{xy} + \int |f_{xz}|\, d\mu_{xz} + \int |f_{yz}|\, d\mu_{yz} \geq \sum_{n=1}^{+\infty} (3n-3)p_n = +\infty$$

Boundedness of a dual solution

Remark

In the classical Monge-Kantorovich problem if the cost function is bounded then there exists a bounded dual solution.

Theorem

Assume $X = Y = Z = \mathbb{N}$; μ_x, μ_y, μ_z are probability measures on X, Y, Z. Suppose $\mu_{xy} = \mu_x \times \mu_y$, $\mu_{xz} = \mu_x \times \mu_z$, $\mu_{yz} = \mu_y \times \mu_z$; c is a cost function such that $0 \leq c \leq 1$. Denote by F a dual solution of $(3, 2)-$problem with projections μ_{xy}, μ_{xz}, μ_{yz} and the cost function c. Then $-12 \leq F$ almost everywhere.

Remark

In the $(3, 2)-$problem for compact metric spaces X, Y, Z, bounded c and almost product μ primal solution is bounded.

Boundedness of a dual solution

Assume $\mu = \mu_x \times \mu_y \times \mu_z$ and opt is the primal solution.
Complementary slackness:

$$\mathrm{opt}(x, y, z) = 0 \text{ или } F(x, y, z) = c(x, y, z).$$

Proposition

For arbitrary probability measure ν there holds $\int F\, d\nu \leq 1$. If the support of ν is a subset of the support of opt then $\int F\, d\nu = \int c\, d\nu \geq 0$.

Lemma

For every z_0 such that $\mu_z(z_0) > 0$ there holds

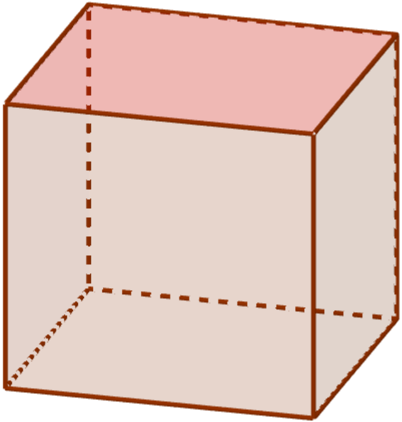
$$\int_{z=z_0} F(x, y, z_0)\, d\mu_x \times \mu_y \geq -1 + \int_{X \times Y \times Z} F\, d\mu.$$

Boundedness of a dual solution

Consider signed μ_0:

$\mu_0(x, y, z) =$

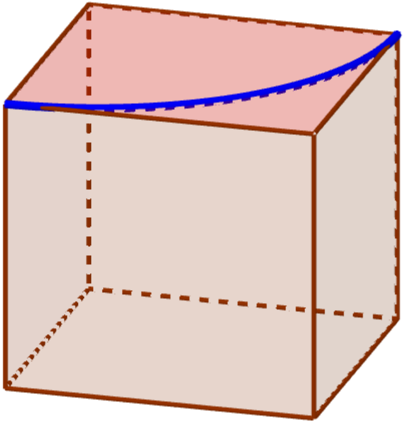
$=$



Boundedness of a dual solution

Consider signed μ_0:

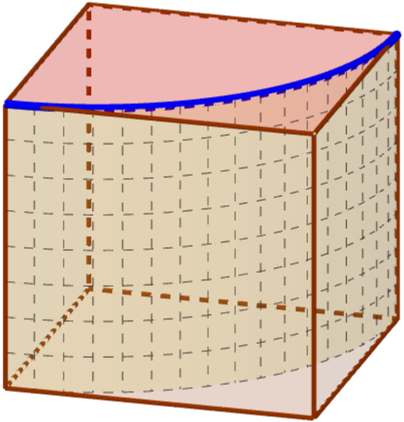
$$\mu_0(x, y, z) =$$
$$= \frac{1}{\mu_z(z_0)} \mathrm{opt}(x, y, z)\delta_{z_0}(z)$$



Boundedness of a dual solution

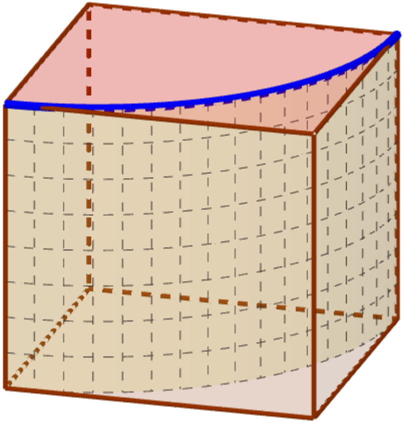
Consider signed μ_0:

$$\mu_0(x, y, z) = \\ = \frac{1}{\mu_z(z_0)} \mathrm{opt}(x, y, z)\delta_{z_0}(z) \\ - \frac{1}{\mu_z(z_0)} \mathrm{opt}(x, y, z_0)\mu_z(z)$$



Consider signed μ_0:

$$\mu_0(x,y,z) = \\ = \frac{1}{\mu_z(z_0)}\mathrm{opt}(x,y,z)\delta_{z_0}(z) \\ - \frac{1}{\mu_z(z_0)}\mathrm{opt}(x,y,z_0)\mu_z(z) \\ + \mu(x,y,z)$$



Boundedness of a dual solution

Easy to check that

$$Pr_{X\times Y}\mu_0 = \mu_x \times \mu_y,$$
$$Pr_{X\times Z}\mu_0 = \mu_x \times \delta_{z_0},$$
$$Pr_{Y\times Z}\mu_0 = \mu_y \times \delta_{z_0},$$

So projections of μ_0 coincide with those of $\mu_x \times \mu_y \times \delta_{z_0}$.
Then

$$\int F(x,y,z_0)\ d\mu_x \times \mu_y = \int F(x,y,z) d\mu_0 \geq 0 - 1 + \int F\ d\mu.$$

Boundedness of a dual solution

Let $\mu(x_0, y_0, z_0) > 0$. Then there exist x_1, y_1, z_1 such that

$$\mathrm{opt}(x_1, y_0, z_0) > 0,\ \mathrm{opt}(x_0, y_1, z_0) > 0,\ \mathrm{opt}(x_0, y_0, z_1) > 0.$$

Consider

$$\begin{aligned}\mu_1 = \delta_{x_1} \times \delta_{y_0} \times \delta_{z_0} + \delta_{x_0} \times \delta_{y_1} \times \delta_{z_0} + \delta_{x_0} \times \delta_{y_0} \times \delta_{z_1} - \\ - (\delta_{y_0} \times \delta_{z_1} + \delta_{y_1} \times \delta_{z_0}) \times \mu_x - (\delta_{x_0} \times \delta_{z_1} + \delta_{x_1} \times \delta_{z_0}) \times \mu_y - \\ - (\delta_{x_0} \times \delta_{y_1} + \delta_{x_1} \times \delta_{y_0}) \times \mu_z + (\delta_{x_0} + \delta_{x_1}) \times \mu_y \times \mu_z + \\ + (\delta_{y_0} + \delta_{y_1}) \times \mu_x \times \mu_z + (\delta_{z_0} + \delta_{z_1}) \times \mu_x \times \mu_y - 2\mu_x \times \mu_y \times \mu_z\end{aligned}$$

Projections of μ_1 coincide with those of $\delta_{x_0} \times \delta_{y_0} \times \delta_{z_0}$. That means

$$F(x_0, y_0, z_0) = \int F\, d\mu_1 \geq -12 + 4 \int F\, d\mu.$$

Corollary

There exists a bounded dual solution.

Discontinuous dual solution

Theorem

Assume $X = Y = Z = [0,1]$*;* $\mu_{xy}, \mu_{xz}, \mu_{yz}$ *are Lebesgue measures on* $[0,1]^2$*. Suppose* $c = \max(0, x+y+3z-3)$ *is a cost function. Then any dual solution of the related dual problem equals*

$$F(x,y,z) = \begin{cases} 0 & \text{if } \ z \le \frac{2}{3}, \\ x+y+3z-3 & \text{if } \ z > \frac{2}{3} \end{cases}$$

almost everywhere. In particular, there is no continuous solution for this problem.

Discontinuous dual solution

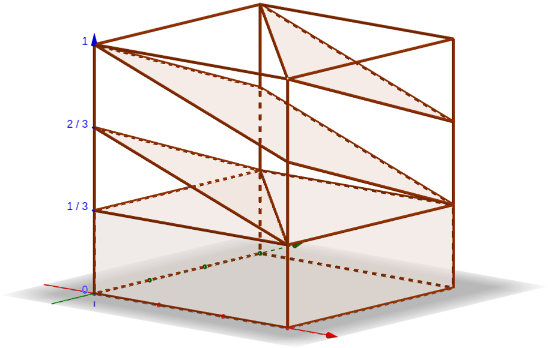


Figure: An optimal measure for the cost function $\max(0, x + y + 3z - 3)$