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ON THE STABILITY OF SYSTEMS WITH MIXED TIME-VARYING PARAMETERS

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SUMMARY

The well-known scaled small gain condition guarantees stability for a linear time invariant system subject to bounded complex nonlinear and/or time-varying perturbations. A polynomial time computable condition is derived that can be substantially less conservative for gain scheduled and other multivariable systems with repeated real time-varying parameters. The proof is a generalization of the purely-complex case given in Andrew Packard's thesis.

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1. INTRODUCTION

Many nonlinear and/or time-varying phenomena encountered in chemical and mechanical processes can be treated as bounded nonlinear and/or time-varying perturbations to a linear time invariant system.[2–4] The optimally scaled small gain theorem provides a sufficient condition for the closed-loop stability of such systems.[2,4,5] Although this condition is also necessary for various sets of complex norm-bounded operators,[6–9] it is well known that it can be an extremely conservative stability condition for other perturbations.

Numerous researchers have exploited additional information to reduce the conservatism in the scaled small gain theorem. The early work focused on systems with a single nonlinearity or time-varying parameter.[2,10–19] Boyd and Yang[20] developed a stability condition for systems with general linear fractional dependence on non-repeated real and complex time-varying parameters in terms of a number of linear matrix inequalities that grow exponentially as a function of the number of real parameters. Becker *et al.*[21] explicitly took into account the real nature of time-varying parameters for multivariable systems with affine parameter dependence. Jönsson and Rantzer[22] and Megretski[23] have derived conditions for system stability for multivariable systems with linear fractional dependence on non-repeated real time-varying parameters with bounded gain and bounded time variation.

This note derives a sufficient stability condition for multivariable systems with general linear fractional dependence on repeated and full block, real and complex, time-varying parameters.

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The condition is computable in polynomial time as a function of the number of parameters and states, and is derived with a minimum of mathematical machinery. The condition is shown to be substantially less conservative for multivariable systems with repeated real time-varying parameters, as occurs in the linear fractional approach to gain scheduling.[3]

The proof of the condition does not require the use of positivity/multiplier theory,[13,24,25] integral quadratic constraints,[22] or the explicit construction of a quadratic Lyaponov function[21,26,27] but follows only from basic properties of the structured singular value. Although the proof does not rely on notions of quadratic stability, the results are closely related, in a similar manner as for the purely-complex case.[27] Packard and Doyle[27] used the quadratic stability approach to derive the condition for the case where all the parameters are complex (the 'real' in the title of Reference 27 refers only to a counter-example that shows that whether a time-varying parameter is real or complex may affect the stability of the system). This note generalizes an alternative proof that appears in Packard's thesis[1] to the case with mixed real and complex real time-varying parameters. This generalization is much more straightforward than directly generalizing the quadratic stability approach taken in Reference 27. To the author's knowledge the results and the method of proof have not before appeared in either proceedings or journal publication.

2. MATHEMATICAL PRELIMINARIES

Below we provide definitions and review prior results.

Linear fractional transformations

System interconnections are characterized using linear fractional transformations (see Figure 1). With

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{1}$$

partitioned to be compatible with Δ_1, the transfer function between the system input $\hat{d}$ and output $\hat{e}$ is given by the *linear fractional transformation* (LFT)

$$F_u(M, \Delta_1) = M_{22} + M_{21}\Delta_1(I - M_{11}\Delta_1)^{-1}M_{12} \tag{2}$$

The LFT $F_u(M, \Delta_1)$ is well defined if and only if the inverse of $I - M_{11}\Delta_1$ exists (the subscript u is used to denote that the upper loop of M is closed by Δ_1). If the lower loop had been closed instead, then the transfer function between inputs and outputs would be the LFT $F_l(M, \Delta_1) = M_{11} + M_{12}\delta_1(I - M_{22}\Delta_1)^{-1}M_{21}$.

Structured singular value

Doyle[28] and Safonov[29] defined the *structured singular value*, μ, as a tool for analysing the robustness of uncertain systems. Without loss of generality we assume that each Δ_i and M is square.[1,30] The definition of μ for mixed real and complex perturbations follows.

Definition 2.1

Let $M \in \mathscr{C}^{n \times n}$ be a square complex matrix and define the set $\mathbf{\Delta}$ by

$$\mathbf{\Delta} \equiv \left\{ \operatorname{diag}\{\delta_1^r I_{r_1}, \ldots, \delta_k^r I_{r_k}, \delta_{k+1}^c I_{r_{k+1}}, \ldots, \delta_m^c I_{r_m}, \Delta_{m+1}, \ldots, \Delta_l\} \mid \delta_i^r \in \mathscr{R}, \delta_i^c \in \mathscr{C}, \Delta_i \in \mathscr{C}^{r_i \times r_i}, \sum_{i=1}^{l} r_i = n \right\} \tag{3}$$

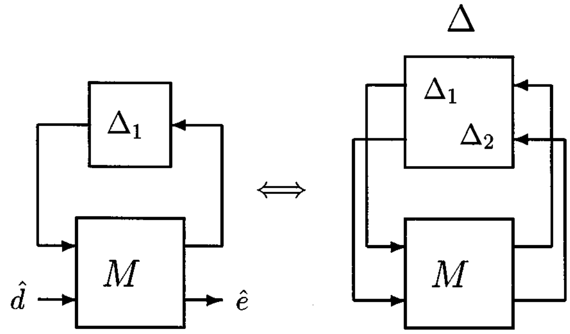

Figure 1. Linear fractional transformations and the M–Δ block structures

where $\mathscr{R}$ is the set of real numbers, $\mathscr{C}$ is the set of complex numbers, $\mathscr{C}^{r \times r}$ is the set of complex $r \times r$ matrices, and I_r is the identity matrix of rank r. Then $\mu_\Delta(M)$ (the structured singular value with respect to the uncertainty structure $\mathbf{\Delta}$) is defined by

$$\mu_\Delta(M) = \begin{cases} 0 \text{ if there does not exist } \Delta \in \mathbf{\Delta} \text{ such that } \det(I - M\Delta) = 0 \\ \left[\min\limits_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) \,|\, \det(I - M\Delta) = 0\}\right]^{-1} \quad \text{otherwise} \end{cases} \tag{4}$$

The following result has proven useful for relating various kinds of robustness problems.[1, 31]

Theorem 2.2. (*Main loop theorem*)

Consider the block diagrams in Figure 1, where M is a complex matrix and Δ_1 has block structure as shown in (3). The following equivalence holds:

$$\bar{\sigma}(F_u(M, \Delta_1)) < 1, \ \forall \bar{\sigma}(\Delta_1) \leqslant 1 \Leftrightarrow \mu_\Delta(M) < 1 \tag{5}$$

where $\mathbf{\Delta} = \mathrm{diag}\{\Delta_1, \Delta_2\}$, and Δ_2 is a full complex square matrix with dimension equal to the number of outputs (in $\hat{e}$).

Upper bound of μ

Define two subsets of $\mathscr{C}^{n \times n}$

$$\mathscr{D} = \{\mathrm{diag}[D_1, \ldots, D_m, d_{m+1} I_{r_{m+1}}, \ldots, d_l I_{r_l}]\colon 0 < D_i = D_i^* \in \mathscr{C}^{r_i \times r_i}, 0 < d_i \in \mathscr{R}\}; \text{ and} \tag{6}$$

$$\mathscr{G} = \{\mathrm{diag}[G_1, \ldots, G_k, O_{r_{k+1}}, \ldots, O_{r_l}]\colon G_i = G_i^* \in \mathscr{C}^{r_i \times r_i}\} \tag{7}$$

where O_r is the $r \times r$ zero matrix, and D^* is the complex conjugate transpose of D. Then

$$\mu_\Delta(M) \leqslant \mu_\Delta^{ub}(M) \equiv \sqrt{\max\left\{0, \inf_{\substack{D \in \mathscr{D} \\ G \in \mathscr{G}}} \bar{\lambda}[\tilde{M}^* \tilde{M} + j(G\tilde{M} - \tilde{M}^* G)]\right\}} \tag{8}$$

where $\tilde{M} \equiv DMD^{-1}$, and $\bar{\lambda}(A)$ is the maximum eigenvalue of A (this result is from Fan, Tits and Doyle[32]).

The computation of the upper bound in (8) can be formulated in terms of a linear matrix inequality, whose solution can be calculated in polynomial time using off-the-shelf software.[33–35] Although a gap may exist between the upper bound and μ, the gap is usually small for complex Δ.[1] The gap may be larger when some of the perturbations are real or repeated. The popular use of the upper bound is motivated by the fact that μ computation is NP-hard (see Reference 36 and the references therein).

3. STABILITY WITH MIXED TIME-VARYING PARAMETERS

Consider the equivalent block diagrams in Figure 2, where the discrete-time nominal transfer function $M(z) = C(zI - A)^{-1}B + D$, and

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{9}$$

Provided that the parameters are bounded, weights can always be chosen so that the parameters are norm bounded by one. Define the set of discrete-time norm-bounded time-varying parameters with the structure of $\mathbf{\Delta}$ (defined in (3)):

$$\mathbf{\Delta}^k \equiv \{\Delta(k) \in \mathbf{\Delta},\ \bar{\sigma}(\Delta(k)) \leqslant 1,\ \forall\, k \geqslant 0\} \tag{10}$$

Note that the parameters in $\mathbf{\Delta}^k$ are allowed to vary arbitrarily fast with time, as long as the unity norm bound is satisfied at each instance in time.

The discrete-time uncertain system $x_{k+1} = F_l(N, \Delta)x_k$ is referred to as being *exponentially stable* if there exist fixed scalars α and $\beta < 1$ such that $\| x_k \| \leqslant \alpha\beta^k \| x_0 \|$ for all initial conditions x_0, all $\Delta \in \mathbf{\Delta}^k$, and all k. The following theorem provides a sufficient condition for exponential stability of the discrete-time interconnected system.

Theorem 3.1. (*Discrete-time stability with mixed time-varying parameters*)

The discrete-time uncertain system $x_{k+1} = F_l(N, \Delta)x_k$ is exponentially stable if $\mu^{ub}_{\tilde{\mathbf{\Delta}}}(N) < 1$, where $\tilde{\Delta} = \begin{bmatrix} \delta^c I_n & \\ & \Delta \end{bmatrix}$, $\delta^c \in \mathscr{C}$, and $\Delta \in \mathbf{\Delta}$.

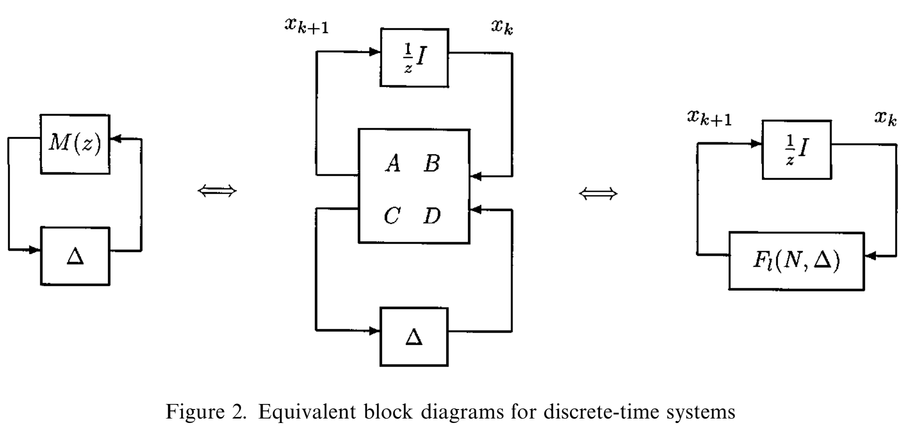

Figure 2. Equivalent block diagrams for discrete-time systems

The stability condition can be formulated in terms of linear matrix inequalities, whose solution can be computed in polynomial time using off-the-shelf software.[33–35] The condition is intuitive, in that the structure of the perturbation matrix in the upper bound to the μ problem is the same as if $1/z$ was treated as a complex parameter δ_c (see middle diagram in Figure 2). We will now consider an example which shows a substantial reduction in conservatism when taking into account the property that time-varying parameters are typically real.

4. NUMERICAL EXAMPLE

Consider the discrete-time 4×4 closed-loop system (N) given by the state space matrices in Appendix B (this example is purely mathematical). The eigenvalues of A are $\{-0{\cdot}1407, 0{\cdot}3961, 0{\cdot}3349, 0{\cdot}1725\}$, which all have magnitude less than one, implying that $M(z)$ is nominally stable. The perturbation $\boldsymbol{\Delta}^k$ consists of a real time-varying parameter repeated four times. If the parameter is treated as being complex, then the computed stability margin is

$$\mu_{\Delta}^{ub}(N) = 3{\cdot}57 > 1 \tag{11}$$

which does not imply the stability of the closed-loop system. If we take the real nature of the parameter into account, then the stability margin is

$$\mu_{\Delta}^{ub}(N) = 0{\cdot}98 < 1 \tag{12}$$

and exponential stability is guaranteed.

5. REVELANCE TO GAIN SCHEDULING

Theorem 3.1 is equivalent to the scaled small gain condition when the perturbation matrix $\boldsymbol{\Delta}$ is complex.[1] Although these results can substantially reduce the conservatism by taking into account the real nature of the time-varying parameters, the following lemma (which follows from results in Reference 31) shows that there is no reduction in conservatism when all the subblocks of Δ are independent scalars.

Lemma 5.1

Theorem 3.1 is no less conservative than the optimally scaled small gain theorem when all the subblocks of Δ are independent scalars.

The example in Section 4 showed that the conservatism can be reduced when the parameters are repeated scalar. Many systems of practical interest have repeated time-varying parameters. One example occurs in the linear fractional transformation approach to gain scheduling.[3] Both the plant and the gain-scheduling controller are treated as LFTs of a linear time invariant system and the same time-varying parameters of the plant (which are assumed to be measured or estimated, see Reference 3 for details). Because both the controller and the plant depend on the parameters, Theorem 3.1 can be applied to analyse the global stability for these systems with less conservatism than provided by the scaled small gain theorem.

6. CONCLUSIONS

Computable conditions were derived that can be substantially less conservative than the scaled small gain theorem for gain scheduled and other multivariable systems with repeated time-varying parameters. The proof is a direct generalization of a proof in Reference 1.

APPENDIX A. PROOF OF THEOREM 3.1

The proof is a direct generalization of arguments given in Chapter 6 of Andy Packard's thesis.[1] Consider the rightmost of the equivalent block diagrams in Figure 3. The system is described by the difference equation $x_{k+1} = F_l(N, \Delta)x_k$. A sufficient condition for stability under norm-bounded time-varying perturbations is that there exists an invertible $T \in \mathscr{C}^{n\times n}$ such that

$$\max_{\substack{\Delta\in\mathbf{\Delta}\\ \bar\sigma(\Delta)\leqslant 1}} \bar\sigma(TF_l(N,\Delta)T^{-1}) = \beta < 1 \tag{13}$$

since in this case the norm of x_k obeys $\| x_k \| \leqslant \kappa(T)\beta^k \| x_0 \|$, where $\kappa(T)$ denotes the condition number of T. The main loop Theorem 2.2 implies

$$\max_{\substack{\Delta\in\mathbf{\Delta}\\ \bar\sigma(\Delta)\leqslant 1}} \bar\sigma(TF_l(N,\Delta)T^{-1}) < 1 \Leftrightarrow \mu_{\hat\Delta}\left[\begin{pmatrix} T & \\ & I\end{pmatrix} N \begin{pmatrix} T^{-1} & \\ & I\end{pmatrix}\right] < 1 \tag{14}$$

where $\hat\Delta = [^{\Delta_1}\ {}_{\Delta}]$, $\Delta_1 \in \mathscr{C}^{n\times n}$, Δ_1 full block, and $\Delta \in \mathbf{\Delta}$. Combining (15) and (16) gives

$$\text{stability} \Leftarrow \inf_{\substack{T\in\mathscr{C}^{n\times n}\\ T\ \text{full}}} \mu_{\hat\Delta}\left[\begin{pmatrix} T & \\ & I\end{pmatrix} N \begin{pmatrix} T^{-1} & \\ & I\end{pmatrix}\right] < 1 \tag{15}$$

To arrive at a computable condition, μ is replaced with its upper bound (8) to give

$$\text{stability} \Leftarrow \max\left\{0, \sqrt{\inf_{\substack{T\in\mathscr{C}^{n\times n}\\ T\ \text{full}}}\ \inf_{\substack{D\in\mathscr{D}\\ G\in\mathscr{G}}} \bar\lambda[\tilde N^*\tilde N + j(G\tilde N - \tilde N^* G)]}\right\} < 1 \tag{16}$$

where

$$\tilde N \equiv D\begin{pmatrix} T & \\ & I\end{pmatrix} N \begin{pmatrix} T^{-1} & \\ & I\end{pmatrix} D^{-1}, D = \begin{pmatrix} d_1 I & \\ & D_2\end{pmatrix}, G = \begin{pmatrix} 0 & 0 \\ 0 & G_2\end{pmatrix} \tag{17}$$

$\bar\lambda(A)$ is the maximum eigenvalue of A, $d_1 \in \mathscr{R}$, $D_2 \in \mathscr{D}_2$, $G_2 \in \mathscr{G}_2$, and the sets $\mathscr{D}_2$ and $\mathscr{G}_2$ are specified by the structure of $\mathbf{\Delta}$. Absorbing d_1 into T and noticing that the structure of $\tilde\Delta$ is appropriate for the new 'D' and 'G' scalings gives the result. □

APPENDIX B. EXAMPLE

$$A = \begin{pmatrix} -1\cdot6662 & -3\cdot2066 & 0\cdot2522 & 4\cdot6348 \\ -3\cdot5907 & -6\cdot5803 & 0\cdot5290 & 9\cdot3770 \\ -10\cdot0332 & -20\cdot5300 & 1\cdot7744 & 27\cdot3046 \\ -2\cdot7552 & -4\cdot9830 & 0\cdot3936 & 7\cdot2349 \end{pmatrix} \tag{18}$$

$$B = \begin{pmatrix} -1\cdot0801 & -0\cdot3601 & -0\cdot7408 & 2\cdot0288 \\ -2\cdot4983 & -0\cdot4873 & -2\cdot0047 & 3\cdot8125 \\ -8\cdot2313 & -2\cdot7842 & -7\cdot3722 & 11\cdot1384 \\ -1\cdot9733 & -0\cdot4020 & -1\cdot5071 & 3\cdot1886 \end{pmatrix} \tag{19}$$

$$C = \begin{pmatrix} -6.0014 & -11.6816 & 1.0568 & 17.1462 \\ -9.0589 & -18.0075 & 1.8934 & 24.7326 \\ -5.3812 & -10.7738 & 1.1081 & 14.9160 \\ -8.5404 & -16.9704 & 1.7568 & 23.4685 \end{pmatrix} \tag{20}$$

$$D = \begin{pmatrix} -4.5765 & -1.1854 & -3.6748 & 7.0926 \\ -7.2120 & -2.2428 & -6.2854 & 10.1094 \\ -4.2841 & -1.3153 & -3.6850 & 6.1183 \\ -6.7764 & -2.0757 & -5.8558 & 9.6079 \end{pmatrix} \tag{21}$$

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REFERENCES

1. Packard, A. K., What's new with μ: structured uncertainty in multivariable control', Ph.D. thesis, University of California, Berkeley, 1988.
2. Desoer, C. A. and M. Vidyasagar, *Feedback Systems: Input–Output Properties*, Academic Press, New York, 1975.
3. Packard, A. 'Gain scheduling via linear fractional transformations', *Systems and Control Letters*, **22**, 79–92 (1994).
4. Safonov, M. G., *Stability and Robustness of Multivariable Feedback Systems*, MIT Press, Cambridge, Massachusetts, 1982.
5. Doyle, J. C. and A. Packard, 'Uncertain multivariable systems from a state space perspective', *Proc. of the American Control Conference*, Minneapolis, MN, 1987, 2147–2152, IEEE, Piscataway, NJ.
6. Megretski, A., 'Necessary and sufficient conditions of stability: A multiloop generalization of the circle criterion', *IEEE Trans. Automat. Control*, **AC-38**, 753–756 (1993).
7. Megretski, A. and S. Triel, 'Power distribution inequalities in optimization and the robustness of uncertain systems', *J. of Math. Systems. Estimation, and Control*, **3**, 301–319 (1993).
8. Shamma, J. S. 'Necessity of the small-gain theorem for time-varying and nonlinear systems', *IEEE Trans. Automat. Control*, **AC-36**, 1138–1147 (1991).
9. Shamma, J. S., 'Robust stability analysis of time-varying systems using time-varying quadratic forms', *Systems and Control Letters*, **24**, 13–17 (1995).
10. Gapski, P. B. and J. C. Geromel, 'A convex approach to the absolute stability problem', *IEEE Trans. Automat. Control*, **AC-39**, 1929–1932 (1994).
11. Narendra, K. S. and J. H. Taylor *Frequency Domain Criteria for Absolute Stability*, Academic Press, New York, 1973.
12. Safonov, M. G. and G. Wyetzner, 'Computer-aided stability analysis renders Popov criterion obsolete', *IEEE Trans. Automat. Control*, **AC-32**, 1128–1131 (1987).
13. Sundareshan, M. K. and M. A. L. Thathacher, 'L_2-stability of linear time-varying systems—conditions involving noncausal multipliers', *IEEE Trans. Automat. Control*, **AC-17**, 504–510 (1972).
14. Yakubovich, V. A., 'The matrix inequality in the theory of the stability of nonlinear control systems—I. The absolute stability of forced vibrations', *Automation and Remote Control*, **25**, 905–917 (1964).
15. Yakubovich, V. A., 'The method of matrix inequalities in the stability theory of nonlinear control systems—II. The absolute stability in a class of nonlinearities with a condition on the derivative', *Automation and Remote Control*, **26**, 577–592 (1965).
16. Yakubovich, V. A., 'The method of matrix inequalities in the theory of stability of nonlinear controller systems—III. Absolute stability of systems with hysteresis nonlinearities', *Automation and Remote Control*, **26**, 753–763 (1965).
17. Zames, G. 'On the input–output stability of time varying nonlinear feedback systems—part I: conditions derived using concepts of loop gain, conicity, and positivity', *IEEE Trans. Automat. Control.*, **AC-11**, 228–238 (1966).

18. Zames, G., 'On the input–output stability of time varying nonlinear feedback systems—part II: conditions involving circles in the frequency plane and sector nonlinearities', *IEEE Trans. Automat. Control*, **AC-11**, 465–476 (1966).
19. Zames, G. and P. L. Falb, 'Stability conditions for systems with monotone and slope-restricted nonlinearities, *SIAM J. on Control and Optim.*, **4**, 89–108 (1968).
20. Boyd, S. and Q. P. Yang, 'Structured and simultaneous Lyapunov functions for systems stability problems', *Int. J. Control*, **49**, 2215–2240 (1989).
21. Becker, G., A. Packard, D. Philbrick and G. J. Balas, 'Control of parametrically-dependent linear systems: a single quadratic Lyapunov approach', *Proc. of the American Control Conference*, San Francisco, California, 1993, pp. 2795–2799, IEEE, Piscataway, NJ.
22. Jonsson, U. and A. Rantzer, 'Systems with uncertain parameters—time-variations with bounded derivatives', *Proc. of the Conference on Decision and Control*, Lake Buena Vista, Florida, 1994, pp. 3074–3079, IEEE, Piscataway, NJ.
23. Megretski, A., 'Frequency domain criteria of robust stability for slowly time varying systems', *IEEE Trans. Automat. Control*, **AC-40**, 153–155 (1995).
24. Balakrishnan, A. V., Y. Huang, A. Packard and J. Doyle, 'Linear matrix inequalities in analysis with multipliers', *Proc. of the American Control Conference*, Baltimore, Maryland, 1994, pp. 1228–1232, IEEE, Piscataway, NJ.
25. Freedman, M. and G. Zames, 'Logarithmic variation criteria for the stability of systems with time-varying gains', *SIAM J. of Control*, **6**, 487–507 (1968).
26. Barmish, B. R., 'Necessary and sufficient conditions for quadratic stabilizability of an uncertain system', *J. Optim. Theory and Applications*, **46**, 399–408 (1985).
27. Packard, A. and J. Doyle, 'Quadratic stability with real and complex perturbations', *IEEE Trans. Automat. Control*, **AC-35**, 198–201 (1990).
28. Doyle, J. C., 'Analysis of feedback systems with structured uncertainties', *IEE Proc. Part D*, **129**, 242–250 (1982).
29. Safonov, M. G., 'Stability margins of diagonally perturbed multivariable feedback systems', *IEE Proc. Part D*, **129**, 251–256 (1982).
30. Lee, J. H., R. D. Braatz, M. Morari and A. Packard, 'Screening tools for robust control structure selection', *Automatica*, **31**, 229–235 (1995).
31. Young, P. M., 'Robustness with parametric and dynamic uncertainties', Ph.D. thesis, California Institute of Technology, Pasadena, 1993.
32. Fan, M. K. H., A. L. Tits and J. C. Doyle, 'Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics', *IEEE Trans. Automat. Control*, **AC-36**, 25–38 (1991).
33. Balas, G. J., J. C. Doyle, K. Glover, A. K. Packard and R. S. R. Smith, *μ-Analysis and Synthesis Toolbox* (*μ-Tools*): *Matlab Functions for the Analysis and Design of Robust Control Systems*, The MathWorks, Inc., Natick, Massachusetts, 1992.
34. Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, volume 15 of Studies in Applied Mathematics, SIAM, Philadelphia, PA, 1994.
35. Young, P.M., M. P. Newlin and J. C. Doyle, 'μ analysis with real parametric uncertainty', *Proc. of the Conference on Decision and Control*, Brighton, U.K., 1991, pp. 1251–1256, IEEE, Piscataway, NJ.
36. Braatz, R. D., P. M. Young, J. C. Doyle and M. Morari, 'Computational complexity of μ calculation', *IEEE Trans. Automat. Control*, **AC-39**, 1000–1002 (1994).