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ROBUSTNESS ANALYSIS FOR SYSTEMS WITH ELLIPSOIDAL UNCERTAINTY

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SUMMARY

This note derives an explicit expression for computing the robustness margin for affine systems whose real and complex coefficients are related by an ellipsoidal constraint. The expression, which is an application of a result by Chen, Fan, and Nett for rank-one generalized structured singular-value problems, extends and unifies previous results on robustness margin computation for systems with ellipsoidal uncertainty. © 1998 John Wiley & Sons, Ltd.

Key words: multiparameter margin; robust stability; real/mixed μ

INTRODUCTION

The generalized structured singular value (μ) introduced by Chen *et al.*[1,2] unified and extended the well-known Kharitonov-like stability conditions. The Kharitonov-like stability conditions correspond to a generalized μ problem of rank one, which can be written as a convex optimization problem that is readily computable, often as an explicit analytical expression. These simple computations are in sharp contrast to the general robustness margin computation problem, which is NP-hard.[3]

As an application of their main result,[1] Chen *et al.* derived explicit conditions for the robustness margins of interval and diamond polynomials whose coefficients are perturbed in an affine fashion.[2] However, ellipsoidal uncertainty descriptions are more naturally obtained from parameter identification procedures.[4] Previously derived conditions for computing robustness margins for systems with ellipsoidal uncertainties apply only to pure real perturbations,[5–8] whereas complex perturbations are needed to represent unmodelled dynamics.

Here we use the generalized structured singular value to derive an analytical expression for the robustness margin for affine systems whose coefficients are related by an ellipsoidal constraint. The condition unifies previous results on the robustness of systems with ellipsoidal uncertainties, and extends these results by addressing both real and complex perturbations.

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BACKGROUND

Scalars and vectors will be represented by lower case and matrices by upper case Roman or Greek. The set of real numbers is $\mathbb{R}$; the set of complex numbers is $\mathbb{C}$; the set of $n \times m$ complex matrices in $\mathbb{C}^{m \times n}$, and I_k is the $k \times k$ identity matrix. A^{T} is the transpose of A, while A^{H} is the complex conjugate transpose of A. Define $\| x \|_2$ as the vector two-norm and $\| A \|_2$ as the induced matrix 2-norm.

As is standard in the structured singular-value framework for robustness analysis, perturbations are collected into a block-diagonal matrix Δ, whose diagonal blocks can be real scalar times identity, complex scalar times identity, or complex full block

$$\Delta := \{\mathrm{diag}(\delta_1^{\mathrm{r}} I_{k_1}, \ldots, \delta_{m_{\mathrm{r}}}^{\mathrm{r}} I_{k_{m_{\mathrm{r}}}}, \delta_1^{\mathrm{c}} I_{k_{m_{\mathrm{r}}+1}}, \ldots, \delta_{m_{\mathrm{c}}}^{\mathrm{c}} I_{k_{m_{\mathrm{r}}+m_{\mathrm{c}}}}, \Delta_1^{\mathrm{C}}, \ldots, \Delta_{m_{\mathrm{C}}}^{\mathrm{C}}): \\ \delta_i^{\mathrm{r}} \in \mathbb{R}; \delta_i^{\mathrm{c}} \in \mathbb{C}; \Delta_i^{\mathrm{c}} \in \mathbb{C}^{k_{m_{\mathrm{r}}+m_{\mathrm{c}}+i} \times k_{m_{\mathrm{r}}+m_{\mathrm{c}}+i}}\} \tag{1}$$

For the block-diagonal matrix $\Delta = \mathrm{diag}(\Delta_i)$, define the matrix norm appropriate for studying systems with ellipsoidal uncertainty descriptions:

$$\| \Delta \| = \left(\sum_{i=1}^{m} \| \Delta_i \|_2^2 \right)^{1/2} \tag{2}$$

The appropriate structured singular value is defined below.

Definition 1

The structured singular value $\mu_{\mathrm{e}}(M)$ of matrix $M \in \mathbb{C}^{n \times n}$ for ellipsoidal uncertainty descriptions is defined to be zero if there is no $\Delta \in \mathbf{\Delta}$ such that $\det(I - M\Delta) = 0$, and otherwise

$$\mu_{\mathrm{e}}(M) = \left(\min_{\Delta \in \mathbf{\Delta}} \{ \| \Delta \| : \det(I - M\Delta) = 0 \} \right)^{-1} \tag{3}$$

The following result provides an analytical expression for $\mu_{\mathrm{e}}(M)$ for M of rank one.

Lemma 1

Suppose that $M = ba^{\mathrm{H}}$, where $a = [a_1^{\mathrm{H}}, \ldots, a_m^{\mathrm{H}}]^{\mathrm{H}}$ and $b = [b_1^{\mathrm{H}}, \ldots, b_m^{\mathrm{H}}]^{\mathrm{H}}$ are partitioned compatibly with $\mathbf{\Delta}$. Define

$$\phi_i = a_i^{\mathrm{H}} b_i, \qquad \eta_i = \| a_i \|_2 \| b_i \|_2 \tag{4}$$

$$\hat{a}^{\mathrm{T}} = [\mathrm{Re}\,\phi_1 \cdots \mathrm{Re}\,\phi_{m_{\mathrm{r}}} \,|\phi_{m_{\mathrm{r}}+1}|\, 0 \cdots |\phi_{m_{\mathrm{r}}+m_{\mathrm{c}}}|\, 0 \,|\eta_{m_{\mathrm{r}}+m_{\mathrm{c}}+1}|\, 0 \cdots |\eta_m|\, 0] \tag{5}$$

$$\hat{b}^{\mathrm{T}} = [\mathrm{Im}\,\phi_1 \cdots \mathrm{Im}\,\phi_{m_{\mathrm{r}}}\, 0 \,|\phi_{m_{\mathrm{r}}+1}| \cdots 0 \,|\phi_{m_{\mathrm{r}}+m_{\mathrm{c}}}|\, 0 \,|\eta_{m_{\mathrm{r}}+m_{\mathrm{c}}+1}| \cdots 0 \,|\eta_m|\,] \tag{6}$$

Then

$$\mu_{\mathrm{e}}(M) = \begin{cases} \| \hat{a} \|_2 & \text{if } \| \hat{b} \| = 0 \\ \sqrt{\hat{a}^{\mathrm{T}}\hat{a} - \dfrac{(\hat{b}^{\mathrm{T}}\hat{a})^2}{\hat{b}^{\mathrm{T}}\hat{b}}} & \text{if } \| \hat{b} \| \neq 0 \end{cases} \tag{7}$$

Proof. It follows from Chen *et al.*[1] for the selection of norm on Δ that $\mu_{\mathrm{e}}(M) = \inf_{x \in \mathbb{R}} \| \hat{a} + x\hat{b} \|_2$. Elementary calculus gives the result. □

Lemma 1 provides a general analytical expression for the robustness margin for systems whose characteristic polynomial is an affine function of parameters which satisfy an ellipsoidal constraint, as such systems are described by $\mu_e(M)$ problems whose M matrix is rank one.[1,2] The robustness margin is equal to the maximum of (7) over the stability boundary. The expression for the robustness margin applies for discrete-time and continuous-time systems with real and complex uncertainties, where the stability boundary can be defined for a general open set in the complex plane (this allows additional criteria to be met, such as sufficient damping or speed of response).

Previously derived conditions for ellipsoidal uncertainties apply only to pure real perturbations.[5–8] Latchman *et al.*'s result holds for only a specialized type of affine perturbations.[7] In the following, we relate Lemma 1 to these prior results.

FIR SYSTEMS WITH AFFINE ELLIPSOIDAL PERTURBATIONS

Consider a system described as a finite impulse response (FIR)

$$H(z) = \sum_{k=1}^{q} h_k z^{-k} \tag{8}$$

with q uncertain parameters defined by $h = [h_1, h_2, \ldots, h_q]^{\mathrm{T}} = h_0 + \delta h$. Then the uncertain ellipsoidal parametric uncertainty description is described by $\delta h = h - h_0 \in \mathbb{D}_h$, where

$$\mathbb{D}_h = \{\delta h \,|\, \delta h^{\mathrm{H}} Q_h^{-1} \delta h \leqslant 1;\, Q_h = Q_h^{\mathrm{H}} > 0;\, \delta h_i \in \mathbb{R} \text{ for } i \in I;\, \delta h_j \in \mathbb{C} \text{ for } j \in J\} \tag{9}$$

where I and J are disjoint sets of integers whose union is $\{1, \ldots, q\}$. Writing the unity negative feedback system in standard $M - \Delta$ form,

$$M = \left(\frac{-1}{1 + [z^{-1} \cdots z^{-q}]\, h_0}\right) [1 \cdots 1]^{\mathrm{T}} [z^{-1} \cdots z^{-q}]\, Q_h^{1/2} = b a^{\mathrm{H}} \tag{10}$$

where $b = (-1/(1 + [z^{-1} \cdots z^{-q}] h_0))\ [1 \cdots 1]^{\mathrm{T}}$, and $a^{\mathrm{H}} = [z^{-1} \cdots z^{-q}] Q_h^{1/2}$. Equation (7) provides an explicit analytical expression for the robustness margin of this system under unity negative feedback (after rearranging the elements of a so that those corresponding to $i \in I$ appear first). This expression is simplified when all the perturbations are real.

Lemma 2

Assume nominal closed-loop stability. Then the unity negative feedback system with open-loop transfer function (8) and real uncertain parameters is stable for all $\delta h \in \mathbb{D}_h$ if and only if

$$\mu_e(\omega) < 1, \quad \forall \omega \tag{11}$$

where

$$\mu_e(\omega) = \begin{cases} \sqrt{\| \mathrm{Re}\, \phi \|_2^2 - \dfrac{((\mathrm{Im}\, \phi)^{\mathrm{T}} (\mathrm{Re}\, \phi))^2}{\| \mathrm{Im}\, \phi \|_2^2}}, & \omega \in (0, \pi) \\ \| \mathrm{Re}\, \phi \|_2, & \omega = 0, \pi \end{cases} \tag{12}$$

$$\phi^{\mathrm{T}} = \left(\frac{-1}{1 + [\mathrm{e}^{-\mathrm{j}\omega} \mathrm{e}^{-\mathrm{j}2\omega} \cdots \mathrm{e}^{-\mathrm{j}q\omega}]\, h_0}\right) [\mathrm{e}^{-\mathrm{j}\omega} \mathrm{e}^{-\mathrm{j}2\omega} \cdots \mathrm{e}^{-\mathrm{j}q\omega}]\, Q_h^{1/2} \tag{13}$$

Basic but somewhat tedious algebra can be used to show that Lemma 2 is equivalent to Theorem 3 of Latchman *et al.*[7]

SYSTEMS WITH STRUCTURED AFFINE ELLIPSOIDAL PERTURBATIONS

Consider a monic polynomial of a complex variable s. Assume that its coefficients are affine functions of a vector k whose m entries are independent real or complex parameters. This polynomial can be written as

$$p(s) = s^n + a_1(k)s^{n-1} + \cdots + a_n(k) = s^n + [s^{n-1}s^{n-2} \cdots 1]\,(Fk + g) \tag{14}$$

where $F \in \mathbb{C}^{n \times m}$ and $g \in \mathbb{C}^n$. This form of perturbations is much more general than those treated in (8). The polynomial is stable if all of its roots are in an open set $\mathscr{S}$. This definition allows us to consider discrete time, continuous time, and more general notions of stability (for example, stability plus a required level of damping). Assume $p(s)$ is stable for $k = 0$. Set $k = \Delta[1 \cdots 1]^{\mathrm{T}}$ where Δ has nonrepeated real or complex scalar blocks. Then

$$s^n + [s^{n-1}\, s^{n-2} \cdots 1](F\Delta[1 \cdots 1]^{\mathrm{T}} + g) = 0 \tag{15}$$

$$\Leftrightarrow 1 + \frac{[s^{n-1}\, s^{n-2} \cdots 1]\, F\Delta\, [1 \cdots 1]^{\mathrm{T}}}{s^n + [s^{n-1}\, s^{n-2} \cdots 1]\, g} = 0 \tag{16}$$

$$\Leftrightarrow \left(I + \frac{1}{s^n + [s^{n-1}\, s^{n-2} \cdots 1]\, g} [1 \cdots 1]^{\mathrm{T}}\, [s^{n-1}\, s^{n-2} \cdots 1]\, F\Delta \right) = 0 \tag{17}$$

This is of the form $\det(I - ba^{\mathrm{H}}\Delta) = 0$, with $b = [1 \cdots 1]^{\mathrm{T}}$ and $a^{\mathrm{H}} = w^{\mathrm{T}}$, where

$$w := \frac{-F^{\mathrm{T}}[s^{n-1}s^{n-2} \cdots 1]^{\mathrm{T}}}{s^n + [s^{n-1}\, s^{n-2} \cdots 1]\, g} \tag{18}$$

and s is evaluated on the stability boundary. Equation (7) provides an explicit analytical expression for the robustness margin of this system (after rearranging the elements of a so that those corresponding to $i \in I$ appear first). For the case where all the perturbations are real, the expression reduces to that reported by several authors:[5,6,8]

$$\mu_e = \begin{cases} \dfrac{\sqrt{\|\operatorname{Im} w\|_2^2\, \|\operatorname{Re} w\|_2^2 - ((\operatorname{Im} w)^{\mathrm{T}}(\operatorname{Re} w))^2}}{\|\operatorname{Im} w\|_2}, & \text{for } \|\operatorname{Im} w\|_2 \neq 0 \\ \|\operatorname{Re} w\|_2, & \text{for } \|\operatorname{Im} w\|_2 = 0 \end{cases} \tag{19}$$

CONCLUSIONS

An analytical expression is derived for computing the robustness margin for affine systems whose real and complex coefficients are related by an ellipsoidal constraint. The expression is an application of a result of Chen, Fan, and Nett on computing robustness margins for rank-one generalized structured singular value problems. The expression extends and unifies previous results on computing robustness margins to ellipsoidal uncertainty.

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