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The average-case identifiability and controllability of large scale systems

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Abstract

Needs for increased product quality, reduced pollution, and reduced energy and material consumption are driving enhanced process integration. This increases the number of manipulated and measured variables required by the control system to achieve its objectives. This paper addresses the question of whether processes tend to become increasingly more difficult to identify and control as the process dimension increases. Tools and results of multivariable statistics are used to show that, under a variety of assumed distributions on the elements, square processes of higher dimension tend to be more difficult to identify and control, whereas the expected controllability and identifiability of nonsquare processes depends on the relative numbers of measured and manipulated variables. These results suggest that the procedure of simplifying the control problem so that only a square process is considered is a poor practice for large scale systems.

Keywords: Process control; Large scale systems; Controllability analysis; Multivariate statistics; Process identification

1. Introduction

Since control is being applied to industrial processes of progressively higher dimension, it is important to determine if processes tend to become more difficult to identify and control as the process dimension increases. There has been some controversy regarding this question. While some academic control engineers have claimed that the identification and control of high dimension processes is no more difficult than for low dimension, many industrial control engineers have strongly disagreed (see, for example, heated discussions at the International Conference on Chemical Process Control [1]). Although some evidence has been provided that paper machines of larger dimension tend to have plant matrices which are more poorly conditioned [2], no proof was available that this trend holds for general processes.

This debate is not whether the control of large scale processes requires more computation, as all engineers agree on this point, which is being addressed [3–9]. Rather, the debate concerns whether the *inherent identifiability* and *controllability* of processes tend to become more difficult as the process dimension increases.

Here tools and results from multivariate statistics are used to show that, under a variety of assumed distributions on the elements, the likelihood that a square process is difficult to identify and control increases with process dimension. For various well-characterized sets of square transfer function matrices, it is shown that: (1) the average condition number increases with the process dimension, and (2) the average minimum singular value decreases with the process dimension. These trends suggest that large scale square processes tend to be more difficult to identify and control than smaller scale processes. It is shown this trend does not hold for nonsquare processes. This suggests that the common procedure of selecting only square transfer functions to use for control purposes, as has been popular in the literature and in industrial practice [10,11], is a poor practice for large scale processes. Similarly, procedures of "squaring down" the process before applying control techniques [12–16] are also inappropriate for large scale processes.

2. Results

Define $\mathcal{R}$ as the field of real scalars, $\mathcal{C}$ as the field of complex scalars, $\mathcal{R}^{m\times n}$ as the set of real matrices with m rows and n columns, and $\mathcal{C}^{m\times n}$ defined similarly. The minimum singular value of a matrix $\mathbf{P}$ is represented as

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$\underline{\sigma}(\mathbf{P})$; the Euclidean condition number of $\mathbf{P}$ is represented as $\kappa(\mathbf{P})$. A random variable is said to be *standard* if it has zero mean and unit variance. The expected value of a function $f(\mathbf{P})$ is represented by $\mathrm{E}\{f(\mathbf{P})\}$.

In physical terms, a real $\mathbf{P}$ corresponds to a steady-state gain matrix (that is, the mapping between manipulated and measured variables), while a complex $\mathbf{P}$ corresponds to the multivariable transfer function evaluated at some frequency of interest, usually some intermediate frequency. It will be assumed throughout that $\mathbf{P}$ has been scaled in the standard manner [17]. The tools and results of multivariate statistics will be used to address the question of whether the identification and control problems are inherently more difficult for large scale processes. The difficulty associated with the identification and control of a process will be quantified using $\underline{\sigma}(\mathbf{P})$ and $\kappa(\mathbf{P})$, which are commonly used in the literature [18–29] as well as in practice [11]. It is generally accepted in the process control community that a process is difficult to identify and control when $\underline{\sigma}(\mathbf{P})$ is small or $\kappa(\mathbf{P})$ is large (our research group prefers to use $\underline{\sigma}(\mathbf{P})$, but $\kappa(\mathbf{P})$ is more commonly used by other process control engineers). In terms of process identification, $\underline{\sigma}(\mathbf{P})$ is the worst-case signal-to-noise ratio in any process input direction, while $\kappa(\mathbf{P})$ is the ratio of the best-case signal-to-noise ratio to the worst-case signal-to-noise ratio [22]. In terms of process control, $\underline{\sigma}(\mathbf{P})$ quantifies the ability of the manipulated variables to reject worst-case disturbances [30], while $\kappa(\mathbf{P})$ quantifies potential sensitivities to model uncertainties experienced by the control system [17].

To mathematically quantify the likelihood that a process is difficult to identify and control, a distribution must be assumed for the elements of the process transfer function. It is difficult to assess what the distribution of all plant transfer functions truly is, as process designs have a variety of constraints such as the satisfaction of material and energy balances, and the interconnections of process units, that suggests that the distribution may not be of a simple form (e.g. uniform or normal). It is reasonable to assume, however, that the expected value of each element of the transfer function is zero, since for any process transfer function, an equivalent transfer function can be defined which has the opposite sign in all of its elements by redefining the signs of the manipulated or measured variables. In other words, the arbitrariness of the sign of the elements implies that the expected value of the elements can be assumed to be zero.

Since the true distribution for the process elements is unknown, the next best strategy would be to derive trends that are true irrespective of the distribution. Although this can be done in some cases (see Lemma 4), in most cases the results will be a function of the distribution. In these cases our approach will be to derive results for two distributions: (i) the normal distribution, which approximates many distributions encountered in practice due to the central limit effect (for a more thorough discussion of this, see pages 43–46 of [31]), and (ii) the uniform distribution, which in many probabilistic matrix analysis problems can be shown to be the "worst-behaved" distribution [32,33]. For these distributions, we will show that the expected values of $\underline{\sigma}(\mathbf{P})$ and $\kappa(\mathbf{P})$ exhibit the same general behavior as the process dimension increases. In the Discussion section, significant literature evidence is provided which indicates that many other matrix properties are weakly dependent on the distribution of the elements, and an explanation for this phenomena is provided in terms of the central limit effect.

Our interest in the properties of large scale systems implies that our focus is on the case where the number of manipulated variables, the number of measured variables, or both numbers become large. The asymptotic results will depend on the relative numbers of manipulated and measured variables. The results are most conveniently separated into three classes:

1. the process transfer function is square;
2. the process transfer function is nonsquare, with a constant ratio of the number of measured variables m to the number of manipulated variables n; and,
3. the process transfer function is nonsquare, with either the number of manipulated variables or the number of measured variables fixed.

These classes will be treated in turn.

3. Expected values for square processes

The following result indicates that the average condition number is proportional to the process dimension, whereas the average minimum singular value has inverse proportionality.

Lemma 1 (expected value asymptotes: square normal case). *Consider a matrix $\mathbf{P} \in \mathcal{R}^{m\times m}$ where each element is an independent standard normal random variable. Then*

$$\lim_{m\to\infty} \mathrm{E}\{\ln\kappa(\mathbf{P})\} \approx \ln m + 1.537 \tag{1}$$

$$\lim_{m\to\infty} \mathrm{E}\left\{\ln\left(m\underline{\sigma}(\mathbf{P})\right)\right\} \approx -1.688 \tag{2}$$

Consider a matrix $\mathbf{P} \in \mathcal{C}^{m\times m}$ where the complex and real part of each element is an independent standard normal random variable. Then

$$\lim_{m\to\infty} \mathrm{E}\{\ln\kappa(\mathbf{P})\} \approx \ln m + 0.982 \tag{3}$$

$$\lim_{m\to\infty} \mathrm{E}\left\{\ln\left(m\underline{\sigma}(\mathbf{P})\right)\right\} \approx 0.116 \tag{4}$$

Proof. The first result is Theorem 6.1; the second result is Corollary 3.2; the third result is Theorem 6.2; and the fourth result is Corollary 3.4 of [34]. □

Lemma 1 indicates that a square process of large dimension is more likely to be difficult to identify and control than a square process of low dimension, whether at steady-state or at higher frequencies. The asymptotes in (1)–(4) are actually relatively accurate for small m as well. As a representative example, the exact values for $E\{\underline{\sigma}(\mathbf{P})\}$ for finite values of m are shown in Fig. 1. The convergence of $E\{\underline{\sigma}(\mathbf{P})\}$ to its asymptote as m increases is quite rapid. Lemma 2 provides another indication that the asymptotic relationship between the expected values and the process dimension is a reasonable approximation for all m.

Lemma 2 (upper and lower bounds). *Consider a matrix* $\mathbf{P} \in \mathcal{R}^{m\times m}$ *where each element is an independent standard normal random variable. Then*

$$\left(\frac{2}{3}-\varepsilon\right)\ln m \leqslant E\{\ \ln\kappa(\mathbf{P})\ \} \leqslant \left(\frac{5}{2}+\varepsilon\right)\ln m \tag{5}$$

where ε *tends to zero as* m *goes to infinity.*

Proof. The lower bound is shown in [35]. The upper bound is shown in [36]. □

The upper and lower bounds hold for all m, and both bounds imply a proportionality between the average condition number and the process dimension.

The general trends for square processes also hold for processes that are nearly square, that is, processes for which $n-m$ is equal to a fixed constant as the process dimension m increases. For example, $E\{\underline{\sigma}(\mathbf{P})\} = 2/m$ for $n = m+1$, while $E\{\underline{\sigma}(\mathbf{P})\} \approx 6/m$ for $n = m+3 \geqslant 10$ [37].

For uniform distributions, Fig. 2 indicates that the average condition number increases and the average minimum singular value decreases with process dimension m, although the dependence on m is not exactly the same as for normal distributions.

4. Expected values for nonsquare processes: fixed relative number of manipulated and measured variables

In contrast to the square case, for processes in which the number of manipulated variables is appreciably different than the number of measured variables, the average condition number approaches a constant and the

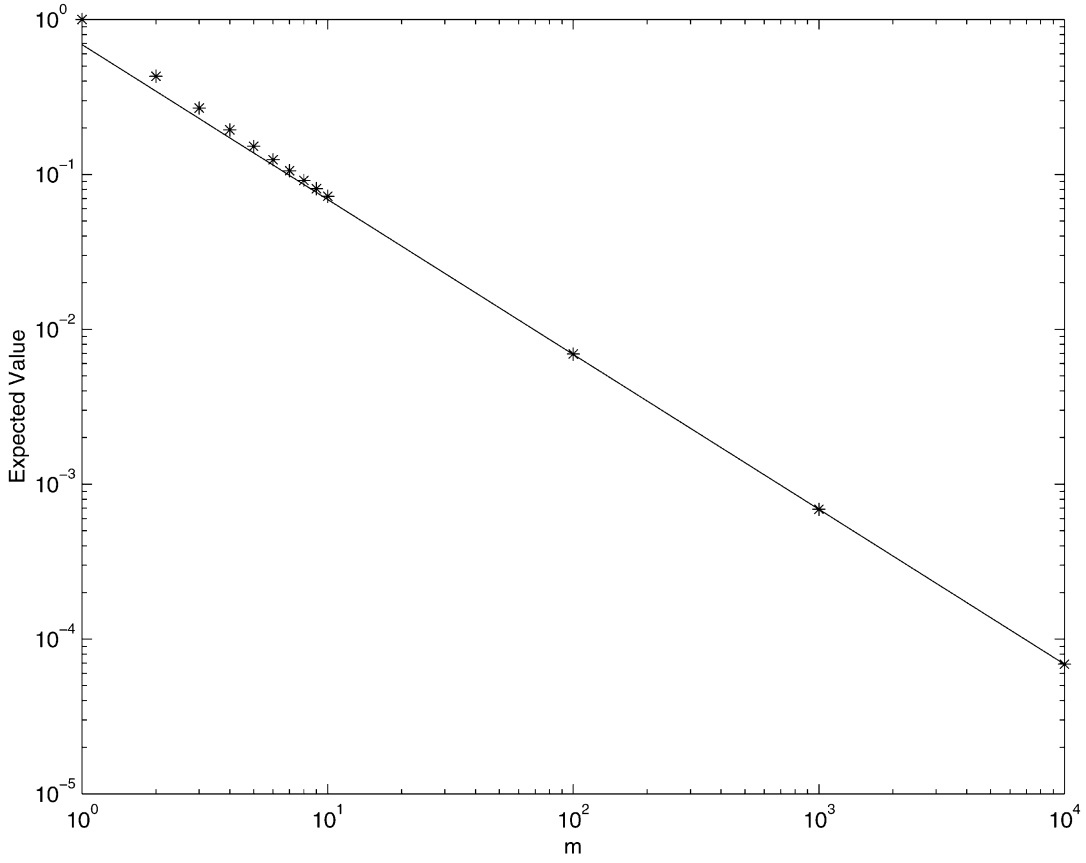

Fig. 1. Expected value of the minimum singular value as a function of m. Each element of $\mathbf{P} \in \mathcal{R}^{m\times m}$ is an independent standard normal random variable. Exact values indicated by asterisks were taken from Table 3 of [37]. The solid line is the asymptote for $m\to\infty$, which is $E\{\underline{\sigma}(\mathbf{P})\} \approx 0.6886/m$.

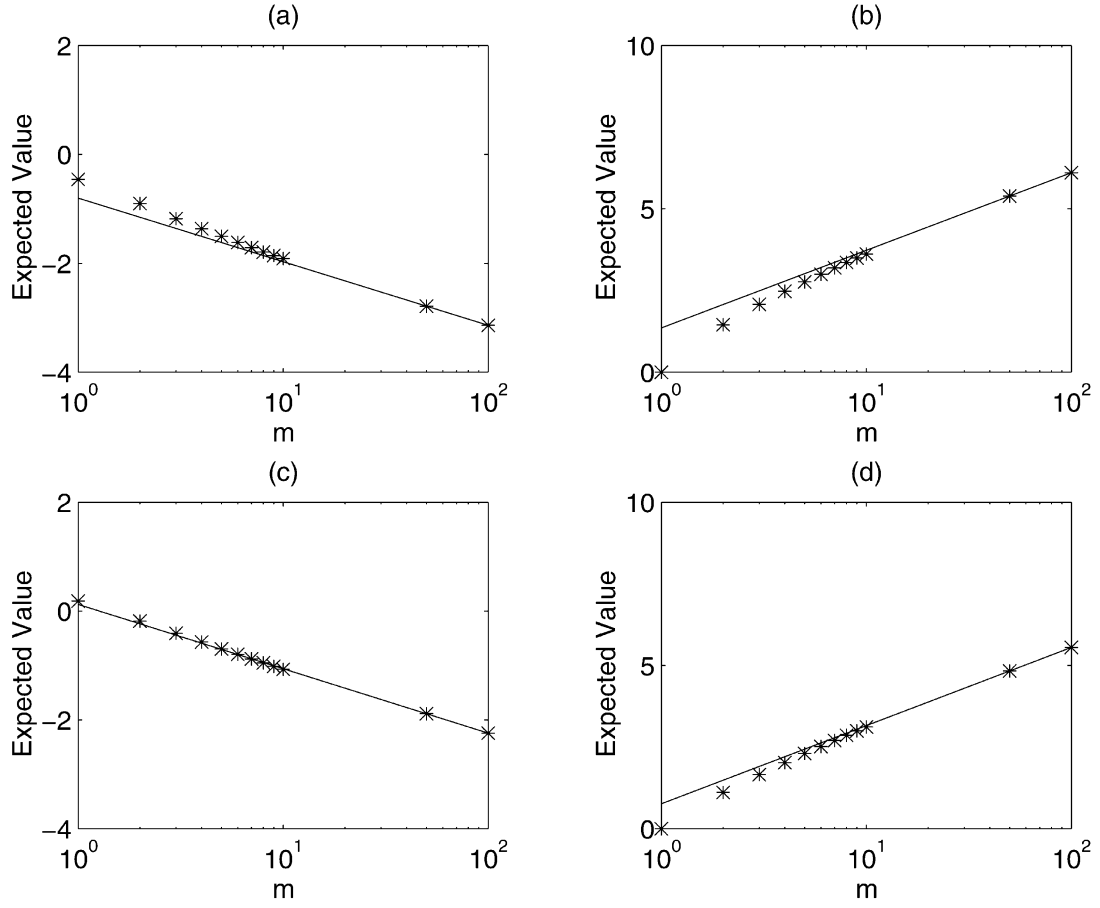

Fig. 2. Expected values of the minimum singular value (a, c) and condition number (b, d) as a function of m. Each element of $\mathbf{P} \in \mathcal{R}^{m\times m}$ in (a) and (b) is an independent standard uniform random variable. The real and imaginary part of each element of $\mathbf{P} \in \mathcal{C}^{m\times m}$ in (c) and (d) is an independent standard uniform random variable. Values indicated by asterisks were computed using Monte Carlo experiments (50,000 matrices). The solid lines are the estimated asymptotes as $m\to\infty$. For (a), $\mathrm{E}\{\ln \underline{\sigma}(\mathbf{P})\}\approx -0.51 \ln m - 0.80$. For (b), $\mathrm{E}\{\ln \kappa(\mathbf{P})\}\approx \ln m + 1.35$. For (c), $\mathrm{E}\{\ln \underline{\sigma}(\mathbf{P})\}\approx -0.51 \ln m + 0.12$. For (d), $\mathrm{E}\{\ln \kappa(\mathbf{P})\}\approx \ln m + 0.77$.

average minimum singular value increases as the process dimension increases.

Lemma 3 (expected value asymptotes: nonsquare normal case, fixed m/n). *Consider a matrix $\mathbf{P} \in \mathcal{R}^{m\times n}$ where each element is an independent standard normal random variable, with $p = m/n \in (0,1)$ equal to a fixed constant. Then*

$$\lim_{m\to\infty} \mathrm{E}\{\ln\kappa(\mathbf{P})\} = \ln\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right) + o(1) \tag{6}$$

$$\lim_{m\to\infty} \mathrm{E}\left\{\ln\underline{\sigma}(\mathbf{P})\right\} = \ln\left(m\left(1-\sqrt{p}\right)^2\right) + o(1) \tag{7}$$

Consider a matrix $\mathbf{P} \in \mathcal{C}^{m\times n}$ where the complex and real part of each element is an independent standard normal random variable, and m/n is equal to a fixed constant $p \in (0,1)$. Then

$$\lim_{m\to\infty} \mathrm{E}\{\ln\kappa(\mathbf{P})\} = \ln\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right) + o(1) \tag{8}$$

$$\lim_{m\to\infty} \mathrm{E}\left\{\ln\underline{\sigma}(\mathbf{P})\right\} = \ln\left(2m\left(1-\sqrt{p}\right)^2\right) + o(1) \tag{9}$$

Proof. The first result is implied by Theorem 6.3; the second result implied by Proposition 5.1; the third result implied by Theorem 6.3; and the fourth result implied by Proposition 5.2 of [34]. □

Lemma 3 implies that large scale nonsquare processes do not tend to be more difficult to identify and control provided that the number of manipulated and measured variables increases in a manner such that their ratio is equal. In fact, the minimum singular value indicates that nonsquare processes are *easier* to identify and control as the process dimension increases. This is in sharp contrast to the case where the processes are square or nearly square. These results suggest that the procedure of selecting only square transfer functions to control, as has been popular in the literature and in industrial practice [10,11], is a poor practice for large scale processes. Similarly, procedures of "squaring down" the process before applying control techniques [12–16] are also inappropriate for large scale processes.

Fig. 3 shows that the general trends for uniform distributions are the same as for normal distributions. The condition number asymptotes are either identical or nearly identical—$E\{\kappa(\mathbf{P})\}\approx 1.71$ for uniform distributions and $E\{\kappa(\mathbf{P})\}\approx 1.76$ for normal distributions (this is computed from Lemma 3 with $p = 1/2$).

5. Expected values for nonsquare processes: fixed *m*

The next lemma, which holds for fixed m, is especially interesting because it is independent of the distribution of the elements of $\mathbf{P}$.

Lemma 4 (expected value asymptotes: nonsquare normal case, fixed *m*). *Consider a matrix $\mathbf{P} \in \mathcal{R}^{m\times n}$ where each element is an independent random variable with zero mean and variance α, and m is fixed.*

Then

$$\lim_{n\to\infty} E\left\{\underline{\sigma}(\mathbf{P})\right\} = n\alpha \tag{10}$$

$$\lim_{n\to\infty} E\left\{\kappa(\mathbf{P})\right\} = 1 \tag{11}$$

Consider a matrix $\mathbf{P} \in \mathcal{C}^{m\times n}$ where the real and imaginary part of each element is an independent random variable with zero mean and variance α, and m is fixed. Then

$$\lim_{n\to\infty} E\left\{\underline{\sigma}(\mathbf{P})\right\} = 2n\alpha \tag{12}$$

$$\lim_{n\to\infty} E\left\{\kappa(\mathbf{P})\right\} = 1 \tag{13}$$

Proof. This is a straightforward generalization of arguments given in Section 7 of [37] (note that only the real case was considered in [37]). □

Lemma 4 indicates that fixing the number of sensors and adding more actuators will tend to make the identification and control problems easier. This also holds when fixing the number of actuators and adding more sensors (by applying Lemma 4 to the transpose of $\mathbf{P}$).

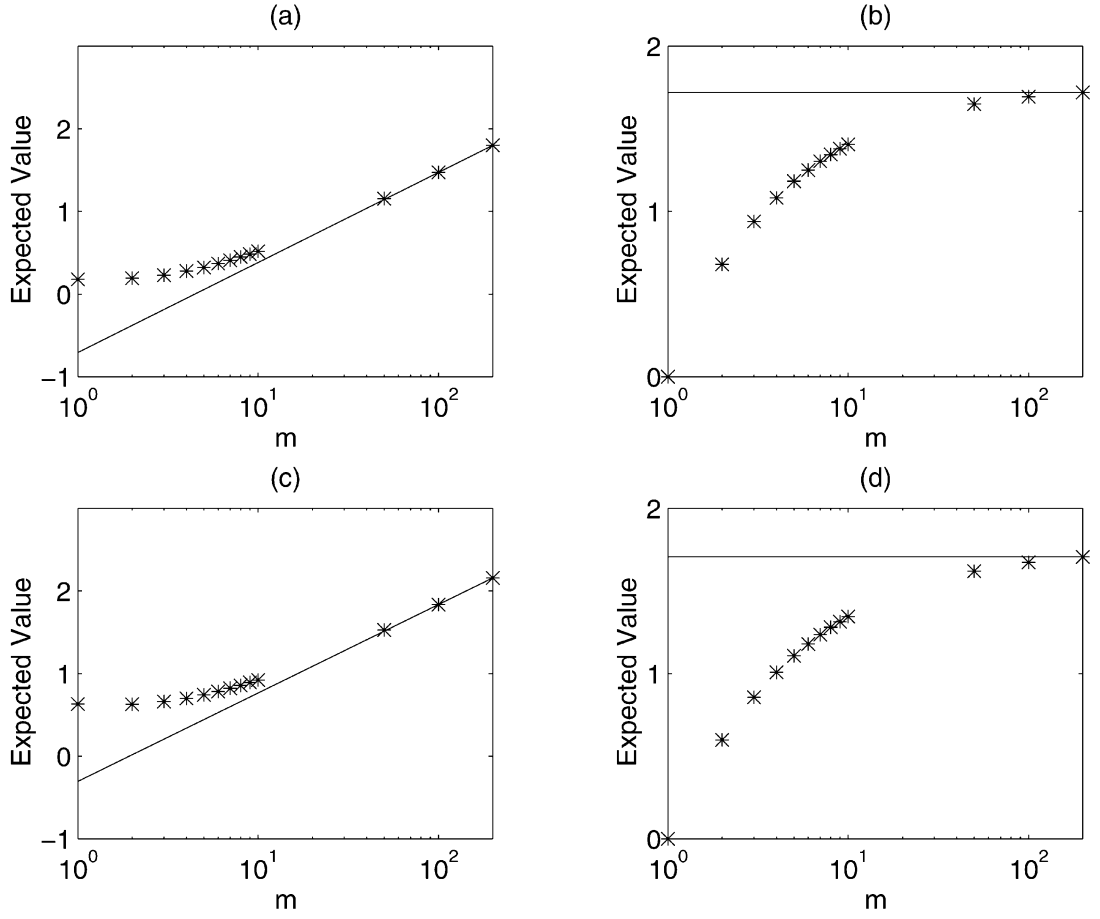

Fig. 3. Expected values of the minimum singular value (a, c) and condition number (b, d) as a function of m. Each element of $\mathbf{P} \in \mathcal{R}^{m\times 2m}$ in (a) and (b) is an independent standard uniform random variable. The real and imaginary part of each element of $\mathbf{P} \in \mathcal{C}^{m\times 2m}$ in (c) and (d) is an independent standard uniform random variable. Values indicated by asterisks were computed using Monte Carlo experiments (50,000 matrices). The solid lines are the estimated asymptotes as $m\to\infty$. For (a), $E\{\ln \underline{\sigma}(\mathbf{P})\}\approx -0.71 + 0.47 \ln m$. For (b), $E\{\ln \kappa(\mathbf{P})\}\approx 1.71$. For (c), $E\{\ln \underline{\sigma}(\mathbf{P})\}\approx -0.31 + 0.46 \ln m$. For (d), $E\{\ln \kappa(\mathbf{P})\}\approx 1.71$.

These statistical results agree with the deterministic conclusions that either more information (more sensors), or more opportunities for manipulation (more actuators) should make the identification and control problems easier [38].

6. Discussion

The general behavior of the average minimum singular value and the average condition number for matrices whose elements are taken from the normal distribution is similar to that for the uniform distribution. The asymptotes in Lemma 4 are completely independent of distribution. Several researchers in mathematics have found that the expected values of many properties of stochastic matrices of large dimension are weakly dependent on the distribution [39,40]. For example, Trefethen and Schreiber [40] computed the dependence of the expected value of various quantities on matrix dimension n for eight classes of stochastic matrices: (1) standard normal distribution of zero mean and unit variance, (2) uniform distribution on [−1, 1], (3) uniform distribution on [0,1], (4) discrete distribution with $p(-1)=p(1)=1/2$, (5) discrete distribution with $p(0)=p(1)=1/2$, (6) symmetric matrices with elements from the standard normal distribution, (7) Toeplitz matrices with elements from the standard normal distribution, and (8) orthogonal matrices distributed by the Haar measure (this is the distribution in which each column or row is uniformly distributed on the $(n-1)$-sphere). It was found that the expected value of several properties were essentially the same irrespective of which of the first seven classes were being considered. Only the eighth class of stochastic matrices, which represents a rather atypical distribution, gave results that were different. A large list of expected values of functions of stochastic matrices that are completely independent of distribution is provided by Stewart [39].

The most well-known distribution-free result is the central limit theorem [31,41], which concerns the probability distribution of the sample mean as the number of points approaches infinity. In essence, the central limit theorem states that, for a sufficiently large sample, the distribution of the sample mean is nearly independent of the distribution used to generate the random sample. This result is closely related to other properties of stochastic matrices of large dimension which are weakly dependent on or independent of the distribution of their elements. For matrices of large dimension, the computation of many matrix properties requires a large number of operations on the elements of the matrices. This has a tendency to reduce the dependence of the matrix property on the distribution of the elements.

The asymptotic results did not explicitly take into account correlations between process elements caused by the requirements of material and energy balances and localization of certain types of measurements and manipulations. These requirements will cause increased collinearity of columns or rows of the transfer function matrix. For example, the transfer function matrix for a process with multiple temperature measurements located on the same well-mixed reactor will have rows that are collinear or nearly collinear. A similar situation can occur for adjacent tray temperature measurements in a distillation column. Material balances can cause flow and composition measurements to be coupled, again causing increased collinearity in the process transfer function matrix. Distributions on the transfer function elements which take into account the increased collinearity due to these effects will have a larger average condition number and a smaller average minimum singular value than for the transfer functions where the distributions on the elements are independent. Hence results that indicate increasingly difficult identification and control problems with increased process dimension are also valid when these correlations are taken into account. This conclusion is supported by studies of paper machines [2], whose transfer function elements are highly correlated.

Another conclusion that will remain valid with correlations are taken into account is that processes with increased number of manipulated or measured variables (with the other number fixed) are easier to identify and control. This will hold because increased information or increased opportunities for manipulation cannot make either an identification or control problem more difficult. A detailed discussion of this for the deterministic case was presented elsewhere [38].

7. Conclusions

The Monte Carlo simulations and theoretical results suggest that identification and control difficulties tend to increase as the dimension of a square process increases, while these trends do not hold for nonsquare processes. In fact, for a fixed number of measured variables, processes tend to be easier to identify and control as the number of manipulated variables is increased. These results imply that removing variables until a square transfer function is achieved is a poor practice for large scale processes.

Acknowledgements

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