PROCESS DESIGN AND CONTROL

On the Computation of Disturbance Rejection Measures

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In any process plant design, a central question is whether it is possible to achieve acceptable control with the available manipulated variables, while taking into account the disturbances that can be expected. Some little known controllability measures have been proposed to address such issues.[1,2] The main reason these controllability measures have received little attention is probably that, while their mathematical formulation was defined, no algorithms were provided to solve the mathematical problems. This paper shows how to calculate these controllability measures and applies the algorithms to some process examples.

1. Introduction

The importance of designing processes that can be acceptably controlled is widely recognized and has been studied by many researchers.[3−12] A significant consideration is whether it is possible to reduce the effect of disturbances to an acceptable level using the available manipulated variables. Three relevant questions in this context are as follows:

1. What is the minimum output error that is obtainable for the worst possible combination of disturbances with the optimal use of the manipulated variables?

2. What is the minimum required magnitude for the manipulated variables to obtain an acceptable output error for the worst possible combination of disturbances?

3. What is the largest possible disturbance for which an acceptable output error is obtainable with the available manipulated variables?

These questions do not require zero output error, as is done in most other disturbance rejection measures.[7,10−12] Zero output error is not achievable in practical systems because of sensor time delays, nonzero sampling times, and other nonminimum phase behavior.

While the mathematical formulation of each of these questions in terms of optimization problems has been provided,[1,2] no explanation was given on how to solve the resulting optimization problems. This paper provides algorithms for computing the solution to these optimization problems.

Obtaining answers to the three questions above is useful when designing processes to be controllable and is also of use in the early stages of control structure design. Answers to these questions can be used to perform the following:

1. Check that acceptable (i.e., not necessarily perfect) control is possible with the available manipulated variables.

2. Check that acceptable control is possible with a specific subset of manipulated variables used for active control.

3. Assess that the sizing of the manipulated variables is appropriate, i.e., whether the range of manipulation is adequate.

4. Identify needs for changing the process design to reduce the effect of disturbances.

1.1. Plant Model. Consider a plant described by a transfer function $\mathbf{P}$ and a disturbance transfer function $\mathbf{P}_d$:

$$\mathbf{y} = \mathbf{P}\mathbf{u} + \mathbf{P}_d\mathbf{d} \tag{1}$$

where $\mathbf{y}$ is the plant output, $\mathbf{u}$ is the vector of manipulated variables, and $\mathbf{d}$ is the vector of disturbances. This paper considers the steady-state case, where $\mathbf{y}$, $\mathbf{u}$, $\mathbf{d}$, $\mathbf{P}$, and $\mathbf{P}_d$ have elements that are real. For the steady state to be well-defined, it is assumed that $\mathbf{P}$ and $\mathbf{P}_d$ are asymptotically stable. The results of the paper also apply to open-loop unstable plants that are stabilized by some lower-level control loops, as long as the lower level loops are in operation and do not saturate. In such cases, $\mathbf{P}$ and $\mathbf{P}_d$ should include the effects of the lower level loops.

1.2. Scaling. To permit consistent evaluation of the results, it is essential that the variables are appropriately scaled. As described in the undergraduate textbook,[13] here it is assumed that the matrices $\mathbf{P}$ and $\mathbf{P}_d$ are scaled such that (1) the largest possible move in the manipulated variables is of magnitude 1, (2) the largest expected disturbance is of magnitude 1, and (3) the largest tolerable offset from the reference value for the outputs is of magnitude 1.

1.3. Organization. This paper deals with the calculation of each of the three problems given above. These problems all take the form of a linear max−min problem. The algorithms are first formulated for the

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minimum output error problem, which consists of an outer maximization problem to find the worst possible combination of disturbances, and an inner minimization problem optimizing the use of the manipulated disturbances for known disturbances. It is shown how to convert the inner minimization problem to a linear program (LP) of standard form. Then duality is used to convert the minimization problem to an equivalent maximization problem and then convert the overall problem to a bilinear programming problem. Because the inner minimization problem can be converted to the standard form of an LP for the other two problems, the conversion to a bilinear program follows similarly. Throughout this paper, the norms of all vectors are measured by the ∞-norm because this is the most useful for most control applications. For example, the ∞-norm is used in nearly all model predictive control formulations for quantifying the allowed magnitude of the manipulated variables.[14−16]

2. Minimum Output Error

In the design of a process, it is of interest to know whether it is possible with the available manipulated variables to achieve acceptably small offset in the outputs y for the worst possible combination of disturbances $\mathbf{d}$, provided the manipulated variables $\mathbf{u}$ are used optimally. The answer to this question is obtained by solving the mathematical problem

$$\max_{||\mathbf{d}||_\infty \le 1} \min_{||\mathbf{u}||_\infty \le 1} ||\mathbf{P}\mathbf{u} + \mathbf{P}_d \mathbf{d}||_\infty \tag{2}$$

When this problem was proposed,[1,2] no indication was provided on how to solve the problem. Solving this max−min problem is nontrivial because the objective function is convex in $\mathbf{d}$, which is the free variable in the maximization. Maximizing a convex function is in essence the same as minimizing a concave function, which in general is difficult. Solving the inner minimization problem is easy though because it is a convex minimization problem. Below we show how to reformulate the max−min problem as a bilinear program. The resulting bilinear program is nonconvex but is computationally tractable for problems of moderate dimension.

We start by rewriting the inner problem in eq 2 as a standard LP for a known disturbance $\mathbf{d}$:

$$\min_{||\mathbf{u}||_\infty \le 1} ||\mathbf{P}\mathbf{u} + \mathbf{P}_d \mathbf{d}||_\infty = \min_{-\mathbf{1} \le \mathbf{u} \le \mathbf{1}} \; \min_{-\gamma\mathbf{1} \le \mathbf{P}\mathbf{u} + \mathbf{P}_d\mathbf{d} \le \gamma\mathbf{1}} \gamma \tag{3}$$

$$= \min_{\substack{\mathbf{u} \le \mathbf{1} \\ -\mathbf{u} \le \mathbf{1} \\ \mathbf{P}\mathbf{u} + \mathbf{P}_d\mathbf{d} - \gamma\mathbf{1} \le 0 \\ -\mathbf{P}\mathbf{u} - \mathbf{P}_d\mathbf{d} - \gamma\mathbf{1} \le 0 \\ \gamma \le 0}} \gamma \tag{4}$$

where $\mathbf{1}$ denotes a column vector with all elements equal to 1. With $\mathbf{v} = \mathbf{u} + \mathbf{1}$ and $\mathbf{w} = [\mathbf{v}^T, \gamma]^T$, the optimization problem is equal to

$$\min_{\substack{\mathbf{w} \ge 0 \\ \begin{bmatrix} I & \mathbf{0} \\ \mathbf{P} & -\mathbf{1} \\ -\mathbf{P} & -\mathbf{1} \end{bmatrix}\mathbf{w} + \begin{bmatrix} 0 \\ \mathbf{P}_d \\ -\mathbf{P}_d \end{bmatrix}\mathbf{d} \le \begin{bmatrix} 2\mathbf{1} \\ P\mathbf{1} \\ -P\mathbf{1} \end{bmatrix}}} \mathbf{f}^T\mathbf{w} \tag{5}$$

The solution to the max−min problem need not be unique. One cause of nonuniqueness is that the input vectors (combinations of manipulated variables and disturbances) $[\mathbf{u}^T, \mathbf{d}^T]^T$ and $[-\mathbf{u}^T, -\mathbf{d}^T]^T$ yield output vectors of the same magnitude. This source of nonuniqueness can be removed by fixing one element of the disturbance vector to +1 because all disturbances can generally be assumed to have a maximum magnitude at the optimal point. Fixing the first element of the disturbance vector to +1 and partitioning $\mathbf{P}_d$ accordingly give

$$\mathbf{d} = \begin{bmatrix} 1 \\ d' \end{bmatrix}; \quad \mathbf{P}_d = [\mathbf{P}_{d1}, \mathbf{P}_d{}'] \tag{6}$$

where $\mathbf{P}_{d1}$ denotes the first column of $\mathbf{P}_d$. Introducing eq 6 into eq 5 gives

$$\min_{\substack{\mathbf{w} \ge 0 \\ \mathbf{A}d' + \mathbf{B}\mathbf{w} \le \hat{b}}} \mathbf{f}^T\mathbf{w} \tag{7}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 \\ \mathbf{P}_d{}' \\ -\mathbf{P}_d{}' \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} I & 0 \\ \mathbf{P} & -\mathbf{1} \\ -\mathbf{P} & -\mathbf{1} \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 2\mathbf{1} \\ P\mathbf{1} - \mathbf{P}_{d1} \\ -P\mathbf{1} + \mathbf{P}_{d1} \end{bmatrix} \tag{8}$$

Thus, the inner program has been written as a standard LP. Now consider the overall problem, with the new variable $\mathbf{x}$ defined by $\mathbf{x} = d' + \mathbf{1}$:

$$\max_{||\mathbf{d}||_\infty \le 1} \min_{||\mathbf{u}||_\infty \le 1} ||\mathbf{P}\mathbf{u} + \mathbf{P}_d\mathbf{d}||_\infty = \max_{\mathbf{0} \le \mathbf{x} \le 2\mathbf{1}} \min_{\substack{\mathbf{w} \ge 0 \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w} \le \mathbf{b}}} \mathbf{f}^T\mathbf{w} \tag{9}$$

where $\mathbf{b} = \hat{b} + \mathbf{A}\mathbf{1}$. There exists a branch-and-bound algorithm for solving this max−min LP that converges in a finite number of steps.[17] Here the max−min LP is converted to a bilinear program that can be solved using several publically available optimization codes. The inner (minimization) problem is replaced by an equivalent maximization problem using duality:[17]

$$\max_{\mathbf{0} \le \mathbf{x} \le 2\mathbf{1}} \min_{\substack{\mathbf{w} \ge 0 \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w} \le \mathbf{b}}} \mathbf{f}^T\mathbf{w} = \max_{\substack{\mathbf{z} \ge \mathbf{0} \\ \mathcal{B}\mathbf{z} \ge \mathcal{D}}} \mathbf{z}^T\mathcal{A}\mathbf{z} + \mathcal{C}^T\mathbf{z} \tag{10}$$

where

$$\mathbf{z} = [\mathbf{x}^T, \boldsymbol{\lambda}^T]^T, \quad \mathcal{C} = [\boldsymbol{0}^T, -\mathbf{b}^T]^T, \quad \mathcal{D} = [2\mathbf{1}^T, -\mathbf{f}^T]^T,$$
$$\mathcal{A} = \begin{bmatrix} 0 & \mathbf{A}^T \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{B}^T \end{bmatrix} \tag{11}$$

This optimization problem is bilinear in the objective and linear in the constraints. Although the optimization problem is nonconvex, available optimization codes can solve this type of problem provided the problem size is moderate.[18] The solution to the optimization problem will give the values for $\mathbf{x}$ and $\boldsymbol{\lambda}$. The worst combination of disturbances can be found directly through the relationship $d' = \mathbf{x} - \mathbf{1}$ (keeping in mind that $d_1 = 1$), whereas the vector $\boldsymbol{\lambda}$ gives the solution to the duality of the original inner minimization problem. The solution to the dual problem can be used to compute the solution to the primal problem in order to obtain the optimal values for the manipulated variables. Such a conversion is described in standard optimization textbooks, e.g., in work by Luenberger.[19] Because the worst-case disturbance now has been calculated, another approach would be to solve the inner minimization problem in eq 5 for a fixed worst-case disturbance to obtain the optimal inputs. Because LPs can be solved very efficiently, the solution of eq 5 can be computed with little effort.

2.1. Solution via a Series of LPs. Because the objective function in eq 2 is convex in $\mathbf{d}$, the maximization over $\mathbf{d}$ can be replaced by a maximization over the vertexes of this set; that is, eq 2 is equivalent to (e.g., see theorem 2 of ref 20)

$$\max_{d_k \in \{-1,+1\}} \min_{||\mathbf{u}||_\infty \le 1} ||\mathbf{P}\mathbf{u} + \mathbf{P}_\mathrm{d}\mathbf{d}||_\infty \tag{12}$$

Because this maximization is over a discrete set, in principle it can be solved by solving the inner (LP) problem for each vertex and taking the maximum. The computational requirements of this approach are reasonable when the number of disturbances is relatively low. The number of vertexes grows exponentially with the dimension of $\mathbf{d}$, resulting in an excessive computational load when the dimension of $\mathbf{d}$ is larger than ~12. This motivated our investigation into the alternative formulation as a bilinear program, which can in principle solve the max−min problem without checking all of the vertexes.

2.2. Connections to Flexibility Analysis. While *flexibility* as commonly defined in the process design literature (e.g., see eq 5 of ref 21) treats different variables and constraints, it has a mathematical structure similar to that of eq 2. Several algorithms have been proposed for computing the solution of the flexibility optimization problems, including heuristic vertex search, implicit enumeration schemes, and active set strategies.[20−22] In the latter strategy and for the case of linear constraints, the Kuhn−Tucker conditions are used to reformulate the max−min flexibility problem as a mixed-integer LP. While the original mathematical problems are somewhat different, this approach for solving the max−min flexibility program is related to our approach to solving eq 2 because both approaches use $\boldsymbol{\lambda}$ variables to reformulate the original optimization problem.

3. Required Input Magnitude

The minimization of the magnitude of the manipulated variables while achieving an acceptable offset for the worst possible combination of disturbances was formulated as[1,2]

$$\max_{||\mathbf{d}||_\infty \le 1} \min_{||\mathbf{P}\mathbf{u}+\mathbf{P}_\mathrm{d}\mathbf{d}||_\infty \le 1} ||\mathbf{u}||_\infty \tag{13}$$

Similarly as before, the inner minimization problem can be converted into a standard LP (for a fixed disturbance $\mathbf{d}$):

$$\min_{||\mathbf{P}\mathbf{u}+\mathbf{P}_\mathrm{d}\mathbf{d}||_\infty \le 1} ||\mathbf{u}||_\infty = \min_{\substack{\left[\begin{smallmatrix} I \\ -I \end{smallmatrix}\right]\mathbf{u} \le \gamma \mathbf{1} \\ \left[\begin{smallmatrix} \mathbf{P} \\ -\mathbf{P} \end{smallmatrix}\right]\mathbf{u} + \left[\begin{smallmatrix} \mathbf{P}_\mathrm{d} \\ -\mathbf{P}_\mathrm{d} \end{smallmatrix}\right]\mathbf{d} \le \mathbf{1}}} \gamma \tag{14}$$

Substituting $\mathbf{v} = \mathbf{u} + \gamma\mathbf{1}$ and $\mathbf{w} = [\mathbf{v}^\mathrm{T}, \gamma]^\mathrm{T}$ and fixing the first element of the disturbance vector to +1 give

$$\min_{\mathbf{w} \ge 0} \mathbf{f}^\mathrm{T}\mathbf{w} \tag{15}$$

$$\begin{bmatrix} I & -2\mathbf{1} \\ -I & \mathbf{0} \\ \mathbf{P} & -\mathbf{P}\mathbf{1} \\ -\mathbf{P} & \mathbf{P}\mathbf{1} \end{bmatrix}\mathbf{w} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{P}_\mathrm{d}' \\ -\mathbf{P}_\mathrm{d}' \end{bmatrix} d' \le \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} - \mathbf{P}_\mathrm{d1} \\ \mathbf{1} + \mathbf{P}_\mathrm{d1} \end{bmatrix}$$

For a known fixed d', eq 15 is of the form of a standard LP and may be written on the same form as eq 7. *Provided that the LP is feasible*, one may then proceed exactly as in the previous section and replace the inner minimization problem with its dual maximization problem. However, whereas the LP in eq 7 is always feasible for a bounded disturbance (the maximum output is always finite because the plant is assumed to be stable), the LP in eq 15 need not be feasible if $\mathbf{P}$ does not have full row rank because it need not be possible to achieve acceptably small output errors even with very large control outputs. A trivial sufficient condition for the feasibility of the LP in eq 15 is that $||\mathbf{P}_\mathrm{d}||_{\mathrm{i}\infty} \le 1$, where $||\cdot||_{\mathrm{i}\infty}$ is the induced ∞-norm, in which case the solution to eq 13 has $\mathbf{u} = 0$. A more useful sufficient condition in most applications is $||(I - \mathbf{P}\mathbf{P}^\dagger)\mathbf{P}_\mathrm{d}||_{\mathrm{i}\infty} \le 1$, where $\mathbf{P}^\dagger$ is the generalized matrix inverse of $\mathbf{P}$. If these conditions are violated, then there may exist a disturbance such that $||\mathbf{y}||_\infty > 1$ irrespective of how the available manipulated variables are used. A practical approach to checking the feasibility is to run a rescaled version of the minimum output error problem. If the norm of the minimum output error is greater than 1 even for very large $\mathbf{u}$, then the required manipulated variable magnitude problem is infeasible for all practical purposes.

4. Acceptable Disturbance Magnitude

The problem of determining the maximum magnitude of the disturbances for which an acceptable output error can be obtained with the available manipulated variables can be formulated as

$$\max_{\substack{||\mathbf{u}||_\infty \le 1 \\ ||\mathbf{P}\mathbf{u}+\mathbf{P}_\mathrm{d}\mathbf{d}||_\infty \le 1}} ||\mathbf{d}||_\infty \tag{16}$$

This can be written in terms of the solution of a set of LPs using mathematical manipulations similar to, but more complex than, those in section 2. First, do not fix the first element of $\mathbf{d}$. Then eq 16 is equal to

$$\begin{aligned} &\max_{\substack{||\mathbf{u}||_\infty \le 1 \\ ||\mathbf{P}\mathbf{u}+\mathbf{P}_\mathrm{d}\mathbf{d}||_\infty \le 1}} \max_j |\mathbf{d}_j| = \\ &\max_j \max_{\substack{||\mathbf{u}||_\infty \le 1 \\ ||P\mathbf{u}+P_\mathrm{d}\mathbf{d}||_\infty \le 1}} |\mathbf{d}_j| = \max_j \max_{\substack{||\mathbf{u}||_\infty \le 1 \\ ||P\mathbf{u}+P_\mathrm{d}\mathbf{d}||_\infty \le 1}} \mathbf{d}_j \end{aligned} \tag{17}$$

The last equality follows because any $\mathbf{d}$ and $\mathbf{u}$ that solves the middle optimization also has that optimization solved by $-\mathbf{d}$ and $-\mathbf{u}$. The rightmost optimization for each j can be written as an LP and solved using mathematical manipulations similar to those in section 2. Then the maximum over j is taken to compute the solution to eq 16.

A problem which is probably of more interest is the smallest disturbance $\mathbf{d}$ that would cause the manipulated variables (when used optimally) to saturate and give the largest acceptable output error. All disturbances smaller than this can be controlled with acceptable output error using optimal manipulated variables. One way to solve this problem is by iteratively rescaling the minimum output error problem in eq 2 with respect to the disturbances.[2] The disturbance magnitude for which the optimal objective function value is equal to 1 is the minimum disturbance magnitude for which acceptable output offset cannot be obtained with the available manipulated variables.

To illustrate the difference between these problems, consider

$$\mathbf{P} = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{P}_d = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix} \tag{18}$$

The largest disturbance in which the output error is acceptable can be selected as $\mathbf{d} = [101, 0]^T$. This indicates that some very large disturbances can be controlled using bounded manipulated variable actions while providing acceptable output error. When the output error problem is rescaled, a minimum magnitude disturbance which provides unacceptable output error even when using optimal bounded manipulated variable actions is $d = [0, 0.02]^T$. This indicates that some very small disturbances cannot be tolerated. As this example illustrates, there can be a large "gray zone" in which the norm of the disturbance is not sufficient to tell whether a disturbance can be adequately rejected. Although the example is purely mathematical, real processes such as high-purity distillation columns,[13] polymer film extruders (see a blown film extrusion example), and paper machines[23,24] can have such a gray zone.

5. Computational Example

In this section the use of the controllability measures is illustrated on an example adapted from the literature. The example deals with a blown film extrusion process. The fact that each element of the disturbance vector can be set to its maximum or minimum value allows the disturbances to be treated as discrete variables, which speeds up the calculations. The computational time was further reduced by scaling the $\mathbf{A}$ matrix such that the scaled $\mathbf{x}$ variables can only take the values 0 or 1. Together with fixing one element of the disturbance vector to +1 (as described above), this typically reduces the computation time by a factor of 40. Computational time and numerical robustness can also be improved by adding artificial maximum value constraints on the $\boldsymbol{\lambda}$ variables. However, such artificial constraints may lead to a solution which is not globally optimal. If such maximum value constraints on $\boldsymbol{\lambda}$ are active at the optimal solution, those constraints should be relaxed and the optimization problem resolved.

5.1. Blown Film Extrusion Process. Here we consider a blown film extrusion process with 15 actuators and sensors.[25,26] Because of the process design and the spatial distribution of manipulated and controlled variables, it is natural to model both the process and disturbance transfer function matrices as *circulant symmetric matrices*. A matrix is *circulant* if row $i + 1$ can be obtained from row i by simply shifting all elements in row i one position to the right and placing the last element of row i as the first element of row $i + 1$. In order for a circulant matrix also to be symmetric, it must be possible to parametrize the first row as follows:

$$[p_1 \; p_2 \; \cdots \; p_{m-1} \; p_m \; \cdots \; p_m \; p_{m-1} \; \cdots \; p_2] \tag{19}$$

For this example, the process matrix $\mathbf{P}$ is defined by the parameters

$$p_1 = 1.0; \quad p_2 = 0.9; \quad p_3 = 0.6; \quad p_4 = 0.2; \quad p_5 = 0.1; \quad p_6 = -0.1; \quad p_7 = 0.05; \quad p_m = p_8 = 0 \tag{20}$$

The plant matrix $\mathbf{P}$ is not full rank, which is common in polymer film extrusion and coating processes, in part because of an overall flow constraint.[27]

The plant disturbance model is an extension of earlier disturbance models[28−30] to blown film extrusion, and the first row of this disturbance model can be found from

$$[\mathbf{P}_d]_{1,i} = kr^{i-1} \tag{21}$$

where the index i is defined in the same way as that in eq 19, with $k = 1$ and $r = 0.7$.

This corresponds to disturbances that have a significant spatial correlation across the polymer film. The minimum output error is $||\mathbf{y}||_\infty \leq 0.783$, which indicates that the manipulated variables are able to achieve an acceptable closed-loop performance for the entire range of norm-bounded disturbances. A worst-case disturbance, the optimal manipulated variables for this disturbance, and the corresponding output vector are shown in Figure 1. It is interesting that the effect of the worst-case disturbance on the output is flat (see Figure 1a).

Now consider the case where the disturbances have a weaker spatial correlation across the polymer film ($k = 1$ and $r = 0.3$). The minimum output error is $||\mathbf{y}||_\infty \leq 0.8935$, which indicates that the manipulated variables are able to achieve an acceptable closed-loop performance for all potential disturbances. The set of disturbances with lower spatial correlation ($r = 0.3$) is harder to suppress than the set of disturbances with high spatial correlation ($r = 0.7$). It makes physical sense that the effect of the worst-case disturbance on the output has a higher spatial frequency variation for the set of disturbances with a weaker spatial correlation (see Figure 1a). Also, this comparison makes clear the importance of correctly modeling the spatial correlation among the disturbances when evaluating the achievable performance of a closed-loop system.

Based on our experience with experimental data collected from an industrial polymer film extruder, disturbances with low spatial correlation tend to have lower magnitude than disturbances with high spatial correlation. In this case, a more realistic model of the disturbances with low spatial correlation ($r = 0.3$) would be to have a plant disturbance matrix with $k = 0.5$. In this case the minimum output error is $||\mathbf{y}||_\infty \leq 0.382$, indicating that the manipulated variables are able to achieve acceptable closed-loop performance for all potential disturbances. The disturbance, manipulated, and output variables are qualitatively similar in shape (though not necessarily in magnitude) for $k = 0.5$ and 1.

It is interesting that the worst-case disturbance effects on the output in Figure 1a are not exactly sinusoidal, as is predicted by other results.[25,26] The reason for this difference is that the results of refs 25 and 26 are based on closed-loop stability issues (which are independent of the signal norm), whereas the results here are based on steady-state performance considerations (with signal norms measured in terms of the ∞-norm). However, the disturbance effects, manipulated variables, and controller variables are nearly sinusoidal for the $r = 0.3$ results.

Using the manipulated variables optimally, disturbances $\mathbf{d}$ up to a magnitude of 1.1 can be controlled while giving an acceptable output error. The maximum magnitude of the disturbances for which an acceptable

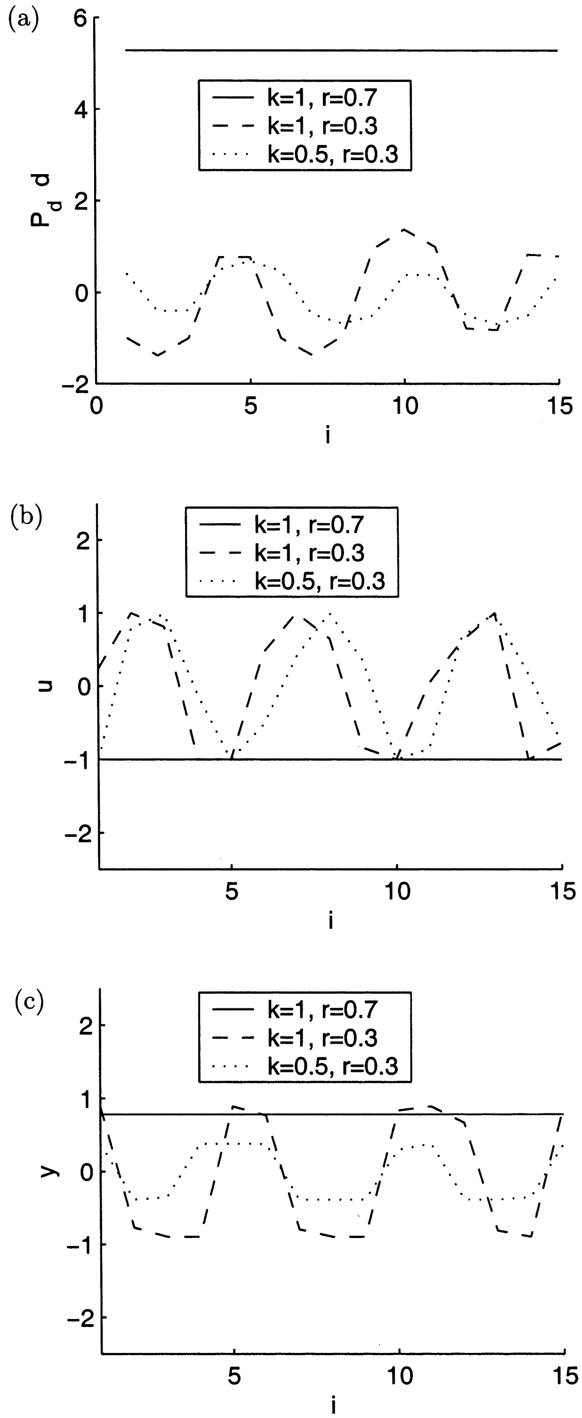

Figure 1. Results for the blown film extrusion process: (a) a worst-case disturbance **d** multiplied by the plant disturbance model $\mathbf{P}_d$; (b) the optimal manipulated variables for this worst-case disturbance; (c) the corresponding output vector for the blown film extrusion process.

output error can be obtained with the available manipulated variables is 5.0 (it is straightforward to use the properties of symmetric circulant matrices[31,32] to show that, for symmetric circulant $\mathbf{P}$ and $\mathbf{P}_\mathrm{d}$, only one of the rightmost optimizations in eq 17 needs to be solved to compute the solution to eq 16). Hence, some rather large disturbances can be controlled with acceptable output error, whereas there are much smaller disturbances (less than 25% of the size) that cannot be controlled with acceptable output error, with the allowed manipulated variables.

6. Discussion and Conclusions

In this paper we have shown how to solve the optimization problems associated with the minimum output error and required input magnitude controllability measures.[1,2] The acceptable disturbance magnitude controllability measure was reformulated, and the reformulated measure can be evaluated using the same mathematical techniques as those for the two other controllability measures.

Solving these controllability measures is computationally demanding but typically less demanding than the alternative method of solving the inner (LP) problem for all vertexes in the disturbance set.

At present, the controllability measures are evaluated at steady state; the evaluation of these measures at nonzero frequencies will be a topic for further research. However, steady-state models are often available relatively early in the process design stage, and evaluating the measures at steady state will therefore make it possible to take important control considerations into account early in the process design.

Skogestad and Wolff[2] advise caution when applying these controllability measures. In particular, they mention that if the process model includes lower level control loops, the manipulated variables in these loops may saturate and the controllability measures may therefore be invalid. This problem can be resolved at the cost of having to solve a larger optimization problem, simply by including the manipulated variables in the lower level control loops as outputs from the model and solving the optimization problems with the appropriate constraints on these additional outputs.

These measures complement the insight that can be gained from other measures for evaluating the effect of disturbances on controllability. Notably, the measures proposed by Cao et al.[33] for selecting subsets of manipulated variables for active control are significantly less computationally intensive. However, these measures are based on the vector 2-norm, which make them poorly suited for assessing hard constraints in manipulated or controlled variables. To illustrate, consider an n-dimensional vector (of manipulated or controlled variables), scaled such that the individual elements should be in the range -1 to 1. Then, for values of the vector 2-norm in the range 1 to $\sqrt{n}$, one cannot determine from the 2-norm whether individual vector elements are at or outside their constraints.

Most controllability measures apply to linear systems only. It is conceptually simple to extend these controllability measures to account for nonlinearities in the process model. However, the method developed here for evaluating the measures would then break down, and solving the corresponding optimization problems for nonlinear processes of modest dimensions would be prohibitively computationally expensive. In practice, these controllability measures are therefore, at present, restricted to linear systems. However, the optimization problems solved when evaluating these controllability measures will identify potentially "difficult" combinations of disturbances. With these difficult combinations of disturbances, the practicing engineer may choose to check using a nonlinear model that acceptable control is indeed achievable. This is computationally a lot less demanding than using the nonlinear model directly to identify the more difficult combinations of disturbances.

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Literature Cited

(1) Wolff, E. A. Studies on Control of Integrated Plants. Thesis, Department of Chemical Engineering, University of Trondheim, Trondheim, Norway, 1994.

(2) Skogestad, S.; Wolff, E. A. Controllability Measures for Disturbance Rejection. In *Preprints of the IFAC Workshop on Interactions between Process Design and Process Control*; Perkins, J. D., Ed.; 1992.

(3) Braatz, R. D.; Lee, J. H.; Morari, M. *Comput. Chem. Eng.* **1996**, *20*, 463−468.

(4) Braatz, R. D.; Morari, M. *SIAM J. Control Optim.* **1994**, *32*, 1763−1768.

(5) Cao, Y.; Owens, D. H.; Rossiter, D. *Chem. Eng. Res. Des.* **1998**, 849−854.

(6) Fisher, W. R.; Doherty, M. F.; Douglas, J. M. *Ind. Eng. Chem. Res.* **1988**, *27*, 597−615.

(7) Hovd, M.; Skogestad, S. *Automatica* **1992**, *28*, 989−996.

(8) Lee, J. H.; Braatz, R. D.; Morari, M.; Packard, A. *Automatica* **1995**, *31*, 229−235.

(9) Piette, R.; Harris, T. J.; McLellan, P. J. *Ind. Eng. Chem. Res.* **1995**, *34*, 4436−4450.

(10) van de Wal, M.; De Jager, B. A. Control structure design: A survey. In *Proceedings of the American Control Conference*; IEEE Press: Piscataway, NJ, 1995.

(11) Lewin, D. R. *Comput. Chem. Eng.* **1996**, *20*, 13−25.

(12) Lewin, D. R. *Comput. Chem. Eng.* **1996**, *20*, 1477−1489.

(13) Skogestad, S.; Postlethwaite, I. *Multivariable Feedback Control: Analysis and Design*; John Wiley & Sons: New York, 1996.

(14) Garcia, C. E.; Prett, D. M.; Morari, M. *Automatica* **1989**, *25*, 335−348.

(15) Rawlings, J. B. Tutorial: Model predictive control technology. In *Proceedings of the American Control Conference*; IEEE Press: Piscataway, NJ, 1999.

(16) VanAntwerp, J. G.; Braatz, R. D. *J. Process Control* **2000**, *10*, 1−8.

(17) Falk, J. E. *Math. Program.* **1973**, *5*, 169−188.

(18) Sahinidis, N. V. *J. Global Optim.* **1996**, *8*, 201−205.

(19) Luenberger, D. G. *Linear and Nonlinear Programming*, 2nd ed.; Addison-Wesley Publishing Co.: Reading, MA, 1984.

(20) Halemane, K. P.; Grossman, I. E. *AIChE J.* **1983**, *29*, 425−433.

(21) Grossmann, I. E.; Floudas, C. A. *Comput. Chem. Eng.* **1987**, *11*, 675−693.

(22) Swaney, R. E.; Grossmann, I. E. *AIChE J.* **1985**, *31*, 631−641.

(23) Featherstone, A. P.; Braatz, R. D. *Ind. Eng. Chem. Res.* **1998**, *37*, 97−106.

(24) VanAntwerp, J. G.; Braatz, R. D. *IEEE Trans. Control Syst. Technol.* **2000**, *8*, 408−417.

(25) Featherstone, A. P.; Braatz, R. D. *AIChE J.* **1997**, *43*, 1989−2001.

(26) Featherstone, A. P.; VanAntwerp, J. G.; Braatz, R. D. *Identification and Control of Sheet and Film Processes*; Springer-Verlag: London, U.K., 2000.

(27) Braatz, R. D.; Tyler, M. L.; Morari, M.; Pranckh, F. R.; Sartor, L. *AIChE J.* **1992**, *38*, 1329−1339.

(28) Bergh, L. G.; MacGregor, J. F. *Can. J. Chem. Eng.* **1987**, *65*, 148−155.

(29) Rawlings, J. B.; Chien, I.-L. *AIChE J.* **1996**, *42*, 753−766.

(30) Tyler, M. L.; Morari, M. *AIChE J.* **1995**, *41*, 846−854.

(31) Hovd, M.; Braatz, R. D.; Skogestad, S. *Automatica* **1996**, *33*, 433−439.

(32) VanAntwerp, J. G.; Featherstone, A. P.; Braatz, R. D. *J. Process Control* **2001**, *11*, 149−178.

(33) Cao, Y.; Rossiter, D.; Owens, D. *Comput. Chem. Eng.* **1997**, S403−S408.

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