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Brief Paper

Screening Tools for Robust Control Structure Selection*

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Key Words—Robust control; measurement selection; distillation columns.

Abstract—Screening tools for control structure selection in the presence of model/plant mismatch are developed in the context of the Structured Singular Value (μ) Theory. The developed screening tools are designed to aid engineers in the elimination of undesirable control structure candidates for which a robustly performing controller does not exist. The screening tools are examined on a multi-component distillation column control problem and compared with previously published methods such as the Condition Number Criterion.

1. *Introduction*

Practical control problems often involve more actuators and sensors than are needed for designing effective, economically viable control systems. On a distillation column, for example, there are at least four actuators and there can be as many temperature measurements as there are trays, possibly hundreds, that can be utilized for composition control. In practice, one does not use all the available actuators and sensors for composition control since two of the four actuators must be used for inventory control and the use of all temperature measurements leads to an unnecessarily complex and expensive control system. An appropriate set of actuators and sensors must be selected from the available candidates, and subsequently, partitioned and paired for decentralized control. Control structure selection refers to both actuator/sensor selection and partitioning/pairing. The partitioning/pairing problem for decentralized control has been studied extensively and many practical tools such as the Relative Gain Array and other interaction measures have been proposed (Bristol, 1966; Niederlinski, 1971; Grosdidier and Morari, 1986). In this paper, we will concentrate on the problem of actuator/sensor selection.

The main question arising in control structure selection is as follows: 'What makes one control structure more desirable than another?' The closed-loop performance achievable for the plant *model* (the achievable *nominal* performance) is clearly an important criterion. It is determined by factors such as right-half plane (RHP) zeros, delays, and signal-to-noise ratios of the measurements. When expressed through quantitative measures like the H_2 or H_∞ norms, it can be easily computed through standard optimization techniques (Doyle *et al.*, 1989). Besides these well-known factors, another outstanding issue contributing to the overall closed-loop performance is model/plant mismatch. Some control structures are inherently more sensitive than others to the mismatch between the model and the real plant. Hence, any practical control structure selection criterion should address not only the achievable *nominal* performance, but also the achievable *robust* performance, that is, the achievable *worst-case* performance in the presence of a prespecified level of model/plant mismatch.

Owing to the combinatorial nature of the problem, the number of potential control structures to be examined (referred to as *control structure candidates* from this point on) can be very large. Naturally, a method which can reduce the number of candidates before applying detailed analysis is of significant practical value. The first step to this should be to eliminate the candidates for which a controller achieving a desired level of robust performance does not exist *regardless* of the controller design method. The criteria that can be used to accomplish this screening will be referred to as *design-independent screening tools.* This screening leaves candidates for which a control system with satisfactory performance potentially exists. After the design-independent screening, an additional screening may be carried out in the context of a particular design method. The criteria that assume a specific controller design approach will be called *design-dependent screening* tools.

Traditionally, most research on control structure selection was carried out in the stochastic optimal control setting. Therefore, all the developed criteria were based on the achievable nominal performance (Kumar and Seinfeld, 1978a, b; Harris *et al.*, 1980). Model/plant mismatch was taken into account in *ad hoc* ways, for example, mimicking it through arbitrarily chosen state-excitation noise. In the late 1970s, there were some efforts to bring rigorous descriptions of model uncertainty into the control structure selection problem. In the context of secondary measurement selection, Brosilow and co-workers (Weber and Brosilow, 1972; Joseph and Brosilow, 1978) suggested what is known as the *Condition Number Criterion,* which is valid for a specific type of norm-bounded uncertainty on the model. This criterion will be examined further in this article. More recently, Skogestad *et al.* (1988) showed that the Relative Gain Array (RGA) can be used as a measure of the sensitivity of a control structure to diagonal input uncertainty. The latest contribution to the control structure selection problem came from Lee and Morari (1991) who suggested a criterion in the context of the Structured Singular Value Theory. The strengths of this criterion were that a more general model uncertainty description (known as *structured uncertainty*) could be used and that the system dynamics could be incorporated. However, all the published criteria either assume a specific design approach or a specific uncertainty description and, therefore, cannot be used as *general* design-independent screening tools. The achievable *nominal* performance (obtained through H_2 or H_∞ optimization) qualifies as a design-independent screening tool since achieving a desired performance level in the absence of model uncertainty is clearly required for achieving the same level of performance in the presence of model uncertainty. However, its practicality is limited since it fails to address

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one of the most important issues in control—model uncertainty.

The purpose of this article is to introduce a set of design-independent screening tools that can be used to reduce the number of control structure candidates. The approach is based on the Structured Singular Value Theory, therefore allowing a general structured norm-bounded uncertainty description.

2. *General framework*

2.1. *Structured Singular Value.* The Structured Singular Value ($\mu : \mathscr{C}^{n\times n} \times \Delta \to \mathscr{R}_{0+}$) is defined as follows:

Definition 1. *Structured Singular Value* (μ). Let $M \in \mathscr{C}^{n\times n}$ and define the set Δ as follows:

$$\Delta = \Big\{ \text{diag}\,[\delta_1 I_{r_1}, \ldots, \delta_m I_{r_m}, \Delta_1, \ldots, \Delta_l];\quad \delta_j \in \mathscr{C}, \Delta_i \in \mathscr{C}^{p_i \times p_i}; \sum_{j=1}^{m} r_j + \sum_{i=1}^{l} p_i = n \Big\}. \quad (1)$$

Then $\mu_\Delta(M)$ (μ of M with respect to the uncertainty structure Δ) is defined as

$$\mu_\Delta(M) = \begin{cases} \Big[\min_{\Delta} \{\bar{\sigma}(\Delta) : \det(I + M\Delta) = 0, \Delta \in \Delta\} \Big]^{-1} \\ 0 \text{ if } \exists \text{ no } \Delta \in \Delta \text{ such that } \det(I + M\Delta) = 0. \end{cases} \quad (2)$$

The structured singular value has the following lower and upper bounds:

$$\max_{Q \in \mathcal{Q}} \rho(QM) = \mu_\Delta(M) \le (\approx) \inf_{D \in \mathscr{D}} \bar{\sigma}(DMD^{-1}), \quad (3)$$

where

$$\mathcal{Q} = \{Q \in \Delta : Q^*Q = I_n\} \quad (4)$$

$$\mathscr{D} = \{\text{diag}\,[D_1, \ldots, D_m, d_1 I_{p_1}, \ldots, d_l I_{p_l}] : D_i \in \mathscr{C}^{r_i \times r_i}, D_i = D_i^* > 0; d_i \in \mathscr{R}_+\}. \quad (5)$$

$\bar{\sigma}(\cdot)$ denotes the maximum singular value, and $\rho(\cdot)$ denotes the spectral radius.

The maximum spectral radius is always equal to μ, but the maximization is nonconvex and computing the global optimum of such functions is in general difficult. In contrast, the upper bound can be formulated as a convex optimization. Though the upper bound is not necessarily equal to μ except when the number of blocks in Δ is three or less (Packard, 1988), the upper bound is almost always very close to μ (within 98–99% for most problems). For this reason the upper bound is used in most tests requiring numerical calculations of μ.

2.2. *Representation of uncertain systems.* We will use the following notations for Linear Fractional Transformations (LFT):

$$\mathscr{F}_u(X, Y) = X_{22} + X_{21} Y (I - X_{11} Y)^{-1} X_{12} \quad (6)$$

$$\mathscr{F}_l(X, Y) = X_{11} + X_{12} Y (I - X_{22} Y)^{-1} X_{21}, \quad (7)$$

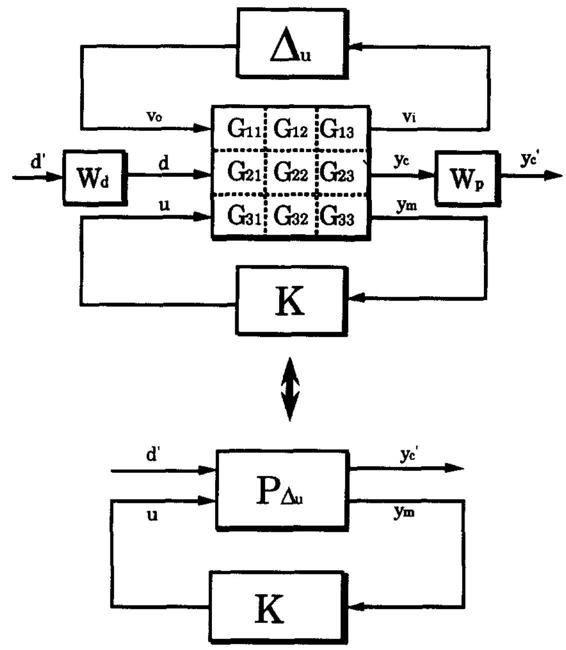
where X is partitioned in such a way that X_{11} has the same dimension as Y^T for the upper LFT ($\mathscr{F}_u$) and X_{22} has the same dimension as Y^T for the lower LFT ($\mathscr{F}_l$). X and Y can be either transfer functions or complex matrices.

Figure 1 represents the general block diagram for linear systems with model uncertainty. The uncertain system is represented as the Linear Fractional Transformation (LFT) of $G(s)$ and the L_∞-norm-bounded block Δ_u. More specifically, the true system can be any system $P_\Delta(s)$ satisfying the following conditions: (1) The frequency response matrix of the system $P_{\Delta_u}|_{s=j\omega}$ for each frequency ω belongs to the set $P_u(\omega)$, where

$$P_u(\omega) = \{(\mathscr{F}_u(G(j\omega), \Delta_u) : \Delta_u \in B\Delta_u\} \quad (8)$$

$$B\Delta_u = \{\Delta \in \Delta_u : \bar{\sigma}(\Delta) \le 1\} \quad (9)$$

$$\Delta_u = \Big\{ \text{diag}\,(\delta_1 I_{r_1}, \ldots, \delta_m I_{r_m}, \Delta_1, \ldots, \Delta_{l-1}) : \Delta_i \in \mathscr{C}^{p_i \times p_i}, \delta_j \in \mathscr{C}, \sum_i p_i + \sum_j r_j = \dim\{v_0\} = \dim\{v_i\}, 1 \le j \le m, 1 \le i \le l-1 \Big\}. \quad (10)$$



d : external signal vector (disturbances, measurement noise, reference signals)
y_c : controlled variable error (controlled variable - reference) vector
d' : normalized external signal vector (disturbances, measurement noise, reference signals)
y_c' : normalized controlled variable error vector
u : manipulated variable vector
y_m: noise-corrupt measured variable vector
v_i, v_o: internal variable vectors

Fig. 1. General block diagram of an uncertain system and a feedback controller.

(2) $P_{\Delta_u}(s)$ has the same number of right-half plane (RHP) poles as the nominal model $P_0(s)$.

We will refer to the set of systems satisfying the above conditions as p_{Π}. The above uncertainty type is said to be *structured* since Δ_u carried a specific block structure as opposed to being a single unstructured block. We assumed that each Δ_i is square without loss of generality since a nonsquare block can always be expressed in terms of a square block through the use of weighting matrices.

2.3. *Robust performance.* The closed-loop system is said to achieve *robust performance* if and only if $\mathscr{F}_l(P_{\Delta_u}, K)$ is stable $\forall P_{\Delta_u} \in P_\Pi$ and satisfies the *worst-case H_∞* performance condition

$$\max_{P_{\Delta_u} \in P_\Pi} \|\mathscr{F}_l(P_{\Delta_u}, K)\|_\infty < 1. \quad (11)$$

It can be shown (Doyle, 1984) that robust performance is achieved if and only if the closed-loop system is nominally stable ($\mathscr{F}_l(P_0, K)$ is stable) and

$$\mu_{\left[\begin{smallmatrix}\Delta_u & \\ & \Delta_p\end{smallmatrix}\right]}(M(j\omega)) < 1 \quad \forall\omega, \quad (12)$$

where

$$M = \begin{bmatrix} I & \\ & W_p \end{bmatrix} \mathscr{F}_l(G, K) \begin{bmatrix} I & \\ & W_d \end{bmatrix} \quad (13)$$

$$\Delta_p = \{\Delta_l : \Delta_l \in \mathscr{C}^{\dim\{d'\} \times \dim\{y_c'\}}\} \quad (14)$$

and W_d, W_p are the frequency dependent weights at the plant input and output, respectively.

Again, without loss of generality, we assume that Δ_p is a square block (i.e. $\dim\{y_c'\} = \dim\{d'\}$).

In this article, we will approximate μ by its upper bound. This is justified not only because the upper bound is very close to μ for most cases, but since it is used in most tests involving the numerical calculation of μ. Hence, expression

(12) is replaced with

$$\inf_{D \in \mathcal{D}_{rp}} \bar{\sigma}(DM(j\omega)D^{-1}) < 1 \quad \forall\omega, \tag{15}$$

where

$$\mathcal{D}_{rp} = \{\text{diag}\,[D_1, \ldots, D_m, d_1 I_{p_1}, \ldots, d_{l-1} I_{p_{l-1}}, d_l I_{\dim\{y_c^i\}}]: d_j \in \mathcal{R}_+; D_i \in \mathcal{C}^{r_i \times r_i}, D_i = D_i^* > 0\}. \tag{16}$$

3. *Design-independent screening tools*

In this section, we develop screening tools that can be used to eliminate control structure candidates for which no LTI controller exists meeting the robust performance requirements. First, we derive a necessary and sufficient (but untestable) condition for the existence of a controller achieving robust performance. Then, by relaxing the causality requirement of the controller, we show that we can derive necessary conditions for the existence of a controller achieving robust performance. These necessary conditions are formulated as convex optimizations and are proposed as screening tools.

3.1. *Test condition for existence of a causal controller achieving robust performance.* Our goal is to test whether or not there exists a controller meeting the robust performance requirement for a given set of actuators and measurements. Mathematically, we test if the following condition is satisfied:

$$\inf_{K \in \mathcal{K}_s} \sup_{\omega} \inf_{D(\omega) \in \mathcal{D}_{rp}} \bar{\sigma}\left(D(\omega)\left(\begin{bmatrix} I & \\ & W_p \end{bmatrix} \mathcal{F}_l(G^{ij}, K) \times \begin{bmatrix} I & \\ & W_d \end{bmatrix}\right)\Big|_{s=j\omega} D^{-1}(\omega)\right) < 1, \tag{17}$$

where G^{ij} denotes the plant model G with the ith set of actuators and the jth set of measurements. For simplicity of notation, we will drop the superscript $\{\cdot\}^{ij}$ from this point on. $\mathcal{K}_s$ represents the set of all stabilizing *causal* controllers. The causality of the controller implies that the controller's current/future inputs do not affect its past outputs; hence causality is required for the controller to be physically realizable. Mathematically, $\mathcal{K}_s$ is expressed as

$$\mathcal{K}_s \equiv \left\{K \in \mathcal{R}_s : \begin{bmatrix} (I - G_{33}K)^{-1} & G_{33}(I - KG_{33})^{-1} \\ K(I - G_{33}K)^{-1} & (I - KG_{33})^{-1} \end{bmatrix} \in \mathcal{RH}_\infty\right\}, \tag{18}$$

where $\mathcal{R}_s$ represents the set of all proper rational transfer functions (of size $\dim\{u\} \times \dim\{y_m\}$) and $\mathcal{RH}_\infty$ represents the set of all proper rational transfer functions (of appropriate size) that are analytic in the closed RHP. Note that K has nonlinear constraints and also enters M in a nonlinear fashion. The following parametrization of $\mathcal{K}_s$ (Youla *et al.*, 1976a, b) yields an affine parametrization of M without any nonlinear constraints:

$$\mathcal{K}_s = \{K : K = (Y - TQ)(X - SQ)^{-1}, Q \in \mathcal{RH}_\infty\} \tag{19}$$

$$= \{K : K = (\tilde{X} - Q\tilde{S})^{-1}(\tilde{Y} - Q\tilde{T}), Q \in \mathcal{RH}_\infty\}, \tag{20}$$

where (S, T) and $(\tilde{S}, \tilde{T})$ are right and left coprime factors of G_{33}, respectively (i.e. $G_{33} = ST^{-1} = \tilde{T}^{-1}\tilde{S}$), and $(X, Y, \tilde{X}, \tilde{Y})$ is a solution to the following Bezout identity:

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{S} & \tilde{T} \end{bmatrix}\begin{bmatrix} T & Y \\ S & X \end{bmatrix} = I. \tag{21}$$

Note that for open-loop stable systems we can choose $T = -\tilde{T} = -I$, $S = -\tilde{S} = -G_{33}$, $X = -\tilde{X} = I$ and $Y = \tilde{Y} = 0$; the parametrization (19) simply becomes

$$\mathcal{K}_s = \{K : K = Q(I + G_{33}Q)^{-1}, Q \in \mathcal{RH}_\infty\}.$$

Using the parametrization (19) and (20), (17) becomes

$$\inf_{Q \in \mathcal{RH}_\infty} \sup_{\omega} \inf_{D(\omega) \in \mathcal{D}_{rp}} \bar{\sigma}[D(\omega)(N_{11} + N_{12}QN_{21})|_{s=j\omega} D^{-1}(\omega)] < 1, \tag{22}$$

where

$$N_{11} = \begin{bmatrix} I & \\ & W_p \end{bmatrix}\left\{\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} + \begin{bmatrix} G_{13} \\ G_{23} \end{bmatrix} T\tilde{Y}[G_{31} \quad G_{32}]\right\} \times \begin{bmatrix} I & \\ & W_d \end{bmatrix} \tag{23}$$

$$N_{12} = \begin{bmatrix} I & \\ & W_p \end{bmatrix}\begin{bmatrix} G_{13} \\ G_{23} \end{bmatrix} T \tag{24}$$

$$N_{21} = \tilde{T}[G_{31} \quad G_{32}]\begin{bmatrix} I & \\ & W_d \end{bmatrix}. \tag{25}$$

Hence, the Youla parametrization leads to a closed-loop expression which is affine in the parameter Q. The only restriction on Q is that it should be analytic in the closed RHP. However, the coupling of the parameters Q and D makes the optimization required in expression (22) nonconvex. There is currently no method of checking condition (22).

It is worthwhile to mention that various methods are available enabling us to test whether nominal performance (i.e. when G_{11}, G_{12}, G_{21}, G_{23}, $G_{13} = 0$) can be achieved. According to the latest method by Doyle *et al.* (1989), testing this essentially amounts to checking if positive semidefinite solutions to two Riccati equations exist and the spectral radius of the product of the two solutions is less than a certain constant. These conditions can be used for design-independent screening, but their practical value is limited since they do not address one of the most important issues in control structure selection, namely model uncertainty.

3.2. *Test condition for existence of an acausal controller achieving robust performance.* At this point, let us consider dropping the *causality* requirement on Q. Hence, we allow the controller parameter Q to be *acausal,* meaning the current/future inputs of parameter Q can affect its past outputs. Clearly the set of all acausal controllers includes all causal controllers.

Mathematically, the relaxation of causality of Q is equivalent to replacing the requirement of $Q \in \mathcal{RH}_\infty$ with $Q \in \mathcal{R}_s$. The condition (22) with $Q \in \mathcal{R}_s$ is equivalent to the following frequency-by-frequency condition:

$$\inf_{Q \in \mathcal{C}^K} \inf_{D \in \mathcal{D}_{rp}} \bar{\sigma}(D(N_{11} + N_{12}QN_{21})|_{s=j\omega} D^{-1}) < 1 \quad \forall\omega. \tag{26}$$

The superscript $\{\cdot\}^K$ in $\mathcal{C}^K$ indicates that it is the set of complex matrices of size $\dim\{u\} \times \dim\{y_m\}$. Another interpretation of replacing $Q \in \mathcal{RH}_\infty$ with $Q \in \mathcal{R}_s$ in the context of a causal controller is that we relax the internal stability requirement.

Relaxation of the causality or stability requirement introduces conservativeness to the condition (i.e. satisfying condition (26) does not imply the existence of a causal K achieving robust performance), but the conservativeness is expected to be significant only around crossover. For example, condition (26) restricted to $\omega = 0$ is a necessary and sufficient condition for the existence of a controller gain matrix meeting the specified worst-case steady state requirement. For most chemical processes, such a condition can be a very useful screening tool since steady state error is often of primary importance.

Defining $\tilde{Q} = TQ\tilde{T} + T\tilde{Y}$ and noting that

$$\{\tilde{Q} : \tilde{Q} \in \mathcal{C}^K\} \equiv \{TQ\tilde{T} + T\tilde{Y}|_{s=j\omega} : Q \in \mathcal{C}^K\}$$

since $T(j\omega)$ is nonsingular for all ω, we arrive at the following necessary and sufficient condition for the existence of an *acausal* Q satisfying condition (22).

Theorem 1. Let N_{11}, N_{12} and N_{21} be defined as in (23)–(25). Then

$$\inf_{Q \in \mathcal{R}_s} \inf_{D \in \mathcal{D}_{rp}} \bar{\sigma}(D(N_{11} + N_{12}QN_{21})|_{s=j\omega} D^{-1}) < 1 \quad \forall\omega \tag{27}$$

if and only if

$$\inf_{\tilde{Q} \in \mathcal{C}^K} \inf_{D \in \mathcal{D}_{rp}} \bar{\sigma}(D(\tilde{N}_{11} + \tilde{N}_{12}\tilde{Q}\tilde{N}_{21})|_{s=j\omega} D^{-1}) < 1 \quad \forall\omega, \tag{28}$$

where

$$\tilde{N}_{11} = \begin{bmatrix} I & \\ & W_p \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} I & \\ & W_d \end{bmatrix} \quad (29)$$

$$\tilde{N}_{12} = \begin{bmatrix} I & \\ & W_p \end{bmatrix} \begin{bmatrix} G_{13} \\ G_{23} \end{bmatrix} \quad (30)$$

$$\tilde{N}_{21} = [G_{31} \quad G_{32}] \begin{bmatrix} I & \\ & W_d \end{bmatrix}. \quad (31)$$

Note that with the above reparametrization there is no need for finding the double coprime factor of G_{22} and solving the Bezout identity (21) since the expression for $\tilde{N}$ involves only G and frequency-dependent weighting matrices.

3.3. *Formulation of test conditions into screening tools.* So far, we have shown that (28) is a necessary condition for the existence of a controller achieving robust performance. In this section, we show that condition (28) can be transformed into two separate conditions which can be addressed via convex optimization.

We first reparametrize $\tilde{Q}$ such that the matrices pre- and post-multiplying $\tilde{Q}$ in condition (28) are both unitary. Note that $\tilde{Q} \in \mathscr{C}^K$ is equivalent to

$$\tilde{Q} \in \{(\tilde{N}_{12}^*\tilde{N}_{12})^{-1/2}\hat{Q}(\tilde{N}_{21}\tilde{N}_{21}^*)^{-1/2}|_{s=j\omega} : \hat{Q} \in \mathscr{C}^K\},$$

where $\{\cdot\}^*$ denotes the adjoint operator (i.e. $N^*(s) = N^T(-s)$). The notation $\{\cdot\}^*$ will also be used to represent the complex conjugate transpose for the case of a constant matrix. The condition (28) can be now transformed into

$$\inf_{\hat{Q} \in \mathscr{C}^K} \inf_{D \in \mathscr{D}_{rp}} \bar{\sigma}(D(\tilde{N}_{11} + \hat{N}_{12}\hat{Q}\hat{N}_{21})|_{s=j\omega} D^{-1}) < \forall\omega, \quad (32)$$

where $\hat{N}_{12} = \tilde{N}_{12}(\tilde{N}_{12}^*\tilde{N}_{12})^{-1/2}$ and $\hat{N}_{21} = (\tilde{N}_{21}\tilde{N}_{21}^*)^{-1/2}\tilde{N}_{21}$ are unitary matrices for all ω. The following theorem shows that the condition (32) can be checked through two conditions each of which is a convex optimization problem.

Theorem 2. Let $\alpha \in \mathscr{R}_+$, $R \in \mathscr{C}^{n \times n}$, $U \in \mathscr{C}^{n \times r}$ and $V \in \mathscr{C}^{t \times n}$. Suppose $U^*U = I_r$, $VV^* = I_t$ and $U_\perp \in \mathscr{C}^{n \times (n-r)}$ and $V_\perp \in \mathscr{C}^{(n-t) \times n}$ are chosen such that $[U \ U_\perp] \in \mathscr{C}^{n \times n}$ and

$$\begin{bmatrix} V \\ V_\perp \end{bmatrix} \in \mathscr{C}^{n \times n}$$

are unitary. Then

$$\inf_{Q \in \mathscr{C}^{r \times t}} \inf_{D \in \mathscr{D}_{rp}} \bar{\sigma}(D(R + UQV)D^{-1}) < \alpha \quad (33)$$

if and only if $\exists X \in \mathscr{D}_{rp}$ such that

$$\lambda_{max}[V_\perp(R^*XR - \alpha^2 X)V_\perp^*] < 0 \quad (34)$$

and

$$\lambda_{max}[U_\perp^*(RX^{-1}R^* - \alpha^2 X^{-1})U_\perp] < 0. \quad (35)$$

Proof. See Appendix.

Comments.

(1) Conditions (34) and (35) are convex with respect to X and X^{-1}, respectively. Each of the two conditions is a necessary condition for the existence of a controller achieving robust performance and can be checked through standard algorithms (Boyd and Barratt, 1991).

(2) Checking the conditions (34) and (35) together is more difficult and is not resolved at the moment except for the following special cases:

- *Full control case.* If U has a full row rank, condition (35) drops out and (34) is necessary *and* sufficient for (33).
- *Full information case.* If V has a full column rank, condition (34) drops out and (35) is necessary *and* sufficient for (33).
- 2 *full-block case.* For the case of 2 full-block Δ, (33) is

$$\inf_{Q \in \mathscr{C}^{r \times t}} \inf_{d_1, d_2 \in \mathscr{R}_+} \bar{\sigma}\left(\begin{bmatrix} d_1 I & \\ & d_2 I \end{bmatrix}(R + UQV)\begin{bmatrix} \frac{1}{d_1} I & \\ & \frac{1}{d_2} I \end{bmatrix}\right) < \alpha. \quad (36)$$

By multiplying and then dividing the expression by d_2 (36) becomes

$$\inf_{Q \in \mathscr{C}^{r \times t}} \inf_{d \in \mathscr{R}_+} \bar{\sigma}\left(\begin{bmatrix} dI & \\ & I \end{bmatrix}(R + UQV)\begin{bmatrix} \frac{1}{d} I & \\ & I \end{bmatrix}\right) < \alpha, \quad (37)$$

where $d = d_1/d_2$. Hence, for 2 full-block cases, conditions (34) and (35) can be expressed as follows:

$$g(d) \equiv \lambda_{max}\left[V_\perp\left(R^*\begin{bmatrix} dI & \\ & I \end{bmatrix}R - \alpha^2\begin{bmatrix} dI & \\ & I \end{bmatrix}\right)V_\perp^*\right] < 0 \quad (38)$$

$$h(1/d) \equiv \lambda_{max}\left[U_\perp^*\left(R\begin{bmatrix} 1/dI & \\ & I \end{bmatrix}R^* - \alpha^2\begin{bmatrix} 1/dI & \\ & I \end{bmatrix}\right)U_\perp\right] < 0. \quad (39)$$

$\mathscr{T}_{FC} \equiv \{s \in \mathscr{R}_+ : g(s) < 0\}$ and $\mathscr{T}_{FI} \equiv \{t \in \mathscr{R}_+ : h(1/t) < 0\}$ are open intervals (since $g(s)$ and $h(t)$ are convex with respect to s and t), so it can easily be checked if they intersect.

Using the results from Theorem 2 with $\alpha = 1$, we now propose the following screening tools:

Design-independent screening tool #1. Elimination control structure candidates for which

$$\mathscr{T}_{FC}(\omega) \cap \mathscr{T}_{FI}(\omega) = \varnothing \quad \text{for some } \omega, \quad (40)$$

where

$$\mathscr{T}_{FC}(\omega) = \left\{s \in \mathscr{R}_+ : \lambda_{max}\left[(\hat{N}_{21})_\perp\left((\tilde{N}_{11})^*\begin{bmatrix} sI & \\ & I \end{bmatrix}\tilde{N}_{11} - \begin{bmatrix} sI & \\ & I \end{bmatrix}\right)(\hat{N}_{21})_\perp^*\Big|_{s=j\omega}\right] < 0\right\} \quad (41)$$

$$\mathscr{T}_{FI}(\omega) = \left\{t \in \mathscr{R}_+ : \lambda_{max}\left[(\hat{N}_{12})_\perp^*\left(\tilde{N}_{11}\begin{bmatrix} \frac{1}{t} I & \\ & I \end{bmatrix}(\tilde{N}_{11})^* - \begin{bmatrix} \frac{1}{t} I & \\ & I \end{bmatrix}\right)(\hat{N}_{12})_\perp\Big|_{s=j\omega}\right] < 0\right\} \quad (42)$$

for any combination of two of the given Δ blocks.

Design-independent screening tool #2. Eliminate control structures for which

$$\inf_{X \in \mathscr{D}_{rp}} \lambda_{max}[(\hat{N}_{21})_\perp(\tilde{N}_{11}^*X\tilde{N}_{11} - X)(\hat{N}_{21})_\perp^*|_{s=j\omega}] \geq 0 \quad \text{for some } \omega. \quad (43)$$

Design independent screening tool #3. Eliminate control structures for which

$$\inf_{X \in \mathscr{D}_{rp}} \lambda_{max}[(\hat{N}_{12})_\perp^*(\tilde{N}_{11}X\tilde{N}_{11}^* - X)(\hat{N}_{12})_\perp|_{s=j\omega}] \geq 0 \quad \text{for some } \omega. \quad (44)$$

We note that the above screening tools, although manageable, are numerically more complex than conventional tools like the RGA or the condition number. However, these other tools do not address the issue of uncertainty in a general rigorous way like the tools above. Examples illustrating the importance of considering uncertainty (and the structure of the uncertainty) when selecting actuators and sensors are given, for example, by Skogestad *et al.* (1988) and Lee and Morari (1991).

4. *Comparison with other screening tools: multicomponent distillation*

We apply the screening tools to a multi-component distillation column control problem studied by Weber and Brosilow (1972). We compare the proposed tools with Brosilow's criteria because these are well-known to many process control researchers, and the papers describing these tools are widely referenced and are considered by many to be classics in the field. We will discuss how Brosilow's criteria (and a generalized version useful for comparison with our criteria) leads to a counter-intuitive result. On the other hand, the new screening tools lead to physically consistent results and are helpful in analyzing the sensitivity of various control structures to uncertainty.

4.1. *Problem description.* The schematic diagram of the column and proposed control configuration is shown in Fig. 2. It is a 16 stage, five component distillation column with a total condenser and a total reboiler. The detailed information on the operating conditions and modeling assumptions can be found in Brosilow and Tong (1978). The control objective is to maintain constant overhead and bottom product compositions (y_D and x_B, respectively) in the presence of feed disturbances. The manipulated variables are the reflux ratio (L) and the vapor boilup rate (V). The temperature measurements are available on the 1st, 3rd, 8th, 14th and 16th trays (T_1, T_3, T_8, T_{14} and T_{16}, respectively) of the column, where T_1 is located at the bottom of the column. The model for the input–output relationships between disturbances/manipulated variables and controlled/measured variables are as follows:

	d_1	d_2	d_3	d_4	d_5	L	V
y_D	$\frac{-0.188}{72s+1}$	$\frac{-0.163}{72s+1}$	$\frac{0.0199}{70s+1}$	$\frac{0.0043}{80s+1}$	$\frac{0.002}{85s+1}$	$\frac{-0.173}{70s+1}$	$\frac{0.0305}{75s+1}$
x_B	$\frac{0.0174}{15s+1}$	$\frac{0.0259}{13s+1}$	$\frac{0.0045}{4s+1}$	$\frac{-0.00029}{3s+1}$	$\frac{-0.00099}{3s+1}$	$\frac{0.015}{18s+1}$	$\frac{-0.00768}{7s+1}$
T_1	$\frac{-7.99}{9s+1}$	$\frac{-9.78}{9s+1}$	$\frac{-5.28}{5s+1}$	$\frac{3.59}{8s+1}$	$\frac{6.09}{5s+1}$	$\frac{7.47}{8s+1}$	$\frac{2.70}{4s+1}$
T_3	$\frac{-11.29}{12s+1}$	$\frac{-15.91}{12s+1}$	$\frac{-4.23}{5s+1}$	$\frac{3.63}{8s+1}$	$\frac{4.75}{5s+1}$	$\frac{9.80}{15s+1}$	$\frac{3.79}{5s+1}$
T_8	$\frac{-18.28}{5s+1}$	$\frac{-16.43}{10s+1}$	$\frac{-0.47}{5s+1}$	$\frac{3.96}{3s+1}$	$\frac{4.60}{1.5s+1}$	$\frac{8.20}{30s+1}$	$\frac{2.30}{18s+1}$
T_{14}	$\frac{-42.02}{50s+1}$	$\frac{-35.92}{70s+1}$	$\frac{4.45}{65s+1}$	$\frac{1.10}{70s+1}$	$\frac{0.46}{75s+1}$	$\frac{36.0}{65s+1}$	$\frac{6.82}{70s+1}$
T_{16}	$\frac{-5047}{25s+1}$	$\frac{-25.26}{75s+1}$	$\frac{3.15}{70s+1}$	$\frac{0.68}{78s+1}$	$\frac{0.32}{80s+1}$	$\frac{30.0}{67s+1}$	$\frac{3.46}{70s+1}$

. (45)

Owing to space constraints, we limit ourselves to the following combinations of two temperature measurements:

$$y_m^1=\begin{pmatrix}T_1\\T_3\end{pmatrix}\quad y_m^2=\begin{pmatrix}T_1\\T_8\end{pmatrix}\quad y_m^3=\begin{pmatrix}T_1\\T_{14}\end{pmatrix}\quad y_m^4=\begin{pmatrix}T_1\\T_{16}\end{pmatrix}$$

$$y_m^5=\begin{pmatrix}T_3\\T_8\end{pmatrix}\quad y_m^6=\begin{pmatrix}T_3\\T_{14}\end{pmatrix}\quad y_m^7=\begin{pmatrix}T_3\\T_{16}\end{pmatrix}\quad y_m^8=\begin{pmatrix}T_8\\T_{14}\end{pmatrix}$$

$$y_m^9=\begin{pmatrix}T_8\\T_{16}\end{pmatrix}\quad y_m^{10}=\begin{pmatrix}T_{14}\\T_{16}\end{pmatrix}.$$

4.2. *Reformulation of Brosilow's criteria.* Without loss of generality, we assume that W_d is chosen as a scalar-times-identity (kI) for the discussion in this section.

Brosilow and co-workers (Weber and Brosilow, 1972; Joseph and Brosilow, 1978) suggested the following two *steady-state* criteria for measurement selection:

(1) *Minimization of projection error (nominal estimation error).* Minimize the projection error ε_∞, where

$$\varepsilon_\infty = \bar{\sigma}(R), \tag{46}$$

where

$$R = G_{y_cd} - G_{y_cd}G_{y_md}^{\mathrm{T}}(G_{y_md}G_{y_md}^{\mathrm{T}})^{-1}G_{y_md}. \tag{47}$$

(2) *Minimization of condition number (sensitivity to modeling error).* Minimize the condition number κ of G_{y_md}, where

$$\kappa(G_{y_md}) = \frac{\bar{\sigma}(G_{y_md})}{\underline{\sigma}(G_{y_md})}. \tag{48}$$

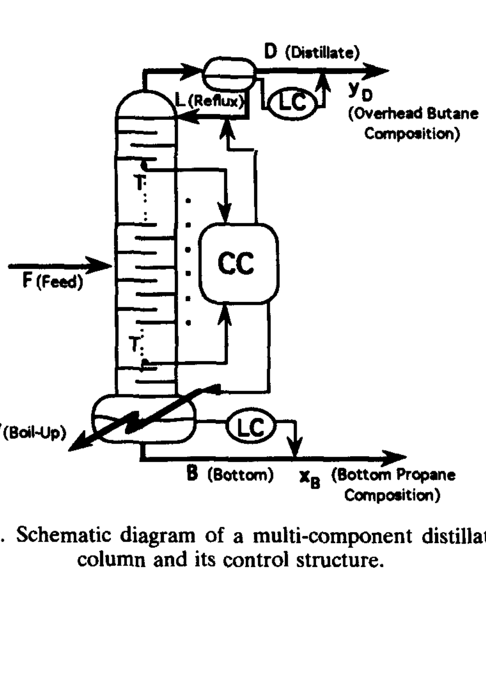


Fig. 2. Schematic diagram of a multi-component distillation column and its control structure.

They indicate that (46) tends to decrease and (48) tends to increase as the number of the measurements is increased, and leave the final tradeoff to engineering judgment. We note that the projection error as originally defined by Brosilow and coworkers is not that of equation (46), but

$$\varepsilon_2 = \sqrt{\frac{\operatorname{trace}\{RR^{\mathrm{T}}\}}{\operatorname{trace}\{G_{y_cd}G_{y_cd}^{\mathrm{T}}\}}}. \tag{49}$$

The original definition of the projection error is appropriate in the stochastic setting since it can be interpreted as the relative ratio between the closed-loop and the open-loop variances of the output when the disturbance vector is a zero-mean random variable with a scalar-times-identity covariance matrix (i.e. $E\{d\}=0$, $E\{dd^{\mathrm{T}}\}=k^2I$). Note that for measurement selection minimizing ε_2 is the same as minimizing $\sqrt{\operatorname{trace}\{RR^{\mathrm{T}}\}}$ since $\sqrt{\operatorname{trace}\{G_{y_cd}G_{y_cd}^{\mathrm{T}}\}}$ is independent of measurements. In the worst-case error setting of H_∞ control, ε_∞ is an appropriate generalization of the term $\sqrt{\operatorname{trace}\{RR^{\mathrm{T}}\}}$ in equation (49), since it is the maximum attainable 2-norm of y_c for all d such that $\|d\|_2 < 1$.

Brosilow's criteria may be justified by deriving the expression for the worst-case uncertainty under a particular uncertainty structure. Suppose that the model error on G_{y_md} can be described as follows:

$$\{G_{y_md}\}_{\mathrm{true}} = (I + w\Delta)G_{y_md};$$
$$\Delta \in \mathbf{\Delta} \equiv \{\Delta \in \mathcal{R}^{\dim\{y_m\}\times\dim\{y_m\}} : \bar{\sigma}(\Delta) \le 1\}, \tag{50}$$

where w is a real positive scalar indicating the magnitude of the uncertainty. Furthermore, assume that the least-squares type controller will be used. More precisely, K is to be designed such that

$$K(0) = Q_{1s}(I + G_{y_m u}(0)Q_{1s})^{-1} \tag{51}$$

$$Q_{1s} = (G_{y_c u})_r^{-1} G_{y_c d} G_{y_m d}^{\mathrm{T}} (G_{y_m d} G_{y_m d}^{\mathrm{T}})^{-1}. \tag{52}$$

The above choice of $K(0)$ minimizes the steady-state error variance of the output y_c in the presence of random step disturbances d (d is an *integrated* white noise of a scalar-times-identity covariance matrix). Here, we assumed that $(G_{y_c u})_r^{-1}$, a right inverse of $G_{y_c u}$, exists. When $G_{y_c u}$ does not have a full column rank, $(G_{y_c u})_r^{-1}$ should be replaced by $(G_{y_c u}^{\mathrm{T}} G_{y_c d})^{-1} G_{y_c u}^{\mathrm{T}}$. However, we do not consider this case in order to simplify the derivation. The closed-loop expression from d to y_c with the above choice of K is as follows:

$$\begin{aligned}\mathscr{F}_{y_c d}(0) = {} & [G_{y_c d} - G_{y_c d} G_{y_m d}^{\mathrm{T}} (G_{y_m d} G_{y_m d}^{\mathrm{T}})^{-1} G_{y_m d}] \\ & - w[G_{y_c d} G_{y_m d}^{\mathrm{T}} (G_{y_m d} G_{y_m d}^{\mathrm{T}})^{-1} \Delta G_{y_m d}].\end{aligned} \tag{53}$$

Hence, the worst possible 2-norm of the output y_c for $\|d\|_2 < 1$ is expressed as

$$\begin{aligned}& \max_{\Delta \in \boldsymbol{\Delta}} \bar{\sigma}(\mathscr{F}_{y_c d}(0)) \\ & \quad \le \varepsilon_\infty + w \max_{\Delta \in \boldsymbol{\Delta}} \bar{\sigma}[G_{y_c d} G_{y_m d}^{\mathrm{T}} (G_{y_m d} G_{y_m d}^{\mathrm{T}})^{-1} \Delta G_{y_m d}] \\ & \quad = \varepsilon_\infty + w\bar{\sigma}[G_{y_c d} G_{y_m d}^{\mathrm{T}} (G_{y_m d} G_{y_m d}^{\mathrm{T}})^{-1}] \bar{\sigma}[G_{y_m d}] \\ & \quad \le \varepsilon_\infty + w\bar{\sigma}[G_{y_c d}] \bar{\sigma}[G_{y_m d}^{\mathrm{T}} (G_{y_m d} G_{y_m d}^{\mathrm{T}})^{-1}] \bar{\sigma}[G_{y_m d}] \\ & \quad = \varepsilon_\infty + w' \kappa(G_{y_m d}) \quad (\text{where } w' = w\bar{\sigma}(G_{y_c d})).\end{aligned} \tag{54}$$

Hence, minimizing a weighted sum of the projection error and the condition number of $G_{y_m d}$ corresponds to minimizing an upper bound of the worst-case closed-loop error. The original derivation of the Condition Number Criterion in a stochastic optimal control setting by Brosilow and co-workers also assumed that the least-squares controller would be used (their uncertainty description, however, is somewhat different). While Brosilow and co-workers left balancing the projection error and condition number to engineering judgement, we have derived here a suitable scalar measure combining the two quantities.

4.3. *Application to the multi-component column.* If we assume the same uncertainty description as above, then the SSV test for robust performance involves 2-block Δ ($\Delta_{2\times 2}$ and Δ_p). Therefore, we can apply the Design-Independent Screening Tool #1 proposed in Section 3.3. To compare with Brosilow's criteria, we will apply the tool at steady state. In this case, the screening tool can be viewed as a necessary and sufficient condition for the existence of K satisfying a given worst-case closed-loop error bound on the output. Instead of simply checking if a specific worst-case error bound can be satisfied for each measurement set, we calculated its achievable worst-case error, that is

$$\inf_K \max_{\Delta \in \boldsymbol{\Delta}} \bar{\sigma}(\mathscr{F}_{y_c d}(0)). \tag{55}$$

This can be easily done by multiplying $G_{y_c d}$ with a real positive scalar c_p and increasing it just enough such that the Screening Tool #1 is no longer satisfied. The achievable worst-case error is the inverse of this particular value of c_p. The results computed for unstructured output uncertainty (50) with $w = 0.1$ are shown in Table 1. The table also shows its upper bound derived from Brosilow's criteria (i.e. $\varepsilon_\infty + w'\kappa(G_{y_m d})$). In comparison, we observe that the new method provides nonconservative measures of the achievable worst-case error. The conservativeness of the upper bound stems not only from the inequalities in the derivation (see expression (54)), but also from the fact that it assumes a least-squares type controller.

This conservatism can cause the upper bound criterion to select measurement sets which are not physically intuitive. This can be seen in Table 1, where the upper bound criterion selects y_m^2 consisting of T_1 and T_8 as the best set. This is counter-intuitive, since the distillate composition estimates are expected to be poor with such a measurement set. Lee *et al.* (1993) show that an additional counter-intuitive prediction which is shared by both the upper bound and Brosilow's Condition Number Criterion is that adding measurements can degrade the predicted achievable closed-loop performance. The main reason for this incorrect prediction is that these criteria require that the controller be least-squares optimal. Least-squares optimal controllers will always be sensitive to model/plant mismatch when sufficiently many temperature measurements are taken.

On the other hand, the new method selects y_m^3 consisting of T_1 and T_{14} as the best measurement set. It is expected that tray temperature measurements must be taken at both ends of the column for good composition estimates. The above uncertainty description (50) was used for comparison purposes. Lee *et al.* (1993) show that applying the new method with a more physically meaningful uncertainty description selects y_m^6 consisting of T_3 and T_{14} as the best set, which agrees with physical intuition and with the usual industrial practice for such a column (see, for example, Lee and Morari, 1991).

5. *Conclusions*

A general framework is formulated for selecting actuators and sensors for control purposes. We proposed that a large number of control structure candidates arising from the combinatorial nature of the problem be reduced down to a manageable level through two-stage screening: *design-independent* screening that is independent of the controller design method and *design-dependent* screening which is tied to a specific type of controller design method. Design-independent screening tools are developed which can be calculated via convex optimization. The tools can be used to eliminate candidates for which no linear time-invariant controller exists satisfying a given H_∞ performance specification under structured uncertainty. The application of the screening tools to a multi-component distillation column

Table 1. The achievable worst-case steady-state error and its upper bound computed from the projection error and condition number of $G_{y_m d}$ for various measurement sets under unstructured output uncertainty with $w = 0.1$

Measurement candidate	Worst-case error $\inf_K \max_{\Delta \in \boldsymbol{\Delta}} \bar{\sigma}(\mathscr{F}_{y_c d}(0))$	Upper bound $\varepsilon_\infty + w'\kappa(G_{y_m d})$	Projection error ε_∞	Condition number $\kappa(G_{y_m d})$
y_m^1	0.1031	0.2805	0.0804	7.960
y_m^2	0.0543	0.1595	0.0341	4.989
y_m^3	0.0227	0.1617	0.0086	6.088
y_m^4	0.0625	0.1976	0.0565	5.613
y_m^5	0.0671	0.2302	0.0530	7.047
y_m^6	0.0260	0.1732	0.0066	6.625
y_m^7	0.0518	0.1793	0.0431	5.149
y_m^8	0.0264	0.2823	0.0103	10.821
y_m^9	0.0538	0.2322	0.0436	7.501
y_m^{10}	0.0331	0.2128	0.0077	8.157

revealed some useful insights while previously existing criteria led to inconsistent results. Although we have not discussed design-dependent screening tools in this paper, several such screening tools are discussed in Lee and Morari (1991, 1994).

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Appendix

Proof of Theorem 2.

$$\inf_{Q \in \mathscr{C}^{r\times t}} \bar{\sigma}[D(R+UQV)D^{-1}] = \inf_{Q \in \mathscr{C}^{r\times t}} \bar{\sigma}[DRD^{-1} + (DU)Q(VD)^{-1}]. \quad \text{(A.1)}$$

We first make the terms pre- and post-multiplying Q unitary by replacing $Q \in \mathscr{C}^{r\times t}$ with

$$Q \in \{[(DU)^*(DU)]^{-1/2}\tilde{Q}[(VD^{-1})(VD^{-1})^*]^{-1/2} \colon \tilde{Q} \in \mathscr{C}^{r\times t}\}.$$

Then,

$$\inf_{Q \in \mathscr{C}^{r\times t}} \bar{\sigma}[D(R+UQV)D^{-1}] = \inf_{\tilde{Q} \in \mathscr{C}^{r\times t}} \bar{\sigma}(DRD^{-1} + \hat{U}\tilde{Q}\hat{V}), \quad \text{(A.2)}$$

where $\hat{U} = (DU)[(DU)^*(DU)]^{-1/2}$ and

$$\hat{V} = [(VD^{-1})(VD^{-1})^*]^{-1/2}(VD^{-1}).$$

We want to find $\hat{U}_\perp$ and $\hat{V}_\perp$ such that $[\hat{U}\ \hat{U}_\perp]$ and $\begin{bmatrix}\hat{V}\\ \hat{V}_\perp\end{bmatrix}$ are both unitary. Simple calculation shows that

$$\hat{U}_\perp = (D^*)^{-1}U_\perp(U_\perp^*(D^*D)^{-1}U_\perp)^{-1/2}$$

and

$$\hat{V}_\perp = (V_\perp D^* D V_\perp^*)^{-1/2} V_\perp D^*.$$

Now

$$\begin{aligned}
&\inf_{\tilde{Q} \in \mathscr{C}^{r\times t}} \bar{\sigma}(DRD^{-1} + \hat{U}\tilde{Q}\hat{V}) \\
&= \inf_{\tilde{Q} \in \mathscr{C}^{r\times t}} \bar{\sigma}\left(DRD^{-1} + [\hat{U}\ \ \hat{U}_\perp]\begin{bmatrix}\tilde{Q} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{V}\\ \hat{V}_\perp\end{bmatrix}\right) && \text{(A.3)}\\
&= \inf_{\tilde{Q} \in \mathscr{C}^{r\times t}} \bar{\sigma}\left([\hat{U}\ \ \hat{U}_\perp]^* DRD^{-1}\begin{bmatrix}\hat{V}\\ \hat{V}_\perp\end{bmatrix}^* + \begin{bmatrix}\tilde{Q} & 0\\ 0 & 0\end{bmatrix}\right) && \text{(A.4)}\\
&= \inf_{\tilde{Q} \in \mathscr{C}^{r\times t}} \bar{\sigma}\left(\begin{bmatrix}\tilde{R}_{11}+\tilde{Q} & \tilde{R}_{12}\\ \tilde{R}_{21} & \tilde{R}_{22}\end{bmatrix}\right), && \text{(A.5)}
\end{aligned}$$

where $\tilde{R}_{11} = \hat{U}^*DRD^{-1}\hat{V}^*$, $\tilde{R}_{12} = \hat{U}^*DRD^{-1}\hat{V}_\perp^*$, $\tilde{R}_{21} = \hat{U}_\perp^*DRD^{-1}V^*$ and $\tilde{R}_{22} = \hat{U}_\perp^*DRD^{-1}\hat{V}_\perp^*$. From Doyle (1984)

$$\inf_{\tilde{Q} \in \mathscr{C}^{r\times t}} \bar{\sigma}\left(\begin{bmatrix}\tilde{R}_{11}+\tilde{Q} & \tilde{R}_{12}\\ \tilde{R}_{21} & \tilde{R}_{22}\end{bmatrix}\right) = \max\left\{\bar{\sigma}([\tilde{R}_{21}\ \ \tilde{R}_{22}]), \bar{\sigma}\left(\begin{bmatrix}\tilde{R}_{12}\\ \tilde{R}_{22}\end{bmatrix}\right)\right\}. \quad \text{(A.6)}$$

Hence, the condition (33) is satisfied if and only if there exists $D \in \mathscr{D}_{rp}$ such that

$$\bar{\sigma}([\tilde{R}_{21}\ \ \tilde{R}_{22}]) < \alpha \quad \text{and} \quad \bar{\sigma}\left(\begin{bmatrix}\tilde{R}_{12}\\ \tilde{R}_{22}\end{bmatrix}\right) < \alpha. \quad \text{(A.7)}$$

Now

$$\begin{aligned}
&\bar{\sigma}([\tilde{R}_{21}\ \ \tilde{R}_{22}])\\
&= \bar{\sigma}(\hat{U}_\perp^* DRD^{-1}[\hat{V}^*\ \ \hat{V}_\perp^*]) && \text{(A.8)}\\
&= \bar{\sigma}(\hat{U}_\perp^* DRD^{-1}) && \text{(A.9)}\\
&= \bar{\sigma}[((D^*)^{-1}U_\perp(U_\perp^*(D^*D)^{-1}U_\perp)^{-1/2})^* DRD^{-1}] && \text{(A.10)}\\
&= \bar{\sigma}[(U_\perp^*(D^*D)^{-1}U_\perp)^{-1/2}U_\perp^* RD^{-1}]. && \text{(A.11)}
\end{aligned}$$

Similarly, one can show that

$$\bar{\sigma}\left(\begin{bmatrix}\tilde{R}_{12}\\ \tilde{R}_{22}\end{bmatrix}\right) = \bar{\sigma}[DRV_\perp^*(V_\perp D^*(DV_\perp^*))^{-1/2}]. \quad \text{(A.12)}$$

Now

$$\begin{aligned}
&\bar{\sigma}[(U_\perp^*(D^*D)^{-1}U_\perp)^{-1/2}U_\perp RD^{-1}] < \alpha && \text{(A.13)}\\
&\leftrightarrow \lambda_{\max}[(U_\perp^*(D^*D)^{-1}U_\perp)^{-1/2}U_\perp^* R(D^*D)^{-1}R^*U_\perp\\
&\qquad \times (U_\perp^*(D^*D)^{-1}U_\perp)^{-1/2} - \alpha^2 I] < 0\\
&\leftrightarrow \lambda_{\max}[U_\perp^* R(D^*D)^{-1}R^*U_\perp - \alpha^2 U_\perp^*(D^*D)^{-1}U_\perp] < 0 && \text{(A.14)}\\
&\leftrightarrow \lambda_{\max}[U_\perp^*(R(D^*D)^{-1}R^* - \alpha^2(D^*D)^{-1})U_\perp] < 0. && \text{(A.15)}
\end{aligned}$$

Likewise

$$\begin{aligned}
&\bar{\sigma}[DRV_\perp^*(V_\perp D^*DV_\perp^*)^{-1/2}] < \alpha\\
&\leftrightarrow \lambda_{\max}[V_\perp(R^*(D^*D)^{-1}R - \alpha^2(D^*D)^{-1})V_\perp^*] < 0. && \text{(A.16)}
\end{aligned}$$

Defining $X = D^*D$ completes the proof. QED