Neural Networks 98 (2018) 251–262

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Standard representation and unified stability analysis for dynamic artificial neural network models[☆]

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ARTICLE INFO

Article history:
Received 28 May 2017
Received in revised form 25 September 2017
Accepted 20 November 2017
Available online 2 December 2017

Keywords:
Dynamic artificial neural networks
Nonlinear stability analysis
Absolute stability
Semidefinite programming
Linear matrix inequalities

ABSTRACT

An overview is provided of dynamic artificial neural network models (DANNs) for nonlinear dynamical system identification and control problems, and convex stability conditions are proposed that are less conservative than past results. The three most popular classes of dynamic artificial neural network models are described, with their mathematical representations and architectures followed by transformations based on their block diagrams that are convenient for stability and performance analyses. Classes of nonlinear dynamical systems that are universally approximated by such models are characterized, which include rigorous upper bounds on the approximation errors. A unified framework and linear matrix inequality-based stability conditions are described for different classes of dynamic artificial neural network models that take additional information into account such as local slope restrictions and whether the nonlinearities within the DANNs are odd. A theoretical example shows reduced conservatism obtained by the conditions.

1. Introduction

Black-box identification of nonlinear dynamical systems using artificial neural network (ANN) models have been investigated since the 1980s, with a strong motivation coming from the ability of ANNs to universally approximate static nonlinear functions (Cybenko, 1989; Funahashi, 1989; Hornik, 1989). Cybenko (1989) and Funahashi (1989) proved that an ANN with only one hidden layer can uniformly approximate any continuous function whereas Hornik (1989) studied the universal approximation property of multi-layer ANNs. Subsequent papers showed that the functional range of an ANN is dense for different activation functions (Park & Sandberg, 1991).

This article[1] starts with considering three popular classes of black-box nonlinear dynamical models:

- Neural State-Space Model (NSSM): This state-space model parameterization for a nonlinear dynamical system has nonlinearities parameterized by multilayer feedforward artificial neural networks (FANNs) with one hidden layer;
- Global Input–Output Model (GIOM): This recursive input–output parameterization for a nonlinear dynamical system has nonlinearities parameterized by FANNs;
- Dynamic Recurrent Neural Network (DRNN): This structure is the same as NSSM except with an additional linear recursive term in the state equation.

Stability analysis and controller synthesis based on robust control theory has been extensively studied for NSSMs (Suykens, Moor, & Vandewalle, 1995; Suykens, Vandewalle, & Moor, 1996), which can be rather parsimonious models for some nonlinear dynamical systems. The GIOM allows the future outputs of the dynamical system to be determined purely from a finite number of past observations of the system's measured inputs and outputs (Billings, Jamaluddin, & Chen, 1992; Levin & Narendra, 1995a; Narendra & Parthasarthy, 1990). Since both inputs and outputs to the network are directly observable at each instant of time, static backpropagation or any other supervised training method of system identification can be used to train the network. The application to the adaptive control has been extensively studied (Antsaklis, 1990; Ge & Wang, 2002; Narendra & Mukhopadhyay, 1997). Although similar to NSSMs, DRNNs are more extensively studied in the literature, including

☆ Preliminary versions of the results in this work were presented in Kim et al. (2011a, 2011b).

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1 Preliminary versions of the results of this work were presented in conference proceedings (Kim, Patrón, & Braatz, 2011a, 2011b).

https://doi.org/10.1016/j.neunet.2017.11.014
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for large FANNs (Hopfield, 1982; Pineda, 1989). DRNNs have been argued as being well suited for modeling associative memories, and the identification and analysis of DRNNs have been extensively investigated in recent years (Fang & Kincaid, 1996; Grujic & Michel, 1991; Jin, Nikiforuk, & Gupta, 1995; Michel, Farrel, & Porod, 1989).

The ability of the three classes of DANNs to universally approximate nonlinear dynamical systems rely on the universal approximation capability of ANNs for static nonlinear functions. While many results have been reported in the literature, this paper presents all three DANN models in a common representation while filling in the theoretical gaps in the literature, which serves as a capstone to the topic. The common representation is argued to be a useful model structure in its own right, due to its inheritance of all of the universal approximation properties with error bounds derived for the three DANN models.

This paper also considers the stability analysis of DANN models, which is a topic that has been investigated by many researchers (see Michel & Liu, 2002; Suykens et al., 1996 and references cited therein). Stability analysis and the convergence of the state trajectories to equilibria have been studied for DRNNs, with sufficient stability conditions derived using diagonal quadratic Lyapunov functions (Michel et al., 1989) and matrix measures (Fang & Kincaid, 1996). NSSMs have been analyzed by reformulation as NL_q systems for which sufficient conditions for global asymptotic stability (g.a.s.) and input–output stability can be applied (Suykens et al., 1996). However, existing stability conditions are problem-dependent in the sense of being applicable only to specific structures of parameterized models, not to a general representation of DANN models. To construct unified analysis tools, we show that any DANN can be represented in a standard nonlinear operator form (SNOF) and we derive polynomial-time sufficient conditions for the stability of a DANN based on its corresponding SNOF. We also show how existing results in literature can be applied to stability analysis, which are compared to the new stability conditions from a theoretical point of view and with a numerical example.

This paper is organized as follows. Section 2 presents some mathematical notation, definitions, and preliminaries on ANNs and DANNs. Section 3 presents the mathematical descriptions for the three different classes of DANNs with a common block diagram representations to help the reader understand their structures and differences. The approximation properties for each model are given with proofs. Section 4 argues that the common block diagram representation could form the basis for the development of new process identification algorithms and shows the representations of DANNs in terms of the SNOF. Section 5 uses a modified Lur'e–Postnikov function to produce less conservative conditions for g.a.s. for the different classes of SNOFs that represent the DANNs. Section 6 discusses several stability conditions that are applicable to the DANNs. Section 7 concludes the paper.

2. Background

2.1. Mathematical notations and definitions

The notation used in this paper is standard: $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of all nonnegative integers and the set of all nonnegative real numbers, respectively; $\|\cdot\|$ is the Euclidean norm for vectors, or the corresponding induced matrix norm for matrices; 0 and I denote the null matrix whose components are all zeros and the identity matrix of compatible dimension, respectively; the superscript T denotes the transpose of a matrix; ℓ_2^n is the set of all measurable essentially bounded functions from $\mathbb{Z}_+$ to $\mathbb{R}^n$ with ℓ_2^n-norm defined by $\|f\|_{\ell_2^n} \triangleq \sum_{k=0}^{\infty}\|f(k)\| < \infty$ and ℓ_∞^n is the set of all measurable essentially bounded functions from $\mathbb{Z}_+$ to $\mathbb{R}^n$ with ℓ_∞^n-norm defined by $\|f\|_{\ell_\infty^n} \triangleq \max_{1\le i\le n}\left\{\sup_{k\ge 0}|f_i(k)|\right\} < \infty$, where the subscript i denotes the ith element of a vector. For a truncated signal, $f_{[0,\kappa]}(k)$ is defined to have the same value as $f(k)$ at $k \in [0,\kappa]$, $\kappa < \infty$, and is zero for all $k > \kappa$; $\mathcal{C}^p(\mathcal{X})$ denotes the set of p-times continuously differentiable functions on an open set $\mathcal{X}$; $\mathcal{L}^1(\mathcal{X})$ denotes the set of integrable functions on an open set $\mathcal{X}$; $\mathcal{X}^p$ denotes the Cartesian product of $\mathcal{X}$ with itself p times; ι^p is the p-Cartesian product of the interval [0, 1]. The maximum singular value of a matrix $M \in \mathbb{R}^{n\times n}$ is denoted by $\bar{\sigma}(M)$ and the spectral radius of M is denoted by $\rho(M)$. If $M = M^{\mathrm{T}}$ then all the eigenvalues of M are real and $\lambda_{\max}(M)$ denoted the largest eigenvalue. If $M = M^{\mathrm{T}}$ then $M > 0$ and $M < 0$ denote the matrix is positive and negative definite, respectively. Let n and m be positive integers and partition $M \in \mathbb{R}^{(n+m)\times(n+m)}$ as $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. For a matrix $\Delta \in \mathbb{R}^{m\times m}$ such that $\mathrm{I}-D\Delta$ is invertible, define the linear fractional transformation $F_l(M,\Delta) \triangleq A+B\Delta(\mathrm{I}-D\Delta)^{-1}C$. This transformation can be used to define an uncertain autonomous discrete-time system: $x_{k+1} = F_l(M,\Delta)x_k$. Similarly, for $\Omega \in \mathbb{R}^{n\times n}$ such that $\mathrm{I}-A\Omega$ is invertible, define $F_u(M,\Omega) \triangleq D + C\Omega(\mathrm{I}-A\Omega)^{-1}B$. This transformation can be used to define a transfer function matrix, e.g., $G(z) := D + C(z\mathrm{I}-A)^{-1}B = F_u(M, \frac{1}{z}\mathrm{I})$. Define the system norm $\|G\|_\infty \triangleq \max_{0\le\theta\le 2\pi}\bar{\sigma}(G(e^{j\theta}))$ induced by signal 2-norms on the input and output vectors.

The notation of Levin and Narendra (1995b) will be used to describe an ANN: An ANN with only forward connections (called a forward ANN, or FANN) containing L layers of neurons with $(L-2)$ hidden layers, each one with $i_2, i_3, \ldots, i_{L-1}$ neurons respectively, is represented by $\mathcal{N}^{L-1}_{i_1,i_2,\ldots,i_L}$. This network has i_1 inputs and i_L outputs. Therefore, a FANN with 3 inputs, 4 neurons at the hidden layer, and 2 outputs is represented by $\mathcal{N}^2_{3,4,2}$.

This paper considers discrete-time nonlinear dynamical systems of the form $x_{k+1} = f(x_k, u_k, k)$, $y_k = g(x_k, u_k, k)$, where f is locally Lipschitz in all of its arguments, g is continuous in all of its arguments such that the existence of unique solution x is guaranteed, and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$ are the state, control input, and output, respectively. The Lipschitz condition for the state transition map f and the continuity of the output map g are common assumptions used to guarantee the existence of unique solution and assumed to hold for all system equations in this paper, although it might appear as different forms. The subscript k denotes the time instant $k \in \mathbb{Z}_+$.

2.2. Architecture of artificial neural networks

An Artificial Neural Network (ANN) is capable of arbitrarily closely approximating nonlinear functional relationships between bounded input and output variables in the sense that the approximation error can be enforced to be measure zero (Cybenko, 1989; Funahashi, 1989; Hornik, 1989; Park & Sandberg, 1991; Sjöberg, Hjalmarsson, & Ljung, 1994). The basic processing elements of an ANN are referred to as *neurons*, a collection of neurons is referred to as a *layer*, and the collection of interconnected layers forms the ANN. The way in which these layers are connected to each other is known as the *architecture* of the ANN. A neuron, as in Fig. 1a, is a processing element of the form: $y = \gamma\left(\sum_{j=1}^m v_j a_j + \beta\right)$ where $\gamma(\cdot)$ represents a nonlinear function known as an *activation function*, $v^{\mathrm{T}} = [v_1, v_2, \ldots, v_m]$ represents a connection parameter vector or weight vector between the neuron and the previous layer, a_j represents the input signals from the previous layer into the neuron, and β represents a bias term. Each neuron is a parameterized mapping $\gamma : \mathbb{R}^m \to \mathbb{R}$. The activation function $\gamma(\cdot)$ is usually chosen to be a monotonic $\mathcal{C}^1$ function bounded in the interval [0, 1] or the interval [−1, 1].

The most common ANN architecture is the Feedforward ANN (FANN) (see Fig. 1b): $y_i = \gamma\left(\sum_{j=1}^h W_{ij}\gamma\left(\sum_{p=1}^m V_{jp}u_p + \beta_j\right)\right)$ for $i = 1, 2, \ldots, l$, where $V \in \mathbb{R}^{h\times m}$, and V_{jp} represents the weight

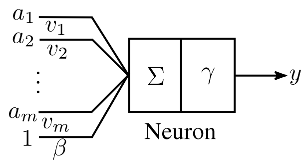

Fig. 1a. Neuron or processing element.

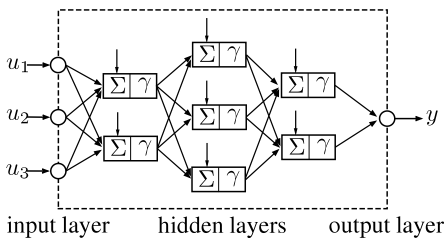

Fig. 1b. Feedforward artificial neural network.

between the pth neuron in the input layer and the jth neuron in the hidden layer, $W \in \mathbb{R}^{l\times h}$, and W_{ij} represents the weight between the jth neuron in the hidden layer and the ith neuron in the output layer, and h is the number of neurons in the hidden layer. FANNs with additional layers are defined in a similar manner. A common practice is to choose the activation function of the neurons in the output layer to be linear (Levin & Narendra, 1995b):

$$y_i = \sum_{j=1}^{h} W_{ij}\gamma\left(\sum_{p=1}^{m} V_{jp}u_p + \beta_j\right) \quad \text{for } i = 1, 2, \ldots, l. \tag{1}$$

Each element of the output vector y is represented by a superposition of nonlinear functions.

2.3. Universal approximation of a static continuous nonlinear function by a FANN

The question of approximation of a nonlinear function by a FANN has been tackled by many researchers (Barron, 1993; Chui & Li, 1992; Cybenko, 1989; Funahashi, 1989; Hornik, 1989, 1991; Kůrková, 1992). The completeness of sigmoidal activation functions with one hidden layer (Cybenko, 1989; Funahashi, 1989; Hornik, 1991), two hidden layers (Kůrková, 1992), and with more than three hidden layers (Hornik, 1989) has been shown. The completeness of polynomials (Chui & Li, 1992) and radial basis functions (Park & Sandberg, 1991) has also been proved. In 1989, Cybenko (1989) showed that the function (1), which is a continuous neural network with only one hidden layer and an arbitrary continuous sigmoidal nonlinearity, can uniformly approximate any continuous function of n real variables with support in the unit hypercube with only mild conditions imposed on the univariate function.

Cybenko (1989) focused only on the existence of a one-hidden layer ANN that is dense in $\mathcal{C}(\iota^n)$. Questions that were addressed later (Barron, 1993) were (i) how many neurons are required to guarantee a specified approximation accuracy? and (ii) what properties of the function being approximated play a role in determining the number of neurons? Barron (1993) examined how the approximation error is related to the number of neurons in the network and showed that the error in the approximation of functions by ANNs is bounded. For an ANN with one hidden layer of h sigmoidal nodes, the squared approximation error, integrated over a bounded subset of d variables, is bounded by C_f/h, where C_f depends on a norm of the Fourier Transform of the function to be approximated. An important observation is that the upper bound on the approximation error does not depend on the dimension d, which is an advantage of sigmoidal activation functions over polynomials (Chui & Li, 1992) and radial basis functions (Park & Sandberg, 1991). The result in Barron (1993) does not consider noise in the output y_i.

2.4. Dynamic Artificial Neural Networks (DANNs)

2.4.1. State-space models

The dynamic behavior of a multivariable process can be represented by input–output or *state-space* equations (Khalil, 2002; Michel, Hou, & Liu, 2008). It is well-known that any input–output equation can be rewritten as state-space equations with appropriately defined state variables. State-space equations also arise naturally from first-principles modeling, where the state variables have physical meaning and may be of independent interest.

Consider multivariable discrete-time nonlinear dynamical systems where all signals are infinite real-valued sequences represented as follows:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + \nu_k; \quad x(0) = x_0 \\ y_k &= g(x_k, u_k) + w_k \end{aligned} \tag{2}$$

where $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $x \in \mathbb{R}^n$, $\nu \in \mathbb{R}^n$, and $w \in \mathbb{R}^l$ represent the input, output, state, process noise, and measurement noise. Both ν and w are assumed to be zero-mean white noise (without loss of generality, since colored noise can be addressed by augmenting the state vector). Although for brevity the terms enter additively in (2) and in all other models described in this article, the generalization to ν and w entering the equation in a nonlinear manner is straightforward. The $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ are assumed to be piecewise continuously differentiable nonlinear maps. Piecewise continuous differentiability is not a restrictive assumption for modeling industrial processes.

2.4.2. Parameterized state-space models

As the real system is unknown, a general parameterized model is required to realize a set of systems and be dense in the set of systems to be identified (Ljung, 1987). Once the existence of such a model has been established, the state and parameters of the model can be estimated by exploiting learning and optimization algorithms in which the available input–output measurements are used. Accurate process identification from input–output data is required for many system and control engineering applications. The simulation (parameterized) models for (2) may be parameterized by the structure:

$$\begin{aligned} \hat{x}_{k+1} &= \hat{f}(\hat{x}_k, u_k; \theta_f) + \hat{\nu}_k; \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y}_k &= \hat{g}(\hat{x}_k, u_k; \theta_g) + \hat{w}_k \end{aligned} \tag{3}$$

where $\hat{y} \in \mathbb{R}^l$ represents the output predictor vector, $\hat{x} \in \mathbb{R}^n$ represents the state predictor vector, θ_f and θ_g represent the (system) parameters to be estimated, and $\hat{x}_0$ denotes the predicted initial condition, and $\hat{\nu}_k$ and $\hat{w}_k$ are modeled as Gaussian random processes with prior known means and standard deviations.

The DANNs can be considered as parameterized model structures (3), with process identification being the determination of parameters θ_f and θ_g such that the parameterized system (3) closely approximates the state-space representation of the real nonlinear system (2) in a proper sense. Here, a *proper sense* means that the closeness of two dynamical nonlinear systems is defined in terms of the norm-bounded trajectories of the difference of solutions for the dynamical nonlinear systems, provided that the solution for each dynamical system exists and is unique. Mathematically, for a space of dynamical systems we might use a metric $d(S, \hat{S}) \triangleq \max_{z\in\mathcal{Z}(S),\, \hat{z}\in\mathcal{Z}(\hat{S})} \|z - \hat{z}\|$, where S and $\hat{S}$ denote the dynamical systems (2) and (3), respectively, and $\mathcal{Z}(\cdot)$ is the set of state solutions or outputs for the dynamical systems.

This paper considers three parameterized model structures using a FANN for the parameterization of nonlinear multivariable systems (3): (a) NSSM, (b) GIOM, and (c) DRNN. The mathematical models and characteristics of each parameterized model are described in Section 3.

3. Identification of nonlinear dynamical systems

This section describes the use of ANNs for modeling nonlinear dynamic systems. The nonlinear dynamical system (2) is approximated in a well characterized way by DANNs with arbitrary accuracy.

The NSSM, GIOM, and DRNN are the most popular nonlinear dynamical models based on ANNs (other DANNs Shaw, Doyle, & Schwaber, 1997 can be addressed in a similar manner). All these DANNs can be trained by dynamic backpropagation algorithms that are readily available (Narendra & Parthasarthy, 1990). NSSM, GIOM, and DRNN all have the ability to accurately model nearly arbitrary nonlinear dynamical systems described by (2). Development of an appropriate structure for the analysis of nonlinear systems is made more convenient by drawing block diagrams for each DANN. The discrete-time versions are considered here, since real data are discrete (the block diagrams of continuous DANNs are nearly identical). The accuracy of the DANNs relies on an accurate approximation of the nonlinear relations by the FANNs. FANNs with one hidden layer, as in (1), can approximate continuous nonlinear functions with arbitrary precision (Cybenko, 1989) provided that (a) the sigmoidal activation function of the neurons at the hidden layer is a fixed univariate function that is discriminatory, (b) the sigmoidal activation function of the output layer is linear, and (c) no constraints are placed on the number of neurons or magnitude of the weights. Although monotonicity is not required in an activation function (Cybenko, 1989), the sigmoid $\sigma(x) = \frac{1}{1+e^{-x}}$ and hyperbolic tangent $\tanh(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$ functions that are widely used in ANN applications are monotonically increasing. Another advantage of these activation functions is the wide availability of training or learning algorithms to determine the estimation parameters. Other activation functions have similar approximation capabilities, such as the sine and cosine functions, but are not monotonic. The remainder of this section demonstrates that any of the three DANN models can approximate, to any degree of precision, any nonlinear dynamical system described by (2), under mild assumptions.

3.1. Neural State-Space Model (NSSM)

3.1.1. Mathematical representation of NSSM

A Neural State-Space Model (NSSM) selects the parameterized nonlinear mappings in (3) as FANNs (Suykens et al., 1995). Usually $\hat{f}(\hat{x}_k, u_k; \theta)$ and $\hat{g}(\hat{x}_k, u_k; \theta)$ are selected to have one hidden layer. With the above definitions, the NSSM for parameterizing the nonlinear dynamics (3) is

$$\begin{aligned} \hat{x}_{k+1} &= W_{AB}\Gamma\left(\begin{bmatrix} V_A & V_B \end{bmatrix}\begin{bmatrix} \hat{x}_k \\ u_k \end{bmatrix} + \beta_{AB}\right) + \nu_k \\ \hat{y}_k &= W_{CD}\Gamma\left(\begin{bmatrix} V_C & V_D \end{bmatrix}\begin{bmatrix} \hat{x}_k \\ u_k \end{bmatrix} + \beta_{CD}\right) + w_k \end{aligned} \tag{4}$$

where $V_A \in \mathbb{R}^{h_x\times n}$, $V_B \in \mathbb{R}^{h_x\times m}$, $W_{AB} \in \mathbb{R}^{n\times h_x}$, $\beta_{AB} \in \mathbb{R}^{h_x}$, $V_C \in \mathbb{R}^{h_y\times n}$, $V_D \in \mathbb{R}^{h_y\times m}$, $\beta_{CD} \in \mathbb{R}^{h_y}$, and $W_{CD} \in \mathbb{R}^{l\times h_y}$ are the parameters to be estimated in FANNs.

3.1.2. Completeness of NSSM

The approximability of (2) by an NSSM follows from mild assumptions on the state-transition map f and the output map g.

Assumption 1 (*Lipschitz Condition*). Let $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{Y} \subset \mathbb{R}^l$ be open sets. Assume that the nonlinear functions $f : \mathcal{X}\times\mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X}\times\mathcal{U} \to \mathcal{Y}$ are uniformly Lipschitz in x on $\mathcal{X}\times\mathcal{U}$ with Lipschitz constants ℓ_f and ℓ_g, respectively, i.e.,

$$\begin{aligned} &\|f(\xi^1, u) - f(\xi^2, u)\| \le \ell_f\|\xi^1 - \xi^2\|, \\ &\|g(\xi^1, u) - g(\xi^2, u)\| \le \ell_g\|\xi^1 - \xi^2\|, \end{aligned} \tag{5}$$

for all $\xi^1, \xi^2 \in \mathcal{X}$ and any $u \in \mathcal{U}$.

The next lemma states that an NSSM (4) can approximate with arbitrary accuracy the state-space equations of any nonlinear dynamical system (2) with bounded inputs during any finite time interval.

Lemma 1 (*Generality of NSSM*). *Under Assumption 1 on the nonlinear functions $f : \mathcal{X}\times\mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X}\times\mathcal{U} \to \mathcal{Y}$, consider x_k as the state and y_k as the output of the system defined by (2) where $\nu = w = 0$. For any $\epsilon > 0$, there exists an NSSM (4), more briefly written as*

$$\Sigma_{\text{nssm}} : \hat{x}_{k+1} = \hat{f}(\hat{x}_k, u_k; \theta_f), \;\; \hat{y}_k = \hat{g}(\hat{x}_k, u_k; \theta_g) \tag{6}$$

with an appropriate initial condition $\hat{x}_0$ such that the estimation error is bounded by ϵ:

$$\|y_{[0,\kappa]} - \hat{y}_{[0,\kappa]}\|_{\ell^l_\infty} < \epsilon, \tag{7}$$

for any $0 < \kappa < \infty$ and any input $u : \mathbb{Z}_+ \to \mathcal{D}_u$ in a compact set $\mathcal{D}_u \subset \mathcal{U}$, where θ_f and θ_g are parameters in (4).

Proof. From the universal approximation property of an ANN in Cybenko (1989), for any $\epsilon_x > 0$ and $\epsilon_y > 0$ there exist $\theta_f \in \mathbb{R}^{N_f}$ and $\theta_g \in \mathbb{R}^{N_g}$, where N_f and N_g are the number of parameters in the state and output ANNs, respectively, such that

$$\|f(\xi, u) - \hat{f}(\xi, u; \theta_f)\| < \epsilon_x, \quad \|g(\xi, u) - \hat{g}(\xi, u; \theta_g)\| < \epsilon_y, \tag{8}$$

for any input $u \in \mathcal{D}_u$. Simple mathematical induction can be used to show that (7) holds for all $0 \le k \le \kappa$ with a finite integer $0 < \kappa < \infty$, where the parameter dependence in the NSSM (4) or (6) is dropped to simplify notation:

$$\begin{aligned} \|y_k - \hat{y}_k\| &= \|g(x_k, u_k) - \hat{g}(\hat{x}_k, u_k)\| \\ &= \|g(x_k, u_k) - g(\hat{x}_k, u_k) + g(\hat{x}_k, u_k) - \hat{g}(\hat{x}_k, u_k)\| \\ &\le \ell_g\|x_k - \hat{x}_k\| + \epsilon_y \end{aligned}$$

Moreover, the state solution trajectories have differences

$$\begin{aligned} \|x_k - \hat{x}_k\| &= \|f(x_{k-1}, u_{k-1}) - \hat{f}(\hat{x}_{k-1}, u_{k-1})\|, \\ &= \|f(x_{k-1}, u_{k-1}) - f(\hat{x}_{k-1}, u_{k-1}) \\ &\quad + f(\hat{x}_{k-1}, u_{k-1}) - \hat{f}(\hat{x}_{k-1}, u_{k-1})\|, \\ &\le \ell_f\|x_{k-1} - \hat{x}_{k-1}\| + \epsilon_x, \\ &\le \ell_f^k\|x_0 - \hat{x}_0\| + (\ell_f^{k-1} + \cdots + \ell_f + 1)\epsilon_x, \\ &= M_k\epsilon_x, \end{aligned}$$

where $M_k = \sum_{i=0}^{k}\ell_f^i$, provided that the initial condition for $\hat{x}$ satisfies $\|x_0 - \hat{x}_0\| \le \epsilon_x$. Therefore, the output approximation residual is bounded for any bounded time interval $0 \le k \le \kappa < \infty$:

$$\|y_k - \hat{y}_k\| \le \ell_g\|x_k - \hat{x}_k\| + \epsilon_y \le \ell_g M_\kappa\epsilon_x + \epsilon_y.$$

Selecting $\epsilon_x = \frac{\epsilon}{2\ell_g M_\kappa}$ and $\epsilon_y - \frac{\epsilon}{2}$ implies that, for any $\epsilon > 0$, there exists a NSSM (6) such that $\|y_k - \hat{y}_k\| \le \epsilon$ for all $k \in [0, \kappa]$.

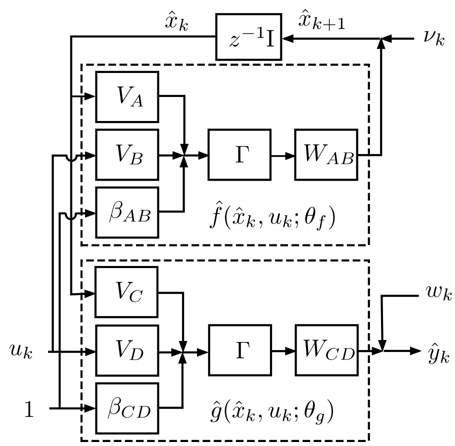

Fig. 2. Neural state-space model.

A similar proof for a closely related DANN is shown in Jin et al. (1995). The next result indicates that the boundedness requirement of the processing time interval can be removed when the vector field $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ has a contraction-mapping property.

Corollary 1. *If the initial condition for the state equation $x(0)$ is known and the unknown nonlinear function $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is Lipschitz in x on $\mathcal{X}$ with a Lipschitz constant $\ell_f < 1$, i.e., f is a contraction mapping on $\mathcal{X}$ for each fixed $u \in \mathcal{D}_u \subset \mathcal{U}$, then there exists a NSSM (6) such that the approximated output is arbitrarily close to the actual output for all $k \in \mathbb{Z}_+$.*

The corollary directly follows from the fact that the sequence M_k is bounded above by $M_\infty \triangleq \frac{1}{1-\ell_f} < \infty$ for all $k \in \mathbb{Z}_+$. Fig. 2 is a block diagram for the NSSM model. The structures enclosed by the dashed boxes are the FANNs that describe the nonlinearities in the state and output equations.

3.2. Global Input–Output Model (GIOM)

3.2.1. Mathematical representation of GIOM

Another form for a DANN is the Global Input–Output Model (GIOM) (Levin & Narendra, 1995a; Narendra & Parthasarthy, 1990):

$$\hat{y}_{k+1} = \hat{g}(u_k, \ldots, u_{k-q+1}, \hat{y}_k, \ldots, \hat{y}_{k-r+1}) + \nu_k \tag{9}$$

where q and r denote the input and output horizon, respectively, and $\hat{y}$ on the right-hand side may be replaced by the actual measurement output y whenever its measurement is available. The existence of this type of approximator for a general class of nonlinear dynamical systems has been investigated by many researchers, with $\hat{g}$ defined by ANNs (Narendra & Parthasarthy, 1990), Volterra series (Boyd, 1985), and other basis functions. For identification purposes, a key property of the system (2) to be identified is its observability, which concerns the uniqueness or invertibility of the system state variables from given pairs of inputs and outputs. To simplify the presentation only, consider the case with $r = q$, i.e., the input and output horizons are equal.

Definition 1 (*Observability (Levin & Narendra, 1995b)*)**.** A dynamic system is said to be *observable* if, for any two state variables x^1 and x^2, there exists an input sequence of length r, $\{u_j\}_{j=0}^{r-1}$, such that $x^1 \neq x^2$ implies $Y^r(x^1, U^r) \neq Y^r(x^2, U^r)$ where Y^r and U^r are the concatenations of the sequences of outputs and inputs up to r time steps, respectively. The minimum length r of the input sequence with which there exists an injective map from $Y^r \in \mathcal{Y}^r$ to $x \in \mathcal{X}$ is called the *degree* of the observability of the dynamic system, which is assumed to be well-defined for any observable system. If r is equal to the order n of the system state, then the system is said to be *strongly observable*.

The generic observability assumption is not restrictive for practical situations and can be tested by relying on the so-called transversality condition (Levin & Narendra, 1995a). For linear time-invariant systems, the equivalence of strong observability to observability follows from the Cayley–Hamilton theorem (Antsaklis & Michel, 1997).

The next proposition shows that any observable system of the form (2) with the degree of observability r can be parameterized as a GIOM.

Proposition 1 (*Prop. 1 in Kim et al., 2011a*)**.** *If a nonlinear dynamical system (2) is causal and observable with degree of observability r then there exists an observability map $h^{[r]} : \mathcal{U}^r \times \mathcal{Y}^r \to \mathcal{Y}$ such that for all $k \in \mathbb{Z}_+$,*

$$y_{k+1} = h^{[r]}(U_k^r, Y_k^r) \tag{10}$$

with $Y_k^r \triangleq \left[y_k^{\mathrm{T}}, \ldots, y_{k-r+1}^{\mathrm{T}}\right]^{\mathrm{T}}$ and $U_k^r \triangleq \left[u_k^{\mathrm{T}}, \ldots, u_{k-r+1}^{\mathrm{T}}\right]^{\mathrm{T}}$.

A FANN parameterization of (9) yields the input–output predictor:

$$\hat{y}_{k+1} = W_A \Gamma \left(V_A u_k + V_B U_{k-1}^{q-1} + V_C \hat{Y}_k^r + \beta\right) + \nu_k \tag{11}$$

where $\hat{Y}_k^r \triangleq \left[\hat{y}_k^{\mathrm{T}}, \ldots, \hat{y}_{k-r+1}^{\mathrm{T}}\right]^{\mathrm{T}}$. The value of the next future output is determined directly by the sequences of the previous inputs and outputs.

3.2.2. Completeness of GIOM

The next lemma states that, for a set of generically observable systems (12), the GIOM is open and dense.

Lemma 2 (*Generality of GIOM*)**.** *Let the system of the form*

$$x_{k+1} = f(x_k, u_k), \quad y_k = g(x_k) \tag{12}$$

be a generically observable system where the set of input sequences for which the system is not observable is of measure zero, and f and g are smooth. For any $\epsilon > 0$, there exists a FANN $\hat{g} : \mathcal{U}^q \times \mathcal{Y}^r \to \mathcal{Y}$ such that

$$\|y_{k+1} - \hat{g}(u_k, \ldots, u_{k-q+1}, y_k, \ldots, y_{k-r+1})\|_\infty < \epsilon \tag{13}$$

for all bounded states and all bounded input sequences except an open set of measure $< \epsilon$ containing the singular input sequences (for all k). Furthermore, both r and q can be selected to be less than $2(n+1)$, where n is the dimension of the state variable x in (12).

Proof. Follows from applying Thms. 5 and 6 of Levin and Narendra (1995a) to each input–output pair.

Fig. 3 is the block diagram for the GIOM (11).

3.3. Dynamic Recurrent Neural Network (DRNN)

3.3.1. Mathematical representation of DRNN

The Dynamic Recurrent Neural Network (DRNN) is presented in Fang and Kincaid (1996) and Jin and Gupta (1996) as follows:

$$\hat{x}_{k+1} = -\alpha \hat{x}_k + W_{AB} \Gamma \left(V_A \hat{x}_k + V_B u_k + \beta_{AB}\right) + \nu_k. \tag{14}$$

The self-feedback parameter α is a fixed constant that controls the state decay (aka the largest Lyapunov exponent) and is chosen to satisfy $|\alpha| \leq 1$. Augmenting the state equation with an output equation results in

$$\begin{aligned} \hat{x}_{k+1} &= -\alpha \hat{x}_k + W_{AB} \Gamma (V_A \hat{x}_k + V_B u_k + \beta_{AB}) + \nu_k \\ \hat{y}_k &= W_{CD} \Gamma (V_C \hat{x}_k + V_D u_k \beta_{CD}) + w_k. \end{aligned} \tag{15}$$

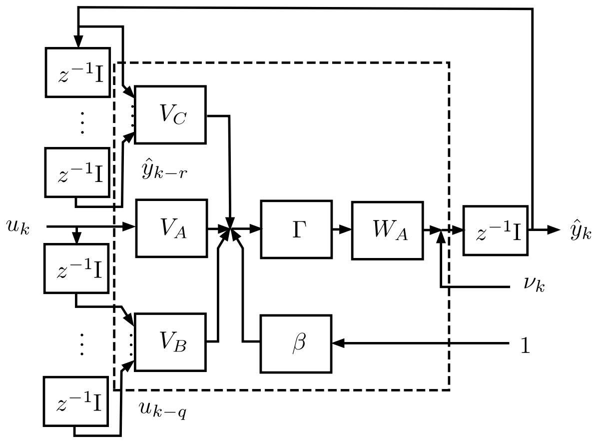

Fig. 3. Global input–output model.

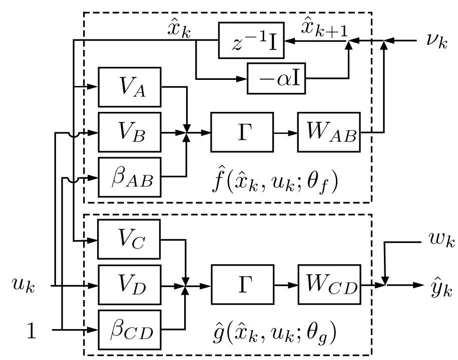

Fig. 4. Dynamic recurrent neural network.

3.3.2. Completeness of DRNN

This section presents the completeness property of a DRNN, adapting some results of Jin et al. (1995).

Lemma 3 (*Generality of DRNN*). *Define $\mathcal{D}_x$ and $\mathcal{D}_u$ as any closed bounded sets, and $\mathcal{X}$ and $\mathcal{U}$ as any open sets within $\mathcal{D}_x$ and $\mathcal{D}_u$, respectively. Fix $\nu_k = w_k = 0$. Given any finite integer κ, any $\epsilon > 0$, any Lipschitz continuous functions f and g in (2) with initial state $x_0 \in \mathcal{X}$ whose solution $x_k \in \mathcal{D}_x$, and an appropriate initial condition $\hat{x}_0$, there exists a DRNN (15) with the hyperbolic tangent as the activation function for which for any bounded input $u_k \in \mathcal{D}_u$ and $0 < \kappa < \infty$,*

$$\|y_{[0,\kappa]} - \hat{y}_{[0,\kappa]}\|_{\ell^l_\infty} < \epsilon. \tag{16}$$

Proof. Under the assumptions and according to Thm. 1 of Jin et al. (1995), there exists a state-space system of the form

$$\eta_{k+1} = -\alpha\eta_k + A\Gamma(\eta_k + Bu_k);\ \hat{x}_k = C\eta_k; \tag{17}$$

where $\eta \in \mathbb{R}^N$, $N \geq n$, represents a dummy state variable and the state $\hat{x} \in \mathbb{R}^n$ of (14) is the output of the new state-space system (17) with an appropriate initial condition η_0, such that for any $\epsilon > 0$ and any bounded input $u \in \mathcal{D}_u$, the error in the state is bounded during any finite time interval $[0, \kappa]$: $\max_{0\leq k\leq\kappa}\|x_k - \hat{x}_k\|_\infty < \frac{\epsilon}{2\ell_g}$, where ℓ_g is the Lipschitz constant in x_k with any fixed u_k for the nonlinear function g. By Lemma 1 of Cybenko (1989), there exist appropriate parameters such that $\max_{0\leq k\leq\kappa}\|g(\hat{x}_k, u_k) - \hat{y}\|_\infty < \epsilon/2$ where $\hat{y}_k := W_{CD}\Gamma(V_C\hat{x}_k + V_Du_k + \beta_{CD})$, which implies $\max_{0\leq k\leq\kappa}\|y_k - \hat{y}_k\|_\infty < \ell_g(\epsilon/2\ell_g) + \epsilon/2 = \epsilon$.

Lemmas 2 and 3 state that, with minor technical conditions, a GIOM (9) and a DRNN (15) can approximate the input–output mapping of any state-space system of the form (2). Lemma 3 can be extended to more general activation functions (Rios-Patron & Braatz). Fig. 4 is the block diagram for (15). DRNNs can be trained by dynamic backpropagation (Narendra & Parthasarthy, 1990). The structures enclosed by the dashed boxes represent the block diagram structures of the DRNN and FANN that approximate the nonlinear functions.

4. Standard nonlinear operator forms of dynamical artificial neural networks

Researchers worked in robust stability analysis for decades before the early 1980s, when a general framework for linear robustness analysis was developed (Doyle, 1982; Safonov, 1982). The developed analysis tools were computable for linear systems of moderate dimension and quickly led to controller design techniques. In approximately 15 years, robust linear control grew to be a relatively mature field. Nonlinear control is not as mature. Numerous nonlinear control techniques have been developed over the past several decades, with no single technique dominating the field. Modern robust linear control has a standard representation, called the Linear Fractional Transformation (LFT), for describing linear systems with uncertainty (Skogestad & Postlethwaite, 1996; Zhou, Doyle, & Glover, 1995). The Standard Nonlinear Operator Form (SNOF) is the standard representation for nonlinear dynamical systems modeled by DANNs.

4.1. The Standard Nonlinear Operator Form (SNOF)

The SNOF in Fig. 5 is a nonlinear model structure that is the interconnection of a linear time-invariant dynamical system in feedback with a bounded static nonlinear operator. The nonlinearities γ_i are collected in a diagonal operator Γ in which each diagonal element is typically continuous, differentiable, odd, monotonically increasing, slope-restricted, and has bounded output. Each of the block diagrams for the DANNs can be rearranged into the SNOF in Fig. 5, with γ_i being the activation functions such as sigmoidal and hyperbolic tangent functions, so that the SNOF is a superset of the DANNs. The matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a linear mapping between the inputs and outputs of the time delay and the operator Γ, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times h}$, $C \in \mathbb{R}^{h\times n}$, $D \in \mathbb{R}^{h\times h}$, and $h \in \mathbb{Z}$ is the total number of static nonlinearities. Denote such a feedback system in Fig. 5 by SNOF($\frac{1}{z}$I, M, Γ). With the vectors q_k and p_k as the input and output of the nonlinear operator, respectively, the SNOF can be written as

$$\text{SNOF}: \begin{bmatrix} \hat{x}_{k+1} \\ q_k \end{bmatrix} = M \begin{bmatrix} \hat{x}_k \\ p_k \end{bmatrix}, \quad p_k = \Gamma(q_k). \tag{18}$$

This representation has been called a *diagonal nonlinear differential inclusion* or a *Lur'e differential inclusion* when the nonlinearities γ_i are written as belonging to some set, such as the set of sector-bounded nonlinearities or monotonic static nonlinearities (Boyd, Ghaoui, Feron, & Balakrishnan, 1994).

4.2. Well-posedness of the SNOF

Definitions of and criteria for well-posedness for a large-scale interconnected system has been heavily studied (Vidyasagar, 1980, 1981). Below are conditions for well-posedness of an LFT and a SNOF.

Definition 2 (*Dullerud & Paganini, 2000*; *Zhou et al., 1995*). An LFT, $F_l(M, \Delta)$, is *well-posed* if $\mathrm{I} - M_{22}\Delta$ is invertible.

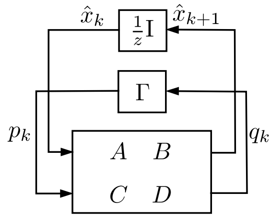

Fig. 5. Standard nonlinear operator form (SNOF).

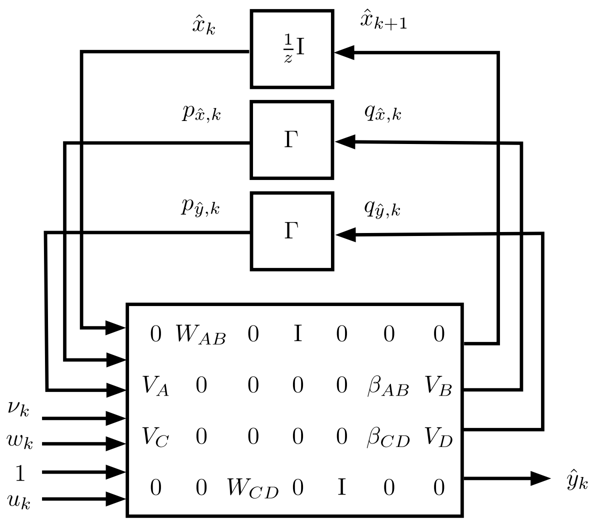

Fig. 6. SNOF of an NSSM with hyperbolic tangent functions.

Proposition 2. *Suppose that the nonlinear operator* $\Gamma : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q}$ *in a SNOF satisfies* $\Gamma(0) = 0$ *and is an injective map. Then there exists a time-varying matrix* $\Delta : \mathbb{Z}_+ \to \mathbb{R}^{n_q \times n_q}$ *such that* $\Gamma(q_k) = \Delta_k q_k$ *holds for any (bounded) signal* $q_k \in \mathbb{R}^{n_q}$, $k \in \mathbb{Z}_+$ *and the SNOF*$(\frac{1}{z}\mathrm{I}, M, \Gamma)$ *is well-posed if the* $F_l(M, \Delta_k)$ *is well-posed for all such* Δ_k.

4.3. Loop transformation in SNOF

In order to exploit the structure of the nonlinear bounded operator Γ, it can be convenient to fix the bounds for the nonlinearity $\gamma(\cdot)$ when developing nonlinear stability analysis tools. If the bounds of a given SNOF do not correspond to those of the stability tool to be used, an equivalent SNOF can be computed via a *loop transformation* (Zhou et al., 1995). (A thorough description of the loop transformation concept is discussed on pages 50–53 of Desoer and Vidyasagar (1975) and page 129 of Boyd et al. (1994).) We also refer the readers to Kim (2009) and Kim et al. (2011b) for more details.

4.4. SNOF for DANN models

Often black-box models for nonlinear dynamical systems are desired for systems that are known to be g.a.s. Writing a DANN as a SNOF enables the application of conditions to assess whether every trajectory of given NSSM, GIOM, or DRNN converges to zero as $k \to \infty$. To simplify notation, assume that $\Gamma(0) = 0$, which is true for the most popular activation function, the hyperbolic tangent. The condition can be forced to hold for other activation functions by employing a loop transformation (an example of this will be given below). Block diagram algebra can be used to rearrange a DANN into the SNOF in Fig. 5.

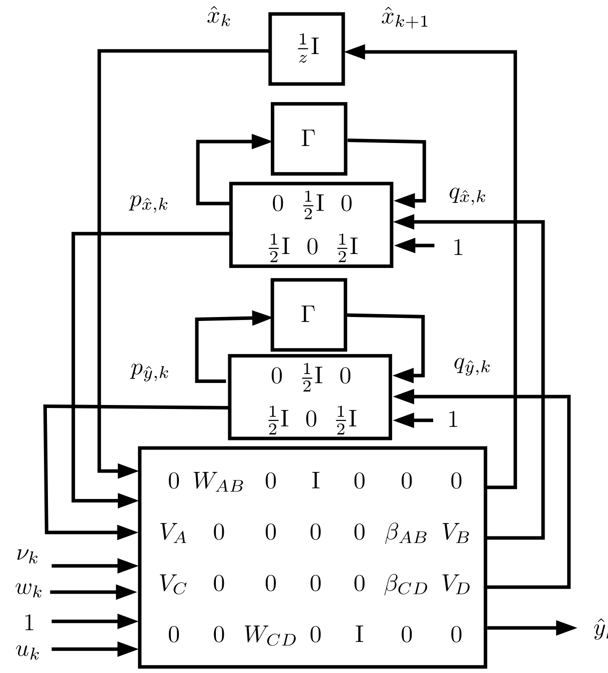

Fig. 7. SNOF of a NSSM with sigmoid functions.

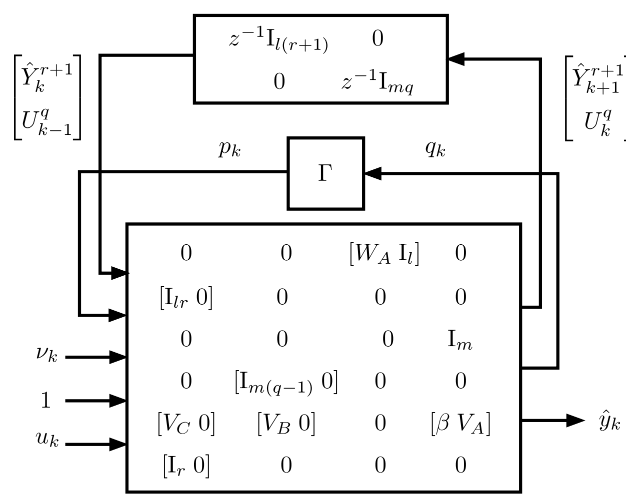

Fig. 8. SNOF of a GIOM with hyperbolic tangent functions.

If the hyperbolic tangent is selected as the activation function, then the output of each nonlinearity (that is, the output of each neuron) is bounded by the sector [0, 1], i.e., $\tanh(\sigma)(\tanh(\sigma) - \sigma) < 0$ for all $\sigma \in \mathbb{R}$. The NSSM block diagram in Fig. 2 when rearranged into the SNOF results in Fig. 6, with the hyperbolic tangent selected as the activation function. If the sigmoid function is selected as the activation function, then the output of each nonlinearity (that is, the output from each neuron) is not sector-bounded since it does not vanish at the origin. In this case, the loop transformation shown in Fig. 7 can be performed in order to satisfy that $\Gamma(0) = 0$ and to be sector-bounded. Matrices G and H are absorbed into M using the Redheffer star product, which is described on pages 531 and 532 of Skogestad and Postlethwaite (1996) (the transformation can also be derived directly through block diagram manipulations). The SNOFs of a GIOM and DRNN are obtained by rearranging Figs. 3 and 4 to produce Figs. 8 and 9.

Since the DRNNs, NSSMs, and GIOMs can all be written in SNOF, the SNOF shares the properties of DRNNs, NSSMs, and GIOMs in

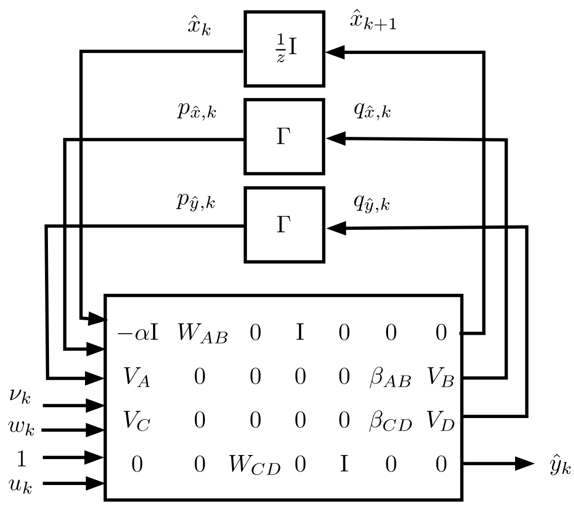

Fig. 9. SNOF of a DRNN with hyperbolic tangent functions.

being able to model the nonlinear dynamics of nearly any real system with arbitrarily small approximation error for more details.

5. Stability analysis for standard nonlinear operator forms of DANNs

While the SNOF is sufficiently general to represent the input–output dynamics of nearly any nonlinear system of practical interest, its simple structure facilitates nonlinear stability analysis. Analysis tools developed for the SNOF in the absolute stability literature (see citations in Liberzon, 2006) and in the extended Lur'e literature (see citations in Kim, 2009) are computable for systems of moderate dimension. Stability analysis tools developed for systems in SNOF are applicable to both open- and closed-loop systems.

5.1. Linear matrix inequalities of absolute stability criteria

Linear matrix inequality-based sufficient stability conditions are presented for some classes of nonlinear systems in SNOF, which are used to analyze the stability of DANNs. The proofs of the main theorems are given in a thesis (Kim, 2009).

Definition 3 (*Some Classes of Nonlinear Operators*)**.** A nonlinear mapping $\Gamma : \mathbb{R}^{n_q} \times \mathbb{Z}_+ \to \mathbb{R}^{n_q}$ is said to be an element of $\Phi_{\rm sb}^{[\underline{\alpha},\overline{\alpha}]}$ if the inequality $\left[\underline{\alpha}_i^{-1}\gamma_i(\sigma,k)-\sigma\right]\left[\overline{\alpha}_i^{-1}\gamma_i(\sigma,k)-\sigma\right] \le 0$ holds for all $\sigma \in \mathbb{R}$, $k \in \mathbb{Z}_+$, and $i = 1,\ldots,n_q$, where the subscript i denotes the ith element of the vector. A nonlinear mapping γ : $\mathbb{R}^{n_q} \times \mathbb{Z}_+ \to \mathbb{R}^{n_p}$ is said to be an element of $\Phi_{\rm sr}^{[\underline{\mu},\overline{\mu}]}$ if $\underline{\mu}_i(\sigma-\hat{\sigma}) \le (\gamma_i(\sigma,k)-\gamma_i(\hat{\sigma},k)) \le \overline{\mu}_i(\sigma-\hat{\sigma})$ holds for all $\sigma \neq \hat{\sigma} \in \mathbb{R}$, $k \in \mathbb{Z}_+$, and $i = 1,\ldots,n_q$. The set of odd functions is denoted by $\Phi_{\rm odd}$.

5.1.1. Discrete-time Lur'e systems with slope-restricted nonlinearities

While conic-sector bounded nonlinearities, which bound the global slope of the nonlinear functions $\gamma_i(\cdot)$, have been heavily studied in the literature, such a description allows the *local* slope of the $\gamma_i(\cdot)$ to vary arbitrarily from one time instance to another. Most practical nonlinearities have a local slope restriction, in which case a less conservative analysis condition may be achieved if these constraints on the nonlinearities are included in the analysis via the S-procedure (Boyd et al., 1994). A local slope restriction also provides an upper bound on the change of the integral term in the Lyapunov function, provided that $\gamma(\cdot)$ is continuous almost everywhere (a.e.).

Theorem 1. *Consider a system of the form the SNOF (18) with the memoryless nonlinearity $\gamma \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$ that is continuous almost everywhere. A sufficient condition for global asymptotic stability is the existence of a positive-semidefinite matrix $P = P^{\rm T}$ with a positive-definite submatrix $P_{11} = P_{11}^{\rm T}$ and diagonal positive-semidefinite matrices $Q, \tilde{Q}, T, \tilde{T}, N \in \mathbb{R}^{n_q\times n_q}$ such that $G < 0$ whose block entries are*

$$\begin{aligned}
G_{11} &= A^{\rm T}(P_{11}+P_{13}C+C^{\rm T}P_{13}^{\rm T}+C^{\rm T}P_{33}C)A-P_{11}\\
&\quad -P_{13}C-C^{\rm T}P_{13}^{\rm T}-C^{\rm T}P_{33}C+A^{\rm T}C^{\rm T}\tilde{Q}\xi CA-C^{\rm T}\tilde{Q}\xi C,\\
G_{12} &= A^{\rm T}(P_{11}+P_{13}C+C^{\rm T}P_{13}^{\rm T}+C^{\rm T}P_{33}C)B-P_{12}\\
&\quad -P_{13}D-C^{\rm T}P_{23}^{\rm T}-C^{\rm T}P_{33}D-C^{\rm T}T+(CA-C)^{\rm T}Q\\
&\quad +A^{\rm T}C^{\rm T}\tilde{Q}\xi CB+(CA-C)^{\rm T}\tilde{Q}-C^{\rm T}\tilde{Q}\xi D+(CA-C)^{\rm T}\mu N\\
G_{13} &= A^{\rm T}P_{12}+A^{\rm T}P_{13}D+A^{\rm T}C^{\rm T}P_{23}^{\rm T}+A^{\rm T}C^{\rm T}P_{33}D\\
&\quad -A^{\rm T}C^{\rm T}\tilde{T}-(CA-C)^{\rm T}Q+A^{\rm T}C^{\rm T}\tilde{Q}\xi D-(CA-C)^{\rm T}\mu N\\
G_{22} &= B^{\rm T}(P_{11}+P_{13}C+C^{\rm T}P_{13}^{\rm T}+C^{\rm T}P_{33}C)B-P_{22}\\
&\quad -P_{23}D-D^{\rm T}P_{23}^{\rm T}-D^{\rm T}P_{33}D-2\xi^{-1}T-TD\\
&\quad -D^{\rm T}T+Q(CB-D)+(CB-D)^{\rm T}Q-\mu^{-1}Q\\
&\quad +B^{\rm T}C^{\rm T}\tilde{Q}\xi CB-D^{\rm T}\tilde{Q}\xi D+(CB-D)^{\rm T}\tilde{Q}+\tilde{Q}(CB-D)\\
&\quad -\mu^{-1}\tilde{Q}-2N+N\mu(CB-D)+(CB-D)^{\rm T}\mu N\\
G_{23} &= B^{\rm T}P_{12}+B^{\rm T}P_{13}D+B^{\rm T}C^{\rm T}P_{23}^{\rm T}+B^{\rm T}C^{\rm T}P_{33}D\\
&\quad -B^{\rm T}C^{\rm T}\tilde{T}+QD+Q\mu^{-1}+\mu^{-1}\tilde{Q}+\tilde{Q}D+B^{\rm T}C^{\rm T}\tilde{Q}\xi D\\
&\quad +2N-(CB-D)^{\rm T}\mu N+N\mu D\\
G_{33} &= P_{22}+P_{23}D+D^{\rm T}P_{23}^{\rm T}+D^{\rm T}P_{33}D-2\xi^{-1}\tilde{T}-\tilde{T}D\\
&\quad -D^{\rm T}\tilde{T}-QD-D^{\rm T}Q-Q\mu^{-1}-\mu^{-1}\tilde{Q}+D^{\rm T}\tilde{Q}\xi D\\
&\quad -2N-N\mu D-D^{\rm T}\mu N
\end{aligned}$$

5.1.2. Discrete-time Lur'e systems with slope-restricted and odd monotonic nonlinearities

Theorem 1 exploits more information on the nonlinear operator than the original Lur'e problem. Below is a less conservative sufficient stability condition for the more restrictive class of nonlinearities that are odd monotonic, derived by introducing additional quadratic constraints.

Theorem 2. *Consider a system of the form (18) with the memoryless nonlinearity $\gamma \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]} \cap \Phi_{\rm odd}$, which is continuous almost everywhere. A sufficient condition for global asymptotic stability is the existence of a positive-semidefinite matrix $P = P^{\rm T}$ with a positive-definite submatrix $P_{11} = P_{11}^{\rm T}$ and diagonal positive-semidefinite matrices $Q, \tilde{Q}, T, \tilde{T}, N, L,$ and $\tilde{L} \in \mathbb{R}^{n_q\times n_q}$ such that $\bar{G} < 0$ whose block entries are*

$$\begin{aligned}
\bar{G}_{11} &= G_{11}\\
\bar{G}_{12} &= G_{12}-(CA-C)^{\rm T}L-(CA+C)^{\rm T}\tilde{L}\\
\bar{G}_{13} &= G_{13}-(CA-C)^{\rm T}L-(CA-C)^{\rm T}\tilde{L}\\
\bar{G}_{22} &= G_{22}-(CB-D)^{\rm T}L-L(CB-D)\\
&\quad -(CB+D)^{\rm T}\tilde{L}-\tilde{L}(CB+D)\\
\bar{G}_{23} &= G_{23}-(CB-D)^{\rm T}L-LD-\xi^{-1}L\\
&\quad -(CB-D)^{\rm T}\tilde{L}-\tilde{L}D+\xi^{-1}\tilde{L}\\
\bar{G}_{33} &= G_{33}-D^{\rm T}L-LD-\tilde{L}D-D^{\rm T}\tilde{L}
\end{aligned}$$

The proofs of the preceding theorems apply a generalized Lur'e–Postinikov function in which a quadratic function $\xi^{\rm T}P\xi$ with $\xi = [x^{\rm T},\ p^{\rm T},\ q^{\rm T}]^{\rm T}$ is combined with integration terms corresponding to sector-boundedness and slope-restriction. The S-procedure is also used to exploit quadratic constraints on p and q so that less conservative conditions can be achieved. The details of this proof are available in Kim (2009) and a solution code built on a MATLAB interface YALMIP (Löfberg, 2004) will be provided upon request.

Theorems 1 and 2 describe LMI feasibility problems, that is, each criterion considers whether the set $\{X \in \mathbb{S}^n | \Psi(X) < 0, X \geq 0\}$ is empty or non-empty. The below lemma shows the equivalence between two sets of matrix decision variables.

Lemma 4 (*Feasibility Considerations*). *The set of symmetric matrices affine in X, $\{X \in \mathbb{S}^n | \Psi(X) < 0, X \geq 0\}$, is nonempty if and only if the set of symmetric matrices $\{X \in \mathbb{S}^n | \Psi(X) < 0, X > 0\}$ is nonempty.*

Proof. The (only if) part is obvious. To prove the (if) part, the LMIs in the set can be written in the form $\Psi(X) = F_0 + \sum_{i=1}^{N} x_i F_i$, where $X = X^T \in \mathbb{R}^{n\times n}$, $N = n(n+1)/2$, and the F_i are of compatible dimension. Suppose that $X^0 \in \{X \in \mathbb{S}^n | \Psi(X) < 0, X \geq 0\}$. With the definition $X^\delta \triangleq X^0 + \delta I, X^\delta > 0$ for any value of positive scalar $\delta > 0$. Now we show that there exists a scalar $\delta > 0$ such that X^δ is in the set $\{X \in \mathbb{S}^n | \Psi(X) < 0, X > 0\}$. Consider $\Psi(X^\delta) = F_0 + \sum_{i=1}^{N}(x_i + \delta_i)F_i$, where $\delta_i \in \{0, \delta\}$ for each $i = 1, \ldots, N$ and the number of nonzero δ_i is n. Since the eigenvalues of $\Psi(X^\delta)$ are continuous for $\delta \to 0$, these eigenvalues approach the eigenvalues of $\Psi(X^0)$ as $\delta \to 0$. For sufficiently small δ, the eigenvalues of $\Psi(X^\delta)$ have negative real part so that $\Psi(X^\delta) < 0$.

From the results of Lemma 4, any LMI solver that is guaranteed to converge for an LMI feasibility problem with strict inequalities will converge for the above LMI feasibility problems.

5.2. Global asymptotic stability criteria for NL_q theory

NL_q theory is an alternative approach to analyze the stability for certain classes of DANNs with hyperbolic tangent activation functions (Suykens et al., 1995; Suykens, Moor, & Vandewalle, 2000; Suykens et al., 1996). Below is the definition of an NL_q system.

Definition 4 (*NL_q Systems (Suykens et al., 1996)*). *An NL_q system has the form*

$$\hat{x}_{k+1} = \Gamma_1^{NL_q}\Big(V_1\Gamma_2^{NL_q}\Big(V_2\cdots\Gamma_q^{NL_q}(V_q x_k + B_q w_k)\cdots + B_2 w_k\Big) + B_1 w_k\Big) \tag{19}$$

where the $\Gamma_i^{NL_q}, i = 1, \ldots, q$, are diagonal matrices with elements that belong to the interval $[0, 1]$, $\hat{x}_k \in \mathbb{R}^n$ is the state vector, $e_k \in \mathbb{R}^l$ is the input vector, and V_i, W_i, B_i, and D_i are constant matrices that contain the weights of the ANN.

The below result enables NL_q theory to be used to analyze NSSMs and GIOMs.

Lemma 5. *Any NSSM (4) and GIOM (11) can be written as an NL_q system (19).*

Proof. The proof for an NSSM described on pages 123, 124, and 207 of Suykens et al. (1996) employs an elementwise notation of the state and output vectors and multiplies and divides by the arguments of the activation functions to obtain the alternating sequence of nonlinear and linear operators in (19). The proof for a GIOM follows by applying the same approach to the state-space realization of the GIOM.

The next result is a sufficient condition for g.a.s. of autonomous and homogeneous NL_q systems ($w_k = 0, \forall k \in \mathbb{Z}_+$) that have $x_{eq} = 0$ as the equilibrium point.

Theorem 3 (*Diagonal Scaling (Suykens et al., 1996)*). *A sufficient condition for g.a.s. of the autonomous NL_q system*

$$x_{k+1} = \left[\prod_{i=1}^{q} \Gamma_i^{NL_q}(x_k)V_i\right] x_k \tag{20}$$

is to find diagonal matrices with nonzero elements D_i such that the LMI

$$V_{tot}^T D_{tot}^2 V_{tot} < D_{tot}^2 \tag{21}$$

holds, where matrices V_{tot} and D_{tot} are

$$V_{tot} = \begin{bmatrix} 0 & V_2 & 0 & \cdots & 0 \\ \vdots & 0 & V_3 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \cdots & 0 & V_q \\ V_1 & 0 & \cdots & \cdots & 0 \end{bmatrix}; \tag{22}$$

$$D_{tot} = \mathrm{diag}\{D_2, D_3, \ldots, D_q, D_1\}. \tag{23}$$

The LMI (21) can be rewritten as the scaled singular value condition $\bar{\sigma}(D_{tot}V_{tot}D_{tot}^{-1}) < 1$ which is not convex in D_{tot}. Theorem 3 is expected to be conservative due to the restrictive choice for the Lyapunov function.

5.3. Review of results from robust and absolute stability tests

Below are some useful results from classical robust control theory and absolute stability tests that are used in Section 6 to compare with Theorem 3 derived from NL_q theory.

5.3.1. Small-gain theorem

Consider a finite-dimensional linear time-invariant system

$$\begin{bmatrix} x_{k+1} \\ e_k \end{bmatrix} = M \begin{bmatrix} x_k \\ d_k \end{bmatrix}, \quad M \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(n_s+n_e)\times(n_s+n_d)} \tag{24}$$

where the Rosenbrock matrix M is given for the transfer function $G(z) := D + C(zI - A)^{-1}B$.

Lemma 6 (*Lemma 8.4 in Dullerud & Paganini, 2000*). *Suppose two signals d_k and e_k are bounded for all $k \in \mathbb{Z}_+$. Then $\|d_k\| \leq \|e_k\|$ if and only if there exists an operator Δ_k with $\|\Delta_k\| \leq 1$ such that $\Delta_k e_k = d_k$.*

Theorem 4 (*Small-Gain Theorem (Well-Known, e.g., Doyle, 1982)*). *Let the system (24) have $\rho(A) < 1$. If the system transfer function satisfies $\|G\|_\infty < 1$ then the origin $x = 0$ is a g.a.s. equilibrium point of*

$$x_{k+1} = F_l(M, \Delta_k)x_k \tag{25}$$

where $\Delta_k \in \mathbf{\Delta}^{[k]} \triangleq \{\tilde{\Delta}_k \in \mathbf{\Delta} : \bar{\sigma}(\tilde{\Delta}_k) \leq 1, \forall k \in \mathbb{Z}_+\}$.

Input–output scaling matrices can be included to reduce conservatism for structured Δ.

Corollary 2 (*Scaled Small-Gain Theorem (Doyle, 1982; Safonov, 1982)*). *Let the system (24) have $\rho(A) < 1$. If there exists a constant scaling matrix $D_s \in \mathcal{S}_c \triangleq \{D \in \mathbb{R}^{n_d\times n_d} : D\Delta = \Delta D, \forall \Delta \in \mathbf{\Delta}^{[k]}\}$ such that the scaled system transfer function satisfies $\|D_s G D_s^{-1}\|_\infty < 1$ then the origin $x = 0$ is a g.a.s. equilibrium point of (25).*

Below is a necessary and sufficient condition for simultaneously satisfying the nominal stability condition $\rho(A) < 1$ and the scaled $\mathcal{H}_\infty$-norm condition $\|D_s G D_s^{-1}\|_\infty < 1$.

Theorem 5 (*Scaled Small-Gain Stability Condition* (Braatz & Morari, 1997)). *The system* (24) *with* $n_e = n_d$ *has* $\rho(A) < 1$ *and* $\|D_s G D_s^{-1}\|_\infty < 1$, *where* $D_s \in \mathcal{S}_c$ *is a constant scaling matrix, if and only if there exists an invertible matrix* $T \in \mathcal{R} \subset \mathbb{R}^{n_s \times n_s}$ *such that*

$$\bar{\sigma}\left(\begin{bmatrix} T & 0 \\ 0 & D_s \end{bmatrix} M \begin{bmatrix} T & 0 \\ 0 & D_s \end{bmatrix}^{-1}\right) < 1. \tag{26}$$

5.3.2. Scaled Popov–Tsypkin criterion

An alternative approach to testing the stability of a DANN is to apply the Popov–Tsypkin criterion to its SNOF.

Theorem 6 (*Scaled Popov–Tsypkin Criterion* (Khalil, 2002)). *Consider the system* (18) *with* $D \equiv 0$. *If there exists* $\Lambda_P := diag\{\lambda_1 I, \ldots, \lambda_{n_d} I\}$, $\lambda_i \in [0, 1)$, $i = 1, \ldots, n_d$, *and constant scaling matrices* $T \in \mathcal{R}$ *and* $D_s \in \mathcal{S}$ *such that the transformed scaled system matrix*

$$\begin{bmatrix} T & 0 \\ 0 & D_s \end{bmatrix} \begin{bmatrix} A & B \\ C + \Lambda_P CA & I + \Lambda_P CB \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & D_s \end{bmatrix}^{-1}$$

yields the strictly positive real transfer function $G_{stP}(z)$, *then the system* (24) *has a g.a.s. equilibrium point* $x = 0$.

6. Discussion

6.1. Comparison of stability criteria: their equivalence and conservatism

This section shows the equivalence of the LFT and NL_q system structures, so that the stability conditions for one structure can be applied to the other.

6.1.1. LFT for NL_q *systems*

An NL_q system (19) can be rewritten as an LFT:

$$x_{k+1} = F_l(M, \Delta_{tot})\, x_k \tag{27}$$

where the Rosenbrock matrix M and the uncertainty block Δ_{tot} are

$$M = \left[\begin{array}{c|ccccc} 0 & 0 & \cdots & \cdots & 0 & I \\ \hline 0 & 0 & V_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & V_{q-1} & 0 \\ V_q & 0 & \cdots & 0 & 0 & 0 \\ 0 & V_1 & \cdots & 0 & 0 & 0 \end{array}\right]; \tag{28}$$

$$\Delta_{tot} = \mathrm{diag}\left\{\Gamma_2^{\mathrm{NL}_q}(x_k), \Gamma_3^{\mathrm{NL}_q}(x_k), \ldots, \Gamma_q^{\mathrm{NL}_q}(x_k), \Gamma_1^{\mathrm{NL}_q}(x_k)\right\}.$$

This result suggests that the stability conditions for NL_q systems are just special cases of more general stability conditions for the corresponding LFT. The next lemma shows the equivalence of the diagonal scaling condition in Theorem 3 and the scaled small gain condition in Theorem 5 for NL_q systems (20).

Lemma 7. *Consider the matrices* V_{tot} *in* (22) *and* M^{NL_q} *in* (28). *Then the LMI* (21) *or an equivalent condition* $\bar{\sigma}(D_{tot} V_{tot} D_{tot}^{-1}) < 1$ *holds for a positive diagonal matrix* $D_{tot} > 0$ *of the form in* (23) *if and only if there exists an invertible matrix* T *and a positive diagonal matrix* D_s *of the same dimension as* V_{tot} *that satisfy*

$$\underbrace{\bar{\sigma}\left(\begin{bmatrix} T & 0 \\ 0 & D_s \end{bmatrix} M^{\mathrm{NL}_q} \begin{bmatrix} T & 0 \\ 0 & D_s \end{bmatrix}^{-1}\right)}_{=:M_{sc}(T, D_s)} < 1. \tag{29}$$

Proof. ($\Rightarrow$) The LHS of (29) is equivalent to $\max\{\max_{i=1,\ldots,q-1}\{\bar{\sigma}(D_{s,i} V_i D_{s,i+1}^{-1})\}, \bar{\sigma}(D_{s,q} V_q T^{-1}), \bar{\sigma}(T D_{s,1}^{-1})\}$ where $D_{s,i}$ denotes the $(i-1)^{\text{th}}$ diagonal block of D_s for $i = 2, \ldots, q$ and $D_{s,1}$ is the qth diagonal block of D_s, i.e., the structure and indices for diagonal blocks are the same as D_{tot}. Since $\bar{\sigma}(D_{tot} V_{tot} D_{tot}^{-1}) < 1$ implies that $\bar{\sigma}(D_i(\prod_{j=0}^{i-1} V_{i-j}) D_{i+1}^{-1}) < 1$ for all $i = 1, \ldots, q$ with $V_{-k} := V_{q-k}$ for $k = 0, \ldots, q-1$, by choosing $D_s = D_{tot}$ and $T = \epsilon D_1$ for some $\epsilon \in \left(\max_{i=1,\ldots,q}\{\bar{\sigma}(D_i(\prod_{j=0}^{i-1} V_{i-j}) D_{i+1}^{-1})\}, 1\right)$ we have the inequality $\bar{\sigma}(M_{sc}(\epsilon D_1, D_{tot})) = \max\{\epsilon^{-1}\bar{\sigma}(D_{tot} V_{tot} D_{tot}^{-1})^q, \epsilon\} < 1$. ($\Leftarrow$) The LHS of (29) can be rewritten as $\bar{\sigma}(M_{sc}(T, D_s)) = \max\left\{\max_{i=1,\ldots,q-1}\{\bar{\sigma}(D_{s,i} V_i D_{s,i+1}^{-1})\}, \bar{\sigma}(D_{s,q} V_q T^{-1}), \bar{\sigma}(T D_{s,1}^{-1})\right\} < 1$, which implies that $\prod_{i=1}^{q-1} \bar{\sigma}(D_i V_i D_{i+1}^{-1}) \bar{\sigma}(D_q V_q T^{-1}) \bar{\sigma}(T D_1^{-1}) < 1$ where D_s is replaced by D_{tot}. Application of the sub-multiplicative property $\bar{\sigma}(AB) \le \bar{\sigma}(A)\bar{\sigma}(B)$ implies that $\prod_{i=1}^{q} \bar{\sigma}(D_i V_i D_{i+1}^{-1}) < 1$ where $D_{q+1} := D_1$, which implies that $\bar{\sigma}(D_{tot} V_{tot} D_{tot}^{-1}) < 1$.

An even less conservative result is obtainable using the scaled Popov–Tsypkin criterion in Theorem 6, but that results in a nonconvex problem for which no polynomial-time algorithm is known. Starting from the zero initial condition for the multiplier Λ_P and solving two LMIs iteratively can produce a less conservative condition for stability. Our sufficient stability conditions in Theorems 1 and 2 can be considered as extensions of the Scaled Popov–Tsypkin stability criterion obtained by generalizing the Popov multipliers and integrating them into the quadratic Lyapunov function to prove stability. An extensive comparison of our stability conditions to several classical absolute stability criteria is available in a Ph.D. thesis (Kim, 2009).

Remark 1. The scaled small-gain theorem (Theorem 5), the scaled Popov–Tsypkin criterion (Theorem 6), and our stability conditions (Theorems 1 and 2) are more flexible than stability conditions (e.g., Theorem 3) derived for NL_q systems.

6.1.2. An alternative of NL_q *systems: SNOF of (multi-layered) NSSM*

The SNOF can exploit more properties (characteristics) of the feedback-connected nonlinear operator, $\Gamma(\cdot)$, than robustness analysis applied to LFTs or NL_q systems. Section 4.2 showed that any SNOF with the sector-bounded nonlinearities can be rewritten as an LFT with norm-bounded uncertainties and that they share the same Rosenbrock matrix M.

Remark 2. The small-gain condition with constant D_s-scales is only a sufficient condition for robust stability. Seiler, Topcu, Packard, and Balas (2009) showed that the satisfaction of this sufficient condition implies that the system is quadratically stable and there exists a parameter-independent Lyapunov function that proves robust stability.

6.2. A numerical example

An example is used to compare the conservatism of the various results.

Example 1. Consider an NSSM with one hidden layer

$$x_{k+1} = \tanh(V x_k) \tag{30}$$

with the weighting matrix

$$V = \begin{bmatrix} 0.5893 & -0.4047 & 0.3142 & 0.3133 & -0.5308 \\ 1.0074 & -0.7935 & 0.7659 & 0.2278 & 0.0204 \\ -1.0197 & -0.0221 & 0.1484 & 0.1643 & 0.8982 \\ 1.1161 & -0.7743 & 0.4514 & -0.8473 & -0.0883 \\ 0.6870 & -1.0181 & 0.0379 & -0.5418 & -0.6798 \end{bmatrix}.$$

The computation of $\min_{D_s \in \mathcal{S}} \bar{\sigma}(D_s V D_s^{-1})$ using SeDuMi (Löfberg, 2004; Sturm, 1998) gave the unique optimal constant scaling $D_s = \text{diag}\{5.2564, 3.9958, 3.4698, 1.4107, 3.4963\}$ and the minimal value of $\bar{\sigma}(D_s V D_s^{-1}) = 2.1831$, which does not satisfy the NL_q stability condition in Theorem 3. Also computed was $\inf_{T \in \mathcal{R}} \min_{D_s \in \mathcal{S}} \bar{\sigma}(\text{diag}\{T, D_s\} M \text{diag}\{T, D_s\}^{-1}) = 1.4775$ with $M = \begin{bmatrix} 0 & I \\ V & 0 \end{bmatrix}$ obtained from (28), which does not satisfy the stability condition in Theorem 5. However, SeDuMi located feasible solutions to the LMIs in Theorems 1 and 2, which prove that the system (30) is g.a.s. As expected, our stability conditions in Theorems 1 and 2 are less conservative.

6.3. Applications of stability tests for DANNs

In many identification applications, the real system is known to be stable and stability must be imposed on the identified model, which can be performed by modifying identification algorithms to include stability constraints (Suykens et al., 1996). Any of the stability conditions for DANNs presented in this paper can be applied for this purpose. The stability criteria can be included as a convex constraint in the identification algorithm to force the identified system to be stable.

7. Conclusion

The characteristics of three classes of dynamic artificial neural network models are reviewed. All three model structures are able to universally approximate very broad classes of nonlinear dynamical systems, with bounds on the approximation error. These model structures are subsets of the standard nonlinear operator form, which is an older model structure first investigated in the 1960s. The SNOF is a promising model structure for black-box identification of nonlinear dynamical systems, due to its smaller approximation error for the same number of states, and that its structure is amenable for the development of efficient identification algorithms that explicitly take known properties of the real system into account. The global asymptotic stability (GAS) of the three classes of dynamic artificial neural network (DANNs) is studied by transforming them into the standard nonlinear operator form (SNOF) used in absolute stability analysis. Two sufficient conditions for GAS for such systems are derived in terms of linear matrix inequalities for which computationally efficient software is available. Theoretical and numerical comparisons are made between these conditions and several stability conditions in the literature that are applicable to DANNs. The proposed stability conditions are observed to be less conservative than several existing conditions.

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1D1A1B03034237).

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