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Robust static and fixed-order dynamic output feedback control of discrete-time parametric uncertain Luré systems: Sequential SDP relaxation approaches

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SUMMARY

Design methods are proposed for static and fixed-order dynamic output feedback controllers for discrete-time Luré systems with sector-bounded nonlinearities in the presence of parametric uncertainties described by polytopes. The derived design conditions are represented in terms of bilinear matrix inequalities, which are nonconvex. By using convex relaxation methods, controller design equations are derived for systems with multiple states, outputs, and nonlinearities in terms of linear matrix inequalities (LMIs) and iterative LMIs, which are associated with semidefinite programs. The proposed design methods are developed from stability conditions using parameter-dependent Lyapunov functions, and existing iterative numerical methods are adapted to solve certain classes of nonconvex optimization problems for controller design. Several numerical examples are provided to illustrate and verify the proposed design methods. Copyright © 2016 John Wiley & Sons, Ltd.

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KEY WORDS: Luré system; static output feedback control; fixed-order dynamic output feedback control; parameter-dependent Lyapunov function; robust control

1. INTRODUCTION

Lyapunov methods are powerful ways to analyze and design stabilizing controllers for nonlinear dynamical systems (e.g., see [1–6] and citations therein). Many stability results have been developed for a well-known benchmark problem known as the *Luré problem* [7–10]. The family of such nonlinear systems consists of a feedback interconnection of a linear system and certain classes of nonlinear functions that are characterized by input and output relations. Many important process models for practical applications can be represented as Luré systems, which include Wiener and Hammerstein models [11], dynamical neural network models [12], and systems with actuator saturation [13] or backlash [14].

Due to the theoretical and practical importance of these nonlinear systems, there have been much research effort to study the stability of Luré systems both in continuous-time and discrete-time cases. In particular, methods based on multiplier theory and positive realness of transfer functions corresponding to the linear system are also extensively studied. The Popov and Circle criteria are sufficient frequency domain conditions for absolute stability of the feedback interconnection of a continuous linear time-invariant system with a sector-bounded nonlinearity [15–20]. Its discrete-time counterparts are known as the Tsypkin criterion [21, 22] and the Jury–Lee

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criterion [23]. More recently, improvements in computing power and convex programming algorithms have resurged interest in stability analysis and control of Luré systems on the basis of convex optimization. In particular, stability analysis and control problems can be represented as conditions of feasibility and/or optimality of linear matrix inequalities (LMIs) [1], or more generally, semidefinite programs. On behalf of the well-known Kalman–Yakubovich–Popov lemma, the positive realness conditions of a transfer function can be equivalently represented as a problem of finding a feasible solution for LMIs. To reduce conservatism by incorporating further structure to the feedback-interconnected nonlinear functions, computationally tractable search for multipliers is important, and many researchers have recently investigated LMI-based conditions for finding multipliers and the associated Lyapunov solutions [24–31]. For relations between Lyapunov and multiplier methods, the readers are referred to [1, 32–35], for which parameters of multipliers can be seen as dual variables corresponding to the S-procedure [36] and integral constraints [37]. In [38], in addition, it was observed that stability of the Luré system with a scalar-valued input and output of feedback connected sector-bounded nonlinear function can be analyzed by checking existence of a common quadratic Lyapunov solution for the associated linear switched systems in which two switching system matrices have rank-one difference.

The discrete-time Luré system consists of the interconnection of a linear time-invariant (LTI) system in feedback with a nonlinear operator:

$$\begin{aligned} x_{[k+1]} &= Ax_{[k]} + B_p p_{[k]}, \\ q_{[k]} &= C_q x_{[k]} + D_{qp} p_{[k]}, \quad p_{[k]} = -\phi(q_{[k]}, k), \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times n_p}$, $C_q \in \mathbb{R}^{n_q \times n}$, $D_{qp} \in \mathbb{R}^{n_q \times n_p}$, and the nonlinear operator $\phi \in \Phi$, where Φ is a set of nonlinear functions that satisfy $\phi(0, k) \equiv 0$ for all $k \in \mathbb{Z}_+$ and have some specified input–output characteristics, such as satisfying a sector bound or having a slope within some specified range (the detailed mathematical descriptions for the nonlinear operators are given in Section 2).

Model uncertainties are typically represented as parametric variations or unmodeled dynamics. A matrix polytope is a standard representation for real parametric uncertainties (for example, see [39, 40]). In applications of Luré-type system models, such parametric uncertainties are ubiquitous and need to be taken into account for robust stability and stabilizing control problems. For example, identification models such as Wiener, Hammerstein, or neural network models that belong to a class of Luré-type systems inevitably include modeling errors, and for more accurate and reliable analysis such modeling errors can be represented as parametric uncertainties that should be considered for analysis and design. A commonly used method for deriving a robust stability test for an uncertain system with state matrices described by polytopes is to use stability conditions based on a single quadratic Lyapunov function for the entire uncertainty set, but this method is known to be conservative in general. The need of less conservative approaches has motivated the reduction of conservatism by using parameter-dependent Lyapunov functions (PDLFs).

Parameter-dependent Lyapunov functions that are quadratic in the state and have an affine dependence on uncertain parameters have been applied to derive LMI-based robust stability conditions for continuous-time linear systems [41–43] and discrete-time linear systems [44]. The robust stability tests involve the solution of parameterized LMIs. The design of stabilizing controllers for systems with nonlinearities and uncertainties is of interest in both control theory and practice, with static output feedback (SOF) being the simplest control to implement (e.g., see the survey paper by [45] and citations therein). The direct application of Lyapunov analysis to SOF design, even for linear time-invariant systems, results in optimization over bilinear matrix inequalities (BMIs), which are not convex. These optimization problems can be solved very slowly using global optimization methods or a local solution can be obtained by iterative linear matrix inequality (ILMI) approaches. Several ILMI-based algorithms have been developed for the SOF controller design of LTI systems (for example, see [46–49]).

The main contribution of this paper is to develop design methods for robust stabilizing SOF controllers for Luré systems with system matrices perturbed by parametric uncertainties. For stability analysis, a scaled Popov criterion and the associated Lyapunov method incorporating the

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S-procedure and matrix algebra are used. Because of existence of bilinear terms corresponding to multiplications of variables incurred by the S-procedure and the control gain parameters, the resultant design problems are nonconvex. Convex relaxation methods on the basis of semidefinite programming (SDP) and sequential SDP are presented for the design conditions. Depending on the specific control objectives and the methods of convex relaxation, the design methods are written in terms of LMIs or ILMIs. For the same control objectives, the results are extended to the computation of fixed-order dynamic output feedback controllers.

As a parallel set of results, in [50], two LMI-based procedures were proposed for the design of observer-based output feedback controllers for a Luré-type system with conic-sector-bounded slope-restricted nonlinearities. Observer design methods were proposed for two different strategies: (a) based on an observer–controller separation and (b) based on simultaneous design derived from the variable reduction lemma (a.k.a. Finsler's lemma). While [50] takes uncertainties in the nonlinear function and disturbance rejection into account, parametric uncertainties in the linear system were not considered, which is the main motivation of this paper.

This paper is organized as follows. Section 2 describes the discrete-time Luré systems and summarizes theoretical results used in the rest of the paper. Section 3 derives stabilizing static and fixed-order dynamic output feedback control designs for nominal Luré systems, and Section 4 derives corresponding results for Luré systems with polytopic parametric uncertainty. The SOF control design methods proposed in this paper are demonstrated and compared in numerical examples in Section 5. Section 6 concludes the paper.

2. MATHEMATICAL PRELIMINARIES

2.1. Notation and definitions

The notation is quite standard. $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of all nonnegative integers and the set of all nonnegative real numbers, respectively. $\|\cdot\|$ is the Euclidean norm for vectors, or the corresponding induced matrix norm for matrices. 0 and I denote the matrix whose components are all zeros and the identity matrix of compatible dimension, respectively. $X \succ 0$ denotes that the matrix X is positive definite, $X \succeq 0$ denotes that X is positive semidefinite, and $X \prec 0$ and $X \preceq 0$ denote negative definite and semidefinite matrices, respectively. $\mathrm{Sym}(X) := X + X^{\mathrm{T}}$ and $X^{\perp}$ denotes a full-rank matrix orthogonal to X. $\mathbb{S}^{n\times n}$ is the set of symmetric matrices in $\mathbb{R}^{n\times n}$. For a given set S, $\mathbf{Co}(S)$ refers to the convex hull of S, which is a minimal convex containing S. Throughout this paper, the nonlinearity ϕ is taken to be a member of some specific classes of nonlinear operators.

Definition 1 (§Definitions of classes of nonlinear operators)
A nonlinearity $\phi : \mathbb{R}^{n_q}\times\mathbb{Z}_+ \to \mathbb{R}^{n_q}$ is of family $\Phi_{\mathrm{sb}}^{|\alpha|}$ if $\left[\alpha_i^{-1}\phi_i(\sigma,k)+\sigma\right]\left[\alpha_i^{-1}\phi_i(\sigma,k)-\sigma\right] \leqslant 0$, and is of family $\Phi_{\mathrm{sr}}^{|\mu|}$ if $-\mu_i \leqslant \frac{\phi_i(\sigma,k)-\phi_i(\hat{\sigma},k)}{\sigma-\hat{\sigma}} \leqslant \mu_i$ for all $\sigma, \hat{\sigma} \in \mathbb{R}^{n_q}$, $k \in \mathbb{Z}_+$, and $i = 1,\ldots,n_q$, where the subscript i denotes the i-th element of the vector and $0/0$ is interpreted as 0. A nonlinearity $\phi : \mathbb{R}^{n_q}\times\mathbb{Z}_+ \to \mathbb{R}^{n_p}$ is of family $\bar{\Phi}_{\mathrm{sb}}^{\alpha}$ if $\|\phi(\sigma,k)\| \leqslant \alpha\|\sigma\|$ holds for all $\sigma \in \mathbb{R}^{n_q}$, $k \in \mathbb{Z}_+$. A nonlinear mapping $\phi : \mathbb{R}^{n_q}\times\mathbb{Z}_+ \to \mathbb{R}^{n_p}$ is of family $\bar{\Phi}_{\mathrm{sr}}^{\mu}$ if $\|\phi(\sigma,k)-\phi(\hat{\sigma},k)\| \leqslant \mu\|\sigma-\hat{\sigma}\|$ for all $\sigma \neq \hat{\sigma} \in \mathbb{R}^{n_q}$, $k \in \mathbb{Z}_+$.

Note that the aforementioned classes of nonlinear functions are allowed to have time dependence, whereas most of existing literature (including textbooks [17, 51]) on the absolute stability analysis and the Luré problems consider time-invariant static functions. Similar definitions for time-dependent sector-bounded nonlinear functions can be found in [52] and considered in [53], for example. Any global input–output characteristic in Definition 1 can be relaxed to its counterpart of local properties for semi-global or local analysis.

§The classes of nonlinear functions $\Phi_{\mathrm{sb}}^{|\alpha|}$ and $\Phi_{\mathrm{sr}}^{|\mu|}$ are also known as sector-bounded and slope-restricted nonlinear functions [7, 20].

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2.2. Lagrange relaxations

The positiveness of a quadratic function $f_0(x)$ in a constraint set expressed in terms of quadratic functions, $f_i(x)$, $i = 1, \ldots, m$, can be implied by a relaxed form with (Lagrange) multipliers [36]. This approach is called *the S-procedure*, which is a special case of Lagrange relaxation in which the constraints are represented in terms of quadratic functions, and the multipliers can be combined into an LMI inequality. For convenience, the form of the S-procedure used in the proofs of this paper is given in the succeeding discussions.

Lemma 1 (S-procedure for quadratic inequalities)
For the symmetric matrices R_i, $i = 0, \ldots, m$, consider the two sets:
(S1) $\zeta^* R_0 \zeta < 0, \ \ \forall \zeta \in \{\zeta \in \mathbb{F}^n | \zeta^* R_i \zeta \leqslant 0, \ \ \forall i = 1, \ldots, m\}$, where $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$;
(S2) $\exists \tau_i \geqslant 0, i = 1, \ldots, m$ such that $R_0 - \sum_{i=1}^{m} \tau_i R_i \prec 0$.
The feasibility of (S2) implies (S1).

2.3. Variable reduction lemma

In LMI-based robust control theory, it is common to transform a set of nonconvex inequalities to an LMI that is either equivalent or a conservative approximation, or to eliminate some decision variables in the original inequalities such that the reduced optimization problem is convex in the remaining variables. In the elimination process, the eliminated variables that satisfy the original non-convex inequalities can be reconstructed from the solution of the reduced LMI. *Finsler's lemma* (a.k.a. the *variable reduction lemma*) is a well-known result for the elimination of parameters to reduce a particular class of BMI to an equivalent LMI.

Lemma 2 (Finsler's lemma [1]*)*
The following statements are equivalent:
(a) $\zeta^{\mathrm{T}} S \zeta > 0$ for all $\zeta \neq 0$ such that $Rx = 0$;
(b) $(R^{\perp})^{\mathrm{T}} S R^{\perp} \succ 0$ for $RR^{\perp} = 0$;
(c) $S + \rho R^{\mathrm{T}} R \succ 0$ for some scalar ρ;
(d) $S + XR + R^{\mathrm{T}} X^{\mathrm{T}} \succ 0$ for some unstructured matrix X.

2.4. Discrete-time Luré systems

This paper considers the design of static and fixed-order dynamic output feedback controllers for some classes of Luré systems with multi-valued nonlinear mappings in a negative feedback interconnection. The global (or local) asymptotic (or exponential) stability of the closed-loop system is guaranteed in the presence of the internal and/or external perturbations. The discrete-time Luré systems

$$\begin{aligned} x_{[k+1]} &= Ax_{[k]} + B_p p_{[k]} + \chi(x_{[k]}, u_{[k]}, k), \\ q_{[k]} &= C_q x_{[k]}, \quad p_{[k]} = -\phi(q_{[k]}, k), \\ y_{[k]} &= C_y x_{[k]}, \end{aligned} \tag{2}$$

are considered where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{n_y}$ denote the state and the measurement vector, respectively, $q \in \mathbb{R}^{n_q}$ and $p \in \mathbb{R}^{n_p}$ are the input and output of the nonlinearity, respectively, and $u \in \mathbb{R}^{n_u}$ is the control input. In addition, the nonlinear function $\chi : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{Z}_+ \to \mathbb{R}^n$ is assumed to be Lipschitz in the first argument. The nonlinear operator $\phi \in \Phi$, where Φ is a set of nonlinear functions that satisfy $\phi(0, k) \equiv 0$ for all $k \in \mathbb{Z}_+$ and have some specified input–output characteristics described in Definition 1. Beyond Luré systems with fixed values of the system matrices, we also consider Luré systems in which the system matrices and control function χ are dependent on uncertain parameters. More specifically, these maps are defined by sets that affinely depend on the uncertain parameter $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$, where the set Θ is assumed to be compact and convex.

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2.5. Stability analysis and state feedback control

The lemma in the succeeding texts provides a sufficient condition for analyzing the stability of the Luré system (2) with $\phi \in \Phi_{\mathrm{sb}}^{|\alpha|}$ or $\phi \in \bar{\Phi}_{\mathrm{sb}}^{\alpha}$ that is later applied to the formulation of control design methods.

Lemma 3
The system (1) with the memoryless nonlinearity $\phi \in \Phi_{\mathrm{sb}}^{|\alpha|}$ and $\chi = 0$ is globally asymptotically stable (GAS) if there exists a positive-definite matrix $Y = Y^{\mathrm{T}}$ and a diagonal positive-definite matrix T such that the LMI

$$\begin{bmatrix} -Y & * & * & * \\ 0 & -T & * & * \\ AY & -B_pT & -Y & * \\ C_qY & 0 & 0 & -S_\alpha T \end{bmatrix} \prec 0, \quad (3)$$

is feasible, where $S_\alpha = \mathrm{diag}\{1/\alpha_1^2, \ldots, 1/\alpha_{n_p}^2\}$. Similarly, the system (1) with the memoryless nonlinearity $\phi \in \bar{\Phi}_{\mathrm{sb}}^{\alpha}$ is GAS if there exists $Y = Y^{\mathrm{T}} \succ 0$ such that the LMI (3) with $S_\alpha = \gamma \mathrm{I}$, $\gamma \equiv 1/\alpha^2$, and $T = \mathrm{I}$ is feasible.

Proof
(Sketch) The LMIs (3) are obtained from applying the S-procedure in Lemma 1 and the Schur complement lemma [1, Chapter 2] to a Luré-type Lyapunov function. The stability condition used in derivation of this LMI condition corresponds to a scaled Popov criterion. Details of the proof are in [35, Chapter 4]. □

Now consider the Luré system (2) with a control affine term $\chi(x_{[k]}, u_{[k]}, k) = B_u u_{[k]}$ with controllable pair (A, B_u) and design objective of determining a linear state feedback controller

$$u_{[k]} = K_s x_{[k]} \quad (4)$$

where K_s is the control gain matrix of compatible dimension. Applying the feedback controller (4) to the system (2) results in the closed-loop system:

$$x_{[k+1]} = (A + B_u K_s)x_{[k]} - B_p\phi(q_{[k]}, k). \quad (5)$$

The lemma in the succeeding texts provides a sufficient LMI condition for the linear state feedback controller (4) to stabilize the closed-loop system (5).

Lemma 4
The closed-loop system (5) with $\phi \in \Phi_{\mathrm{sb}}^{|\alpha|}$ is globally asymptotically stabilized by the state feedback controller $u_{[k]} = K_s x_{[k]}$ with $K_s = WY^{-1}$ if the LMI

$$\begin{bmatrix} -Y & * & * & * \\ 0 & -T & * & * \\ AY + B_uW & -B_pT & -Y & * \\ C_qY & 0 & 0 & -S_\alpha T \end{bmatrix} \prec 0, \quad (6)$$

is feasible for $Y = Y^{\mathrm{T}} \succ 0$, a diagonal matrix $T > 0$, and W, where $S_\alpha = \mathrm{diag}\{1/\alpha_1^2, \ldots, 1/\alpha_{n_p}^2\}$. If the LMI (6) with $S_\alpha = \gamma\mathrm{I}$, $\gamma \equiv 1/\alpha^2$, and $T = \mathrm{I}$ is feasible, then the closed-loop system (5) with $\phi \in \bar{\Phi}_{\mathrm{sb}}^{\alpha}$ is globally asymptotically stabilized by the state feedback control law (4) with $K_s = WY^{-1}$.

Proof
(Sketch) Similarly to (3), the LMI (6) is obtained from applying the S-procedure in Lemma 1 and the Schur complement lemma [1, Chapter 2] to a Luré-type Lyapunov function. The stability

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condition used in the derivation of this LMI condition corresponds to a scaled Popov criterion. Details of the proof are available in [35, Chapter 4]. □

2.6. *Parameter-dependent Lyapunov functions*

Robustness analysis and synthesis have been extensively studied for linear systems with polytopic uncertainty. A widely used approach to these problems is to search for a common quadratic Lyapunov function that is reformulated into a sufficient condition written in terms of matrix inequalities. The use of a single quadratic Lyapunov function can result in highly conservative results, which motivated subsequent efforts that reduce conservatism by using PDLFs. To illustrate the use of PDLFs while presenting some theoretical results used later in the paper, consider the uncertain system

$$x_{[k+1]} = A(\theta_{[k]})x_{[k]}, \tag{7}$$

where A is affine in $\theta_{[k]} \in \Theta \subset \mathbb{R}^{n_\theta}$, $k \in \mathbb{Z}_+$. Consider a Lyapunov matrix that is also affine in the parametric uncertainty vector θ, that is, $X(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]})X_j$, where $\sum_{j=1}^{n_v} \rho_j(\theta_{[k]}) = 1$, $\rho_j(\theta_{[k]}) \in [0, 1]$ for all $\theta_{[k]} \in \Theta \subset \mathbb{R}^{n_\theta}$, and $X_j = X_j^{\mathrm{T}}$ is real for each $j = 1, \ldots, n_v$. In addition, suppose that Θ is a convex hull with a finite set of vertices Θ_v, that is, $\Theta = \mathbf{Co}(\Theta_v)$. It is straightforward to apply Lyapunov analysis to show that if the matrix inequality

$$A^{\mathrm{T}}(\theta_{[k]})X(\theta_{[k+1]})A(\theta_{[k]}) - X(\theta_{[k]}) \prec 0 \tag{8}$$

holds for all $\theta_{[k]}, \theta_{[k+1]} \in \Theta \subset \mathbb{R}^{n_\theta}$ at each $k \in \mathbb{Z}_+$, then the origin of the uncertain system (7) is GAS. Because the parameter-dependent matrix (8) is not jointly convex in $\theta_{[k]}$ and $\theta_{[k+1]}$, it is desirable to find an equivalent LMI condition to (8). To do this, the next lemma is adapted from a similar result in [44].

Lemma 5 (Polytopic parameter-dependent systems)
The origin of the uncertain system (7) is GAS for any time-varying uncertain vector $\theta_{[k]} \in \Theta \subset \mathbb{R}^{n_\theta}$ if any of the following inequalities holds for the specified variables:

1. There exists a Lyapunov matrix $X(\theta_{[k]}) = X^{\mathrm{T}}(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]})X_j \succ 0$ such that

$$A^{\mathrm{T}}(\theta_{[k]})X(\theta_{[k+1]})A(\theta_{[k]}) - X(\theta_{[k]}) \prec 0, \ \forall \theta \in \Theta; \tag{9}$$

2. There exists a Lyapunov matrix $Y(\theta(k)) = Y^{\mathrm{T}}(\theta(k)) = \sum_{j=1}^{n_v} \rho_j(\theta(k))Y_j \succ 0$ such that

$$\begin{bmatrix} Y(\theta_{[k]}) & Y(\theta_{[k+1]})A^{\mathrm{T}}(\theta_{[k]}) \\ A(\theta_{[k]})Y(\theta_{[k+1]}) & Y(\theta_{[k+1]}) \end{bmatrix} \succ 0 \tag{10}$$

for every $\theta \in \Theta$;

3. There exists a Lyapunov matrix $X(\theta_{[k]}) = X^{\mathrm{T}}(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]})X_j \succ 0$ and G of compatible dimensions such that

$$\begin{bmatrix} X(\theta_{[k]}) & A^{\mathrm{T}}(\theta_{[k]})G^{\mathrm{T}} \\ GA(\theta_{[k]}) & \mathrm{Sym}(G) - X(\theta_{[k+1]}) \end{bmatrix} \succ 0 \tag{11}$$

for every $\theta \in \Theta$, or equivalently,

$$\begin{bmatrix} X_j & A_j^{\mathrm{T}}G^{\mathrm{T}} \\ GA_j & \mathrm{Sym}(G) - X_i \end{bmatrix} \succ 0, \ \forall i, j = 1, \ldots, n_v; \tag{12}$$

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4. There exists a Lyapunov matrix $Y(\theta_{[k]}) = Y^{\mathrm{T}}(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]})Y_j \succ 0$ and H of compatible dimensions such that

$$\begin{bmatrix} \mathrm{Sym}(H) - Y(\theta_{[k]}) & H^{\mathrm{T}}A^{\mathrm{T}}(\theta_{[k]}) \\ A(\theta_{[k]})H & Y(\theta_{[k+1]}) \end{bmatrix} \succ 0 \tag{13}$$

for every $\theta \in \Theta$, or equivalently,

$$\begin{bmatrix} \mathrm{Sym}(H) - Y_j & H^{\mathrm{T}}A_j^{\mathrm{T}} \\ A_j H & Y_i \end{bmatrix} \succ 0, \ \forall i, j = 1, \dots, n_v. \tag{14}$$

The inequalities (11) and (13) are jointly affine in the uncertain parameter vectors $\theta_{[k]}$ and $\theta_{[k+1]}$. The LMIs (12) and (14) that check feasibility only at vertices follow from the next standard lemma, which follows from the convexity of linear matrix inequalities. The positive-definite matrix X is used to refer to a primal Lyapunov solution, while the positive-definite matrix Y is used to refer to a dual variable for the associated dual Lyapunov solution. The matrix inequalities in terms of the Lyapunov matrix X have a different structure than the matrix inequalities in terms of the Lyapunov matrix Y, as seen by comparing (11)–(12) with (13)–(14). As seen in Section 3, the matrix inequalities in terms of X in (11)–(12) have a structure that enables the derivation of design methods for systems with uncertainties in the output channel, and the matrix inequalities in terms of Y in (13)–(14) have a structure that enables the derivation of design methods for systems with uncertainties in the input channel.

Lemma 6
Let Θ be a convex hull and Θ_v be the set of its vertices, each vertex having a finite number of elements. For a given matrix-valued function $F : \mathbb{R}^m \times \Theta \to \mathbb{S}^{N \times N}$ that is affine in the second argument, the set $\mathcal{S}(F, \Theta) \triangleq \{x \in \mathbb{R}^m : F(x, \theta) \prec 0, \ \forall \theta \in \Theta\}$, whose cardinality is not necessarily finite, is nonempty if and only if the finite set $\mathcal{S}(F, \Theta_v)$ is nonempty.

Remark 1
Note that no product terms of X_j, Y_j, and A_j appear in (11)–(14), which is indispensable to reducing the corresponding controller synthesis problems in the next sections to LMIs or ILMIs.

2.7. *Static output feedback for LTI systems*

The closed-loop LTI system

$$x_{[k+1]} = Ax_{[k]} + B_u u_{[k]}, \quad y_{[k]} = C_y x_{[k]}, \tag{15}$$

is GAS with an output feedback controller $u_{[k]} = K_o y_{[k]}$ if and only if the matrix $A + B_u K_o C_y$ is Schur stable, that is, the eigenvalues of $A + B_u K_o C_y$ are inside the open unit circle in the domain of complex variables. This condition for a stabilizing controller is equivalent to the existence of a gain matrix K_o that satisfies discrete-time Lyapunov inequality

$$(A + B_u K_o C_y)^{\mathrm{T}} Y (A + B_u K_o C_y) - Y \prec 0 \tag{16}$$

for some $Y = Y^{\mathrm{T}} \succ 0$, which can be rewritten in terms of the dual version of an equivalent continuous-time Lyapunov inequality

$$(A_d + B_{u,d} K_o C_{y,d}) X_d + X_d (A_d + B_{u,d} K_o C_{y,d})^{\mathrm{T}} \prec 0, \tag{17}$$

where

$$A_d \triangleq \begin{bmatrix} -0.5\mathrm{I} & 0 \\ A & -0.5\mathrm{I} \end{bmatrix}, \ B_{u,d} \triangleq \begin{bmatrix} 0 \\ B_u \end{bmatrix}, \ C_{y,d} \triangleq \begin{bmatrix} C_y & 0 \end{bmatrix},$$

and $X_d = \mathrm{diag}\{X, X\}$ with $X = Y^{-1}$.

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The LMI (17) contains bilinear terms in the unknown (decision) matrices X and K_o, separated by constant system matrices. Checking the feasibility of the inequality (17) is a nonconvex problem that is known to be NP-hard [54, 55]. This nonconvex inequality (17) can be reduced to a simpler set of coupled linear matrix inequalities [54] given in the next lemma, which follows from applying Finsler's lemma (Lemma 2) to (17).

Lemma 7
The matrix inequality (17) holds for some K_o and X (or Y) if and only if X (or Y) satisfies the two matrix inequalities:

$$B_{u,d}^{\perp}\left(A_d X_d + X_d A_d^{\mathrm{T}}\right)\left(B_{u,d}^{\perp}\right)^{\mathrm{T}} \prec 0, \tag{18}$$

$$\left(C_{y,d}^{\mathrm{T}}\right)^{\perp}\left(A_d^{\mathrm{T}} Y_d + Y_d A_d\right)\left(\left(C_{y,d}^{\mathrm{T}}\right)^{\perp}\right)^{\mathrm{T}} \prec 0, \tag{19}$$

where $XY = YX = \mathrm{I}$ such that $X_d Y_d = Y_d X_d = \mathrm{I}$.

Finding $X = X^{\mathrm{T}} \succ 0$ and $Y = Y^{\mathrm{T}} \succ 0$ that jointly satisfy the two matrix inequalities (18) and (19) with $YX = XY = \mathrm{I}$ is still a nonconvex problem, but search methods to obtain a local suboptimal solution have been developed based on iterative sequential solutions of the two LMI problems with respect to X and Y (6). By substituting a solution X (or Y) for (18) and (19) into (17) or (16), a stabilizing static output feedback control gain matrix K_o can be computed for the system (15).

3. OUTPUT FEEDBACK CONTROL OF DISCRETE-TIME LURÉ SYSTEMS

This section derives static and fixed-order dynamic output feedback controller design equations for discrete-time Luré systems with either $\phi \in \bar{\Phi}_{\mathrm{sb}}^{\alpha}$ or $\phi \in \Phi_{\mathrm{sb}}^{|\alpha|}$.

3.1. Static output feedback controller design

Consider the SOF control problem for the Luré system

$$\begin{aligned} x_{[k+1]} &= Ax_{[k]} + B_u u_{[k]} - B_p \phi(q_{[k]}, k) \\ y_{[k]} &= C_y x_{[k]}, \quad q_{[k]} = C_q x_{[k]}, \end{aligned} \tag{20}$$

where $\phi(\cdot,\cdot)$ is in a specific class $\bar{\Phi}_{\mathrm{sb}}^{\alpha}$ or $\Phi_{\mathrm{sb}}^{|\alpha|}$. The triplet (A, B_u, C_y) is assumed to be stabilizable and detectable.

For SOF controller synthesis for the nominal Luré system (20), replacing A by $A + B_u K_o C_y$ in the LMI (3) results in the optimization

$$\begin{aligned} &\min_{Y,K_o} \ \gamma \\ &\text{s.t. } Y \succ 0, \quad \begin{bmatrix} -Y & * & * & * \\ 0 & -\mathrm{I} & * & * \\ AY + B_u K_o C_y Y & -B_p & -Y & * \\ C_q Y & 0 & 0 & -\gamma \mathrm{I} \end{bmatrix} \prec 0. \end{aligned} \tag{21}$$

The LMI constraint (21) can be rewritten in the same form as (17):

$$\left(\bar{A}_{\gamma} + \bar{B}_u K_o \bar{C}_y\right)\bar{Y} + \bar{Y}\left(\bar{A}_{\gamma} + \bar{B}_u K_o \bar{C}_y\right)^{\mathrm{T}} \prec 0, \tag{22}$$

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where

$$\bar{A}_\gamma \triangleq \begin{bmatrix} -0.5\mathrm{I} & 0 & 0 & 0 \\ 0 & -0.5\mathrm{I} & 0 & 0 \\ A & -B_p & -0.5\mathrm{I} & 0 \\ C_q & 0 & 0 & -0.5\gamma\mathrm{I} \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} 0 \\ 0 \\ B_u \\ 0 \end{bmatrix},$$
$$\bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix}, \quad \bar{Y} \triangleq \mathrm{diag}\{Y, \mathrm{I}, Y, \mathrm{I}\}.$$

A sufficient LMI condition for the nonconvex problem (17) derived by [56] is used below to derive suboptimal static output feedback controller design methods.

Theorem 1
Consider the system (20) with C_y of full row-rank and the SDP

$$\begin{aligned} &\min_{Y,N} \ \gamma \\ &\text{s.t. } Y \succ 0, \\ &\qquad \bar{A}_\gamma \bar{Y} + \bar{Y}\bar{A}_\gamma^{\mathrm{T}} + \bar{B}_u N \bar{C}_y + \bar{C}_y^{\mathrm{T}} N^{\mathrm{T}} \bar{B}_u^{\mathrm{T}} \prec 0. \end{aligned} \tag{23}$$

The closed-loop system (20) with $\phi \in \Phi_{\mathrm{sb}}^{\alpha^*}$, where $\alpha^* = 1/\sqrt{\gamma^*}$ and γ^* is the optimal value of (23), is globally asymptotically stabilized by the static output feedback controller $K_o = NM^{-1}$ where the full rank matrix M satisfies $MC_y = C_y Y$.

Proof
The proof is straightforward from the results of Theorem 1 in [56]. As C_y is a full row-rank matrix, there exists a unique solution $M \in \mathbb{R}^{n_y \times n_y}$ satisfying the linear matrix equation $M\bar{C}_y = \bar{C}_y \bar{Y}$, which is equivalent to $MC_y = C_y Y$. Thus, if N and Y solve the SDP (23), then setting $K_o = NM^{-1}$ solves the inequality (22) and gives a stabilizing output-feedback control law $u = K_o y$ for the system (20) where the nonlinear operator is $\phi \in \Phi_{\mathrm{sb}}^{1/\sqrt{\gamma^*}}$. □

Theorem 2
Consider the system (20) with B_u of full column-rank and the SDP

$$\begin{aligned} &\min_{X,N} \ \gamma \\ &\text{s.t. } X \succ 0, \\ &\qquad \bar{X}\bar{A}_\gamma + \bar{A}_\gamma^{\mathrm{T}} \bar{X} + \bar{B}_u N \bar{C}_y + \bar{C}_y^{\mathrm{T}} N^{\mathrm{T}} \bar{B}_u^{\mathrm{T}} \prec 0, \end{aligned} \tag{24}$$

where $\bar{X} = \mathrm{diag}\{X, \mathrm{I}, X, \mathrm{I}\}$. The closed-loop system (20) for $\phi \in \Phi_{\mathrm{sb}}^{\alpha^*}$ where $\alpha^* = 1/\sqrt{\gamma^*}$ and γ^* is the optimal value of (23) is globally asymptotically stabilized by the static output feedback controller $K_o = M^{-1}N$ where the full rank matrix M satisfies $B_u M = XB_u$.

Proof
The proof is similar to the proof of Theorem 1. As B_u is a full column-rank matrix, there exists a unique solution $M \in \mathbb{R}^{n_u \times n_u}$ satisfying the linear matrix equation $\bar{B}_u M = \bar{X}\bar{B}_u$, which is equivalent to $B_u M = XB_u$. Thus, if N and X solve the SDP (24), then setting $K_o = M^{-1}N$ solves the inequality (22) and gives a stabilizing output-feedback control law $u = K_o y$ for the system (20) where the nonlinear operator is $\phi \in \Phi_{\mathrm{sb}}^{1/\sqrt{\gamma^*}}$. □

Apart from the previous design methods to compute stabilizing SOF controllers based on parameterization of the control gain K_o, the next theorem proposes another approach using the Finsler's lemma to solve the inequality (22) from which a stabilizing SOF control gain K_o can be obtained.

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Theorem 3
There exists a stabilizing SOF controller gain matrix K_o for the system (20) with $\phi \in \bar{\Phi}_{\rm sb}^{\alpha}$ and upper sector bound $\alpha \triangleq 1/\sqrt{\gamma}$ if there exists $Y = Y^{\rm T} \succ 0$ such that

$$\bar{B}_u^{\perp} \left(\bar{A}_\gamma \bar{Y} + \bar{Y}\bar{A}_\gamma^{\rm T}\right)\left(\bar{B}_u^{\perp}\right)^{\rm T} \prec 0, \tag{25}$$

$$\left(\bar{C}_y^{\rm T}\right)^{\perp}\left(\bar{X}\bar{A}_\gamma + \bar{A}_\gamma^{\rm T}\bar{X}\right)\left(\left(\bar{C}_y^{\rm T}\right)^{\perp}\right)^{\rm T} \prec 0, \tag{26}$$

where $\bar{Y} = \text{diag}\{Y, \text{I}, Y, \text{I}\}$ and $\bar{X} = \bar{Y}^{-1}$.

Proof
The proof directly follows from (22) and Lemma 7. □

The SDPs in Theorems 1–3 can be reformulated to compute stabilizing SOF controllers of the Luré system whose nonlinear operator is described by a more general class of sector conditions, $\phi \in \Phi_{\rm sb}^{|\alpha|}$, which is defined as being componentwise. This reformulation is accomplished by replacing the matrices $\bar{A}_\gamma$, $\bar{Y}$, and $\bar{X}$ in the LMIs of each design criterion by

$$\begin{aligned}
\bar{A}_{S_\alpha} &\triangleq \begin{bmatrix} -0.5\text{I} & 0 & 0 & 0 \\ 0 & -0.5\text{I} & 0 & 0 \\ A & -B_p & -0.5\text{I} & 0 \\ C_q & 0 & 0 & -0.5S_\alpha \end{bmatrix}, \\
\bar{Y}_T &\triangleq \text{diag}\{Y, T, Y, T\}, \\
\bar{X}_T &\triangleq \text{diag}\{X, T, X, T\},
\end{aligned}$$

respectively, where $S_\alpha = \text{diag}\{1/\alpha_1^2, \ldots, 1/\alpha_{n_p}^2\}$ and $T \succ 0$ is a diagonal matrix.

3.2. Fixed-order dynamic output feedback control

When the desired order of a dynamic output feedback controller is less than or equal to the order of the nominal system, that is, $n_c \leqslant n$, the design problem can be reformulated as an equivalent static output feedback design problem, in the same manner as for LTI systems [57]. Consider a state-space realization of the dynamical output feedback controller

$$\zeta_{[k+1]} = A_c\zeta_{[k]} + B_c y_{[k]}, \quad u_{[k]} = C_c\zeta_{[k]} + D_c y_{[k]}, \tag{27}$$

which has the transfer function $C(z) = C_c(z\text{I} - A_c)^{-1}B_c + D_c$, where

$$u_z = C(z)y_z, \tag{28}$$

and u_z and y_z are the z-transformations of $u_{[k]}$ and $y_{[k]}$, respectively. The closed-loop Luré system obtained with (27) can be written as

$$\begin{aligned}
\bar{x}_{[k+1]} &= \bar{A}\bar{x}_{[k]} + \bar{B}_u u_{[k]} - \bar{B}_p\phi(\bar{q}_{[k]}, k), \\
\bar{y}_{[k]} &= \bar{C}_y\bar{x}_{[k]}, \quad \bar{q}_{[k]} = \bar{C}_q\bar{x}_{[k]},
\end{aligned} \tag{29}$$

where $\bar{A} \triangleq \text{diag}\{A, 0\}$, $\bar{B}_u \triangleq \text{diag}\{B_u, \text{I}\}$, $\bar{B}_p^{\rm T} \triangleq \left[B_p^{\rm T}\, 0\right]$, $\bar{C}_y \triangleq \text{diag}\{C_y, \text{I}\}$, $\bar{C}_q \triangleq \left[C_q\, 0\right]$, $\bar{x} \triangleq (x^{\rm T}, \zeta^{\rm T})^{\rm T}$ is the concatenated state, and $u_{[k]}$ is the output of a static output feedback controller

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$$u_{[k]} = K_{\text{dof}} \bar{y}_{[k]}; \quad K_{\text{dof}} \triangleq \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \tag{30}$$

Hence, all of the static output feedback control results presented in this paper can be applied to dynamic fixed-order feedback control problems when $n_c \leqslant n$ by using the transformed state-space realization

$$\bar{G}(z) \triangleq \left[\begin{array}{c|cc} \bar{A} & \bar{B}_u & \bar{B}_p \\ \hline \bar{C}_y & 0 & 0 \\ \bar{C}_q & 0 & 0 \end{array}\right]. \tag{31}$$

4. OUTPUT FEEDBACK CONTROL FOR POLYTOPIC DISCRETE-TIME LURÉ SYSTEMS

This section derives static output feedback controller design equations for polytopic uncertain discrete-time Luré systems with either a nonlinear function $\phi \in \bar{\Phi}_{\text{sb}}^{\alpha}$ or $\phi \in \Phi_{\text{sb}}^{|\alpha|}$ in Figure 1.

4.1. With parametric uncertainties in the output channel

Consider the system

$$\begin{aligned} x_{[k+1]} &= A(\theta_{[k]})x_{[k]} + B_u u_{[k]} - B_p \phi(q_{[k]}, k), \\ y_{[k]} &= C_y(\theta_{[k]})x_{[k]}, \quad q_{[k]} = C_q x_{[k]}, \end{aligned} \tag{32}$$

where $x_{[k]} \in \mathbb{R}^n$ is the state and $u_{[k]} \in \mathbb{R}^{n_u}$ is the control input at time $k \in \mathbb{Z}_+$, and $\theta_{[k]} \in \Theta$ specifies the parametric uncertainty, where $\Theta \subset \mathbb{R}^{n_\theta}$ is a polytope that is closed and compact. Assume that the mappings $A : \Theta \to \mathbb{R}^{n \times n}$ and $C_y : \Theta \to \mathbb{R}^{n_y \times n}$ are continuous in $\theta(k) \in \Theta$ which is Lebesgue measurable for all $k \in \mathbb{Z}_+$. The parametric uncertainty is described in terms of a polytopic linear differential inclusion (PLDI) [1] in which the state and output matrices in (32) are affinely dependent on the time-varying parameter vector $\theta : \mathbb{Z}_+ \to \Theta$,

$$\begin{bmatrix} A(\theta) & C_y(\theta) \end{bmatrix} \in \Omega_{AC} \triangleq \mathbf{Co}\left(\Omega_{AC}^v\right), \quad \forall \theta \in \Theta, \tag{33}$$

where $\Omega_{AC}^v \triangleq \left\{\begin{bmatrix} A_1 & C_{y,1} \end{bmatrix}, \ldots, \begin{bmatrix} A_{n_v} & C_{y,n_v} \end{bmatrix}\right\}$ and $n_v = 2^{n_\theta}$.

Two different SOF controller design schemes are proposed for the system (32) with PDLFs given in Section 2.6.

Theorem 4

Consider the system (32) with $(A(\theta_{[k]}), C_y(\theta_{[k]}))$ represented as a PLDI (33) and assume that B_u is of full column-rank. If there exist matrices $X(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]}) X_j$, G, M_g, and N_g such that the LMIs

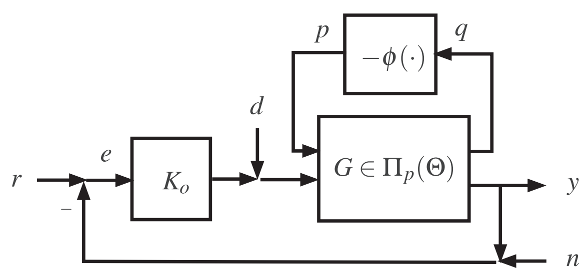

Figure 1. Polytopic uncertain Luré system controlled by output feedback.

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$$
\begin{gathered}
B_u M_g = G B_u \\
\begin{bmatrix} X_j & * & * & * \\ 0 & \mathrm{I} & * & * \\ GA_j + B_u N_g C_{y,j} & -GB_p & \mathrm{Sym}(G) - X_i & * \\ C_q & 0 & 0 & \gamma \mathrm{I} \end{bmatrix} \succ 0 \\
X_j \succ 0
\end{gathered}
\tag{34}
$$

are feasible for all $i, j = 1, \ldots, n_v$, then $K_o = M_g^{-1} N_g$ is a stabilizing SOF control gain, that is, the feedback control signal

$$
u_{[k]} = M_g^{-1} N_g y_{[k]} \tag{35}
$$

stabilizes the system (32) whose uncertain model is represented by the PLDI (33).

Proof
Consider the LMI condition (3) and replace the matrix A by $A(\theta_{[k]}) + B_u K_o C_y(\theta_{[k]})$. From the equivalent conditions in (11) and (12), if there exist $X_j = X_j^{\mathrm{T}} \succ 0$, G, and K_o such that the matrix inequality

$$
\begin{bmatrix} X_j & * & * & * \\ 0 & \mathrm{I} & * & * \\ GA_j + GB_u K_o C_{y,j} & -GB_p & \mathrm{Sym}(G) - X_i & * \\ C_q & 0 & 0 & \gamma \mathrm{I} \end{bmatrix} \succ 0 \tag{36}
$$

holds for all $i, j = 1, \ldots, n_v$, then $u_{[k]} = K_o y_{[k]}$ is a stabilizing controller for the system (32) whose uncertain model is represented by the PLDI (33). Consider a parameterization of SOF control gains $K_o = M_g^{-1} N_g$ where M_g solves the matrix equality $B_u M_g = G B_u$ for the full column-rank matrix B_u. Then the inequality (36) reduces to the conditions in (34). □

Theorem 5
There exists a stabilizing SOF control gain matrix K_o for the system in Theorem 4 if there exist matrices G, H, and $X(\theta_{[k]}) = X^{\mathrm{T}}(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]}) X_j \succ 0$ that satisfy the LMIs

$$
\eta_1 \begin{bmatrix} \mathrm{Sym}(H) - X_j^{-1} & * & * & * & * \\ 0 & \mathrm{I} & * & * & * \\ A_j H & -B_p & \mathrm{Sym}(G^{-1}) & * & * \\ C_q H & 0 & 0 & \gamma \mathrm{I} & * \\ 0 & 0 & G^{-T} & 0 & X_i^{-1} \end{bmatrix} \eta_1^{\mathrm{T}} \succ 0, \tag{37}
$$

$$
\eta_2^{(j)} \begin{bmatrix} X_j & * & * & * \\ 0 & \mathrm{I} & * & * \\ GA_j & -GB_p & \mathrm{Sym}(G) - X_i & * \\ C_q & 0 & 0 & \gamma \mathrm{I} \end{bmatrix} \left(\eta_2^{(j)}\right)^{\mathrm{T}} \succ 0, \tag{38}
$$

for all $i, j = 1, \ldots, n_v$, where $\eta_1 \triangleq \mathrm{diag}\{\mathrm{I}, \mathrm{I}, B_u^{\perp}, \mathrm{I}, \mathrm{I}\}$ and $\eta_2^{(j)} \triangleq \mathrm{diag}\{(C_{y,j}^{\mathrm{T}})^{\perp}, \mathrm{I}, \mathrm{I}, \mathrm{I}\}$.

Proof
See Appendix B.1. □

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4.2. With parametric uncertainties in the input channel

Consider the system

$$\begin{aligned} x_{[k+1]} &= A(\theta_{[k]})x_{[k]} + B_u(\theta_{[k]})u_{[k]} - B_p\phi(q_{[k]}, k), \\ y_{[k]} &= C_y x_{[k]}, \quad q_{[k]} = C_q x_{[k]}, \end{aligned} \tag{39}$$

where $x_{[k]} \in \mathbb{R}^n$ is the state and $u_{[k]} \in \mathbb{R}^{n_u}$ is the control input at time $k \in \mathbb{Z}_+$, and the parameter vector $\theta_{[k]} \in \Theta$ where $\Theta \subset \mathbb{R}^{n_\theta}$ is a polytope that is closed and compact. The mappings $A : \Theta \to \mathbb{R}^{n\times n}$ and $B_u : \Theta \to \mathbb{R}^{n\times n_u}$ are assumed to be affine and continuous in $\theta \in \Theta$ which is Lebesgue measurable:

$$\left[\, A(\theta) \;\; B_u(\theta) \,\right] \in \Omega_{AB} \triangleq \mathbf{Co}\left(\Omega^v_{AB}\right), \quad \forall \theta \in \Theta, \tag{40}$$

where $\Omega^v_{AB} \triangleq \left\{\left[\, A_1 \;\; B_{u,1} \,\right], \cdots, \left[\, A_{n_v} \;\; B_{u,n_v} \,\right]\right\}$, $n_v = 2^{n_\theta}$, and $\theta : \mathbb{Z}_+ \to \Theta$ is a time-varying vector.

Two SOF controller design schemes are proposed for the system (39) by considering PDLFs given in Section 2.6.

Theorem 6

Consider the system (39) where $(A(\theta_{[k]}), B_u(\theta_{[k]}))$ are within a PLDI (40) and C_y is assumed to be full row-rank. If there exist the matrices $Y(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]})Y_j$, G, M_g, and N_g such that the LMIs

$$\begin{gathered} M_g C_y = C_y G \\ \begin{bmatrix} \mathrm{Sym}(G) - Y_j & * & * & * \\ 0 & \mathrm{I} & * & * \\ A_j G + B_{u,j} N_g C_y & -B_p & Y_i & * \\ C_q G & 0 & 0 & \gamma \mathrm{I} \end{bmatrix} \succ 0 \\ Y_j \succ 0 \end{gathered} \tag{41}$$

are feasible for all $i, j = 1, \ldots, n_v$, then $K_o = N_g^{-1} M_g$ is a stabilizing SOF control gain, that is, the feedback control signal

$$u_{[k]} = N_g^{-1} M_g y_{[k]} \tag{42}$$

stabilizes the system (39) whose uncertain model is represented by the PLDI (40).

Proof

The proof is similar to Theorem 4. Consider the LMI condition (3) and replace the matrix A in (3) by $A(\theta_{[k]}) + B_u K_o C_y(\theta_{[k]})$. From the equivalent conditions in (13) and (14), we have that if there exist $Y_j = Y_j^{\mathrm{T}} \succ 0$, G, and K_o such that the LMIs

$$\begin{bmatrix} \mathrm{Sym}(G) - Y_j & * & * & * \\ 0 & \mathrm{I} & * & * \\ A_j G + B_{u,j} K_o C_y G & -B_p & Y_i & * \\ C_q G & 0 & 0 & \gamma \mathrm{I} \end{bmatrix} \succ 0 \tag{43}$$

hold for all $i, j = 1, \ldots, n_v$, then $u_{[k]} = K_o y_{[k]}$ is a stabilizing controller for the system (32) whose uncertain model is represented by the PLDI (33). Consider a parameterization of SOF control gains $K_o = M_g^{-1} N_g$, where M_g solves the matrix equality $M_g C_y = C_y G$ for the full row-rank matrix C_y. Then the inequality (43) reduces to the conditions in (41). □

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Theorem 7
There exists a stabilizing SOF control gain matrix K_o for the system in Theorem 6 if there exist matrices G, $Y(\theta_{[k]}) = Y^{\mathrm{T}}(\theta_{[k]}) = \sum_{j=1}^{n_v} \rho_j(\theta_{[k]})Y_j \succ 0$, and H such that the matrix inequalities

$$\eta_3^{(j)} \begin{bmatrix} \mathrm{Sym}(G) - Y_j & * & * & * \\ 0 & \mathrm{I} & * & * \\ A_j G & -B_p & Y_i & * \\ C_q G & 0 & 0 & \gamma \mathrm{I} \end{bmatrix} \left(\eta_3^{(j)}\right)^{\mathrm{T}} \succ 0, \tag{44}$$

$$\eta_4 \begin{bmatrix} \mathrm{Sym}(G^{-1}) & * & * & * & * \\ 0 & \mathrm{I} & * & * & * \\ HA_j & -HB_p & \mathrm{Sym}(H) - Y_i^{-1} & * & * \\ C_q & 0 & 0 & \gamma \mathrm{I} & * \\ G^{-1} & 0 & 0 & 0 & Y_j^{-1} \end{bmatrix} \eta_4^{\mathrm{T}} \succ 0, \tag{45}$$

hold for all $i, j = 1, \dots, n_v$, where $\eta_3^{(j)} \triangleq \mathrm{diag}\{\mathrm{I}, \mathrm{I}, B_{u,j}^{\perp}, \mathrm{I}\}$ and $\eta_4 \triangleq \mathrm{diag}\{(C_y^{\mathrm{T}})^{\perp}, \mathrm{I}, \mathrm{I}, \mathrm{I}, \mathrm{I}\}$.

Proof
See Appendix B.2. □

The numerical algorithms that implement the results in this section are described in Appendix A.

5. NUMERICAL EXAMPLES

This section applies the results of the previous section to design static output feedback controllers for some uncertain Luré systems. The numerical examples are intended for comparisons, especially in terms of conservatism of the different design methods. The LMIs were solved using off-the-shelf software [58, 59].

Example 1
To enable a comparison of all of the design methods in Section 4, this example has uncertainty only in the A-matrix. Consider the system (32) or (39) with

$$A(\theta_{[k]}) \in \mathbf{Co}(\{A_1, A_2\}),\ B_u = \begin{bmatrix} 0.0758 \\ 0.7576 \end{bmatrix},\ B_p = \begin{bmatrix} -0.6711 \\ -0.4003 \end{bmatrix},$$
$$C_q = \begin{bmatrix} -0.0071 & 0.2107 \end{bmatrix},\ C_y = \begin{bmatrix} 0.3939 & 0.0303 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 0.9697 & 0.1515 \\ -0.3030 & 0.5152 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.9753 & 0.1235 \\ -0.2469 & 0.2346 \end{bmatrix},$$

and nonlinearities are within the set $\phi \in \bar{\Phi}_{\mathrm{sb}}^{\alpha}$. Suppose that the control objective is to maximize the upper bound α on the sector such that the closed-loop system (32) or (39) is stabilized by the static output feedback controller $u_{[k]} = K_o y_{[k]}$. The values for α^* and K_o^* computed from using the results in Theorems 4–7 are reported in Table I. This example indicates that the different design methods can produce controllers with different levels of conservatism for systems that only have uncertainty in the state matrix A. The least conservatism was obtained by the design methods presented in Theorems 5 and 7.

Example 2
This example compares the two different design methods in Section 4.1. Consider the system (32) with

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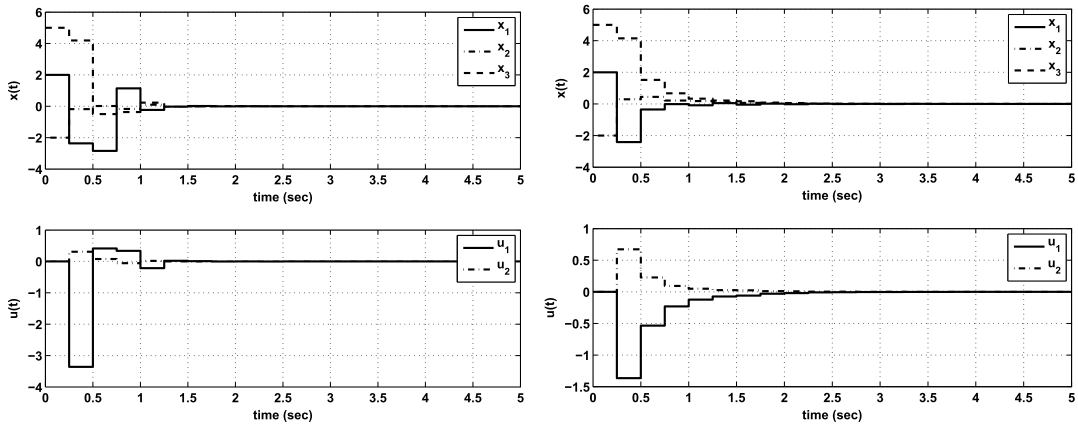
Table I. The maximal upper bound on the sector and optimal SOF control gains for Example 1.

Design methods	α^*	K_o^*
Theorem 4	1.0827	−0.6507
Theorem 5	1.4497	−0.2701
Theorem 6	1.3412	−0.8133
Theorem 7	1.4497	−0.9091

Table II. The maximal upper bound on the sector and optimal SOF control gains for Example 2.

Design methods	α^*	K_o^*
Theorem 4	0.6277	$\begin{bmatrix} 0.3858 & -0.2707 \\ -0.0245 & -0.0195 \end{bmatrix}$
Theorem 5	0.8972	$\begin{bmatrix} 0.1572 & -0.0695 \\ -0.0695 & -0.0038 \end{bmatrix}$



(a) Theorem 4 (b) Theorem 5

Figure 2. Trajectories of system states and control signals with two different design schemes for Example 2.

$$A(\theta_{[k]}) = \begin{bmatrix} -0.12 & 1 & 0 \\ 0 & 0.1 + \theta_{1,[k]} & 0 \\ 0 & 0 & 0.6 + \theta_{2,[k]} \end{bmatrix},$$

$$B_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.6 & 0.4 \\ -0.4 & -0.6 \\ -0.35 & -0.65 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_y(\theta_{[k]}) = \begin{bmatrix} 1 + 1.4\theta_{1,[k]} & 0 & -2 \\ 1 & 1 + \theta_{2,[k]} & 0 \end{bmatrix},$$

nonlinearities within the set $\phi \in \bar{\Phi}_{\text{sb}}^{\alpha}$, and bounds for the uncertain parameters as $\theta_{1,[k]} \in [-0.5, 0]$ and $\theta_{2,[k]} \in [0, 0.5]$ for all $k \in \mathbb{Z}_+$. Suppose that the control objective is to maximize the upper bound α on the sector such that the closed-loop system (32) is stabilized by the static output feedback controller $u_{[k]} = K_o y_{[k]}$. The values of α^* and K_o computed from Theorems 4 and 5 are shown in Table II. As in Example 1, the design method in Theorem 5 achieved the larger value of α^* than for the design method in Theorem 4. Figure 2 shows the state and control input trajectories for the closed-loop system (32) with $\phi(q) = \alpha^* \tanh(q)$, where the uncertain parameters are randomly generated in Θ with a uniform distribution. The main differences in closed-loop state trajectories is

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that one design method has faster response for x_1 and the other design method has faster response for x_3.

Example 3

This example compares the two different design methods in Section 4.2. Consider the system (39) with

$$A(\theta_{[k]}) = \begin{bmatrix} -0.12 & 1 & 0 \\ 0 & 0.1 + \theta_{1,[k]} & 0 \\ 0 & 0 & 0.6 + \theta_{2,[k]} \end{bmatrix},$$

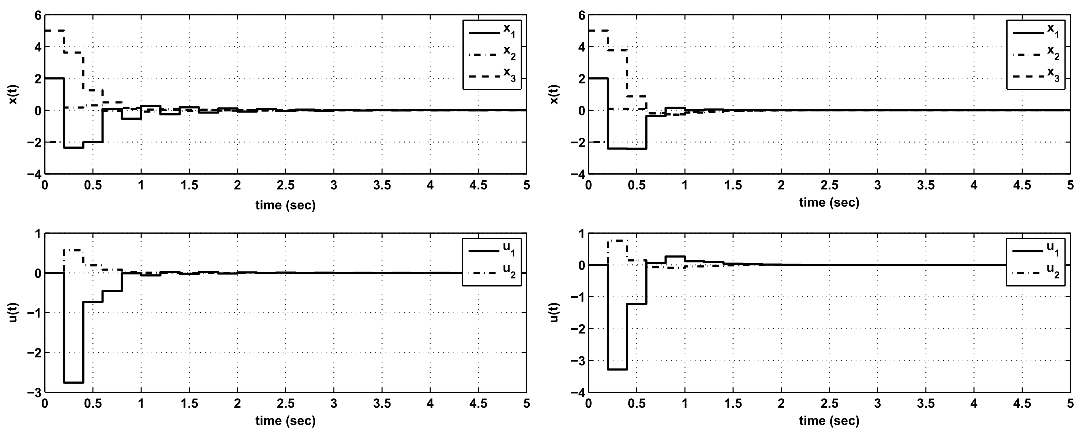
$$B_u(\theta_{[k]}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + 1.4\theta_{1,[k]} \\ 1 + 1.2\theta_{2,[k]} & -1 \end{bmatrix},$$

$$B_p = \begin{bmatrix} 0.6 & 0.4 \\ -0.4 & -0.6 \\ -0.35 & -0.65 \end{bmatrix}, \; C_q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; C_y = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix},$$

nonlinearities within the set $\phi \in \bar{\Phi}^{\alpha}_{\text{sb}}$, and bounds for the uncertain parameters as $\theta_{1,k} \in [-0.5, 0]$ and $\theta_{2,k} \in [0, 0.5]$ for all $k \in \mathbb{Z}_+$. Suppose that the control objective is to maximize the upper bound α on the sector such that the closed-loop system (39) is stabilized by the static output feedback controller $u_{[k]} = K_o y_{[k]}$. The values of α^* and K_o computed from Theorems 6 and 7 are shown in Table III. As in Example 1, the design method in Theorem 7 achieved the larger value of α^* than the design method in Theorem 6. Figure 3 shows the state and control input trajectories for the closed-loop system (39) with $\phi(q) = \alpha^* \tanh(q)$, where the uncertain parameters are randomly generated in the given bound Θ with uniform distribution.

Table III. The maximal upper bound on the sector and optimal SOF control gains for Example 3.

Design methods	α^*	K_o^*
Theorem 6	0.5808	$\begin{bmatrix} 0.4782 & -0.8366 \\ -0.0852 & 0.1144 \end{bmatrix}$
Theorem 7	0.8833	$\begin{bmatrix} 0.3556 & -0.1069 \\ -0.1069 & 0.1295 \end{bmatrix}$



(a) Theorem 6 (b) Theorem 7

Figure 3. Trajectories of system states and control signals with two different design schemes for Example 3.

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In Examples 1–3, the design methods in Theorems 5 and 7 had larger values for the achieved maximum upper sector bound for which the closed-loop system is GAS than for the design methods in Theorems 4 and 6.

6. CONCLUSIONS

Static and fixed-order dynamic output feedback control design methods are derived for polytopic uncertain Luré systems with sector-bounded nonlinearities. The nonconvex matrix inequality formulations for output feedback controller design are provided in a mathematical form for which iterative numerical algorithms have been developed. Each iteration of the numerical algorithms is formulated in terms of linear matrix inequalities that are solved using off-the-shelf software. The design methods are compared in three numerical examples.

APPENDIX A: NUMERICAL METHODS FOR OUTPUT FEEDBACK CONTROLLER DESIGN AND CONVERGENCE ANALYSIS OF ALGORITHMS

All of the output feedback control design equations considered in this paper reduce to tests of feasibility for two matrix inequalities of the forms (18) and (19). This section summarizes two numerical algorithms for finding feasible solutions X and Y for (18) and (19) that satisfy a nonconvex condition $XY = YX = \mathrm{I}$. The *min/max algorithm* has demonstrated good convergence in numerical examples, although its convergence properties have not been well established, whereas the *alternating projection algorithm* has well-understood convergence properties [60–62]. A *cone complementary linearization algorithm* that is originally developed to solve the cone complementary slackness condition of positive-semidefinite matrices [63] can be also used to solve the same nonconvex optimization. These computational methods can be seen as sequential SDP relaxations, that is, methods to iteratively solve semidefinite programs to obtain a suboptimal solution to the original nonconvex problem. Some properties such as convergence of these numerical algorithms are also discussed in the succeeding discussions. This appendix is not a new contribution and is only included to provide a self-contained and concise overview of the algorithms and their convergence.

For notational convenience, define the two convex sets of positive-definite matrices:

$$\mathscr{C}_1 = \{X \in \mathbb{S}^n : X \succ 0, (18)\}, \tag{46}$$

$$\mathscr{C}_2 = \{Y \in \mathbb{S}^n : Y \succ 0, (19)\}. \tag{47}$$

A.1. The min/max algorithm [46, 47]

In the min/max algorithm, the optimization problems

$$X_n = \operatorname{argmin}\{\ell^+ : X \in \mathscr{C}_1, \mathrm{I} \preceq Y_n^{1/2} X Y_n^{1/2} \preceq \ell^+ \mathrm{I}\},$$
$$Y_{n+1} = \operatorname{argmax}\{\ell^- : Y \in \mathscr{C}_2, \ell^- \mathrm{I} \preceq X_n^{1/2} Y X_n^{1/2} \preceq \mathrm{I}\}.$$

are solved iteratively to compute the best X or Y at each step.

A.2. Alternating projection algorithm

Successive projection mappings for (18) and (19) can be formulated as alternating projection problems in which an optimization is solved at each projection step. As stated more formally in the succeeding discussions, the optimization at each projection step is a minimum distance problem in a metric space equipped with the Frobenius norm, which is a Hilbert space with the inner product defined by $\langle A, B\rangle = \mathrm{Tr}(A^{\mathrm{T}}B) = \mathrm{Tr}(BA^{\mathrm{T}})$.

Lemma 8

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the convex sets described in (46) and (47). Then the projections $X_n = \mathcal{P}_{\mathscr{C}_1}(Y_n)$ and $Y_{n+1} = \mathcal{P}_{\mathscr{C}_2}(X_n)$ can be characterized as the unique solutions to the SDPs

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$$X_n = \mathcal{P}_{\mathscr{C}_1}(Y_n) := \mathrm{argmin}_{X \in \mathscr{C}_1} ||Y_n^{-1} - X||_F,$$
$$Y_{n+1} = \mathcal{P}_{\mathscr{C}_2}(X_n) := \mathrm{argmin}_{Y \in \mathscr{C}_2} ||Y - X_n^{-1}||_F, \tag{48}$$

where $||\cdot||_F$ indicates the Frobenius norm, that is, $||A||_F \triangleq \sqrt{\mathrm{Tr}(AA^{\mathrm{T}})}$ for a matrix A of compatible dimension.

The objective of solving the sequential optimization (48) is to find a solution $X \in \mathscr{C}_1 \cap \mathscr{C}_2^{-1}$ or, equivalently, $Y \in \mathscr{C}_1^{-1} \cap \mathscr{C}_2$, where $\mathscr{C}^{-1}$ denotes a set of inverse matrices from the set $\mathscr{C}$. The algorithm can be described as finding the limit of a sequence, and the existence of the limit is guaranteed if the set of feasible solutions $\mathscr{C}_1 \cap \mathscr{C}_2^{-1}$ or $\mathscr{C}_1^{-1} \cap \mathscr{C}_2$ is nonempty.

Corollary 1

Consider the sequences of feasible solutions $\{X_n\}$ and $\{Y_n\}$ for (48). Define a sequence of matrices $\{Z_n\}_{n\in\mathbb{N}}$ that is given as

$$Z_n := \begin{cases} X_{\frac{n+1}{2}} & \text{for } n \text{ odd,} \\ Y_{\frac{n}{2}} & \text{for } n \text{ even} \end{cases} \tag{49}$$

Then the limit $X_\infty := \lim_{n\to\infty} Z_{2n-1}$ exists if and only if the set $\mathscr{C}_1 \cap \mathscr{C}_2^{-1}$ is nonempty. Equivalently, the limit $Y_\infty := \lim_{n\to\infty} Z_{2n}$ exists if and only if the set $\mathscr{C}_1^{-1} \cap \mathscr{C}_2$ is nonempty. Furthermore, they satisfy the relation $X_\infty Y_\infty = Y_\infty X_\infty = \mathrm{I}$.

Proof

This result follows from the monotone convergence theorem (for example, [64]). □

A.3. Cone complementary linearization algorithm

The same nonconvex optimization can be solved using the *cone complementary linearization method*. Its SDP formulation [49] is summarized in the succeeding discussions. First, introduce the optimization

$$\begin{aligned} &\min \mathrm{Tr}(XY) \\ &\text{s.t. } (X, Y) \in \mathscr{M}_{X,Y}^{1,2} \triangleq \mathscr{C}_1 \times \mathscr{C}_2 \cap \mathscr{M}_{X,Y}, \end{aligned} \tag{50}$$

which is equivalent to (48), where

$$\mathscr{M}_{X,Y} \triangleq \left\{ (X, Y) \in \mathbb{S}_+^{N\times N} \times \mathbb{S}_+^{N\times N} : \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \right\}.$$

The objective function of the aforementioned optimization is nonconvex but can be solved by the *linearization algorithm* [49]:

1. Choose the initial guess $(X_0, Y_0) \in \mathscr{M}_{X,Y}^{1,2}$ and set the iteration index n as 0.
2. Solve the SDP to move one step forward:

$$\begin{aligned} (X_{n+1}, Y_{n+1}) &:= \mathrm{argmin} \left\{ \mathrm{Tr}(XY_n + X_nY) : (X, Y) \in \mathscr{M}_{X,Y}^{1,2} \right\}, \\ t_{n+1} &:= \mathrm{Tr}(X_{n+1}Y_n + X_nY_{n+1}). \end{aligned} \tag{51}$$

3. If the stopping criterion

$$|t_{n+1} - t_n| < \epsilon \tag{52}$$

is satisfied for a specified error tolerance $\epsilon > 0$, then the algorithm has converged. Otherwise, go to step 2 with the increased iteration index $n = n + 1$.

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This sequential optimization can be described as finding the limit of a sequence, and it follows from the monotone convergence theorem [64] that the existence of the limit is guaranteed if the set of feasible solutions $\mathscr{C}_1 \cap \mathscr{C}_2^{-1}$ or $\mathscr{C}_1^{-1} \cap \mathscr{C}_2$ is assumed to be nonempty.

Corollary 2
Consider the sequential SDP (51). There exists a limit point (X_∞, Y_∞) for every initial condition $(X_0, Y_0) \in \mathscr{M}_{X,Y}^{1,2}$. A limiting point (X_∞, Y_∞) achieves $t_\infty = 2\mathrm{Tr}(X_\infty Y_\infty) = 2N$ if and only if $X_\infty Y_\infty = Y_\infty X_\infty = \mathrm{I}$.

APPENDIX B: PROOFS OF THEOREMS 5 AND 7

B.1. Proof of Theorem 5

Proof
Consider the matrix inequality (36) that is a BMI for decision variables $(X_i, X_j, G, K_o, \gamma)$. This inequality can be rewritten as

$$\bar{G}\left(\mathscr{A}_{i,j}\left(X_i, X_j, G^{-1}\right) + \bar{B}_u K_o \bar{C}_{y,j}\right) + \left(\mathscr{A}_{i,j}\left(X_i, X_j, G^{-1}\right) + \bar{B}_u K_o \bar{C}_{y,j}\right)^{\mathrm{T}} \bar{G}^{\mathrm{T}} \succ 0 \tag{53}$$

where

$$\mathscr{A}_{i,j}\left(X_i, X_j, G^{-1}\right) \triangleq \begin{bmatrix} \frac{1}{2}X_j & 0 & 0 & 0 \\ 0 & \frac{1}{2}\mathrm{I} & 0 & 0 \\ A_j & -B_p & \mathrm{I} - \frac{1}{2}G^{-1}X_i & 0 \\ C_q & 0 & 0 & \frac{1}{2}\gamma \mathrm{I} \end{bmatrix}, \ \bar{G} \triangleq \mathrm{diag}\{\mathrm{I}, \mathrm{I}, G, \mathrm{I}\},$$
$$\bar{B}_u \triangleq \begin{bmatrix} 0 & 0 & B_u^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}, \ \bar{C}_{y,j} \triangleq \begin{bmatrix} C_{y,j} & 0 & 0 & 0 \end{bmatrix}.$$

From Finsler's lemma (Lemma 2), the existence of a feasible solution K_o solving the matrix inequality (53) is equivalent to the feasibility of two LMIs

$$\begin{aligned} &\bar{B}_u^{\perp}\left(\mathscr{A}_{i,j}\left(X_i, X_j, G^{-1}\right)\bar{G}^{-\mathrm{T}} + \bar{G}^{-1}\mathscr{A}_{i,j}\left(X_i, X_j, G^{-1}\right)\right)\left(\bar{B}_u^{\perp}\right)^{\mathrm{T}} \succ 0, \\ &(\bar{C}_{y,j}^{\mathrm{T}})^{\perp}\left(\bar{G}\mathscr{A}_{i,j}\left(X_i, X_j, G^{-1}\right) + \mathscr{A}_{i,j}\left(X_i, X_j, G^{-1}\right)\bar{G}^{\mathrm{T}}\right)\left((\bar{C}_{y,j}^{\mathrm{T}})^{\perp}\right)^{\mathrm{T}} \succ 0. \end{aligned} \tag{54}$$

The second matrix inequality in (54) can be rewritten as (38). The first matrix inequality in (54) can be rewritten as

$$\bar{B}_u^{\perp}\begin{bmatrix} X_j & 0 & A_j^{\mathrm{T}} & C_q^{\mathrm{T}} \\ 0 & \mathrm{I} & -B_p^{\mathrm{T}} & 0 \\ A_j & -B_p & \mathrm{Sym}(G^{-1}) - G^{-1}X_i G^{-\mathrm{T}} & 0 \\ C_q & 0 & 0 & \gamma\mathrm{I} \end{bmatrix}\left(\bar{B}_u^{\perp}\right)^{\mathrm{T}} \succ 0. \tag{55}$$

Applying the Schur complement lemma and a congruence transformation with an invertible matrix results in the equivalences:

$$\eta_1 \begin{bmatrix} X_j & 0 & A_j^{\mathrm{T}} & C_q^{\mathrm{T}} & 0 \\ 0 & \mathrm{I} & B_p^{\mathrm{T}} & 0 & 0 \\ A_j & -B_p & \mathrm{Sym}(G^{-1}) & 0 & G^{-1} \\ C_q & 0 & 0 & \gamma\mathrm{I} & 0 \\ 0 & 0 & G^{-\mathrm{T}} & 0 & X_i^{-1} \end{bmatrix} \eta_1^{\mathrm{T}} \succ 0, \tag{56}$$

$$\Leftrightarrow \eta_1 \begin{bmatrix} X_j^{-1} & 0 & X_j^{-1}A_j^{\mathrm{T}} & X_j^{-1}C_q^{\mathrm{T}} & 0 \\ 0 & \mathrm{I} & -B_p^{\mathrm{T}} & 0 & 0 \\ A_j X_j^{-1} & -B_p & \mathrm{Sym}(G^{-1}) & 0 & G^{-1} \\ C_q X_j^{-1} & 0 & 0 & \gamma \mathrm{I} & 0 \\ 0 & 0 & G^{-\mathrm{T}} & 0 & X_i^{-1} \end{bmatrix} \eta_1^{\mathrm{T}} \succ 0. \tag{57}$$

From Lemma 5, feasibility of (57) is equivalent to (37) in which a dummy variable H is introduced. □

B.2. Proof of Theorem 7

Proof

Consider the matrix inequality (43), which is a BMI for the decision variables $(Y_i, Y_j, G, K_o, \gamma)$. This inequality can be rewritten as

$$\left(\mathscr{A}_{i,j}(Y_i, Y_j, G^{-1}) + \bar{B}_u K_o \bar{C}_{y,j}\right) \bar{G} + \bar{G}^{\mathrm{T}} \left(\mathscr{A}_{i,j}(Y_i, Y_j, G^{-1}) + \bar{B}_u K_o \bar{C}_{y,j}\right)^{\mathrm{T}} \succ 0 \tag{58}$$

where

$$\mathscr{A}_{i,j}(Y_i, Y_j, G^{-1}) \triangleq \begin{bmatrix} \mathrm{I} - \frac{1}{2} Y_j G^{-1} & 0 & 0 & 0 \\ 0 & \frac{1}{2}\mathrm{I} & 0 & 0 \\ A_j & -B_p & \frac{1}{2} Y_i & 0 \\ C_q & 0 & 0 & \frac{1}{2}\gamma \mathrm{I} \end{bmatrix}, \ \bar{G} \triangleq \mathrm{diag}\{G, \mathrm{I}, \mathrm{I}, \mathrm{I}\},$$
$$\bar{B}_{u,j} \triangleq \begin{bmatrix} 0 & 0 & B_{u,j}^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}, \ \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix}.$$

From Finsler's lemma (Lemma 2), the existence of a feasible solution K_o solving the matrix inequality (58) is equivalent to the feasibility of two LMIs

$$\begin{aligned} \bar{B}_{u,j}^{\perp} \left(\mathscr{A}_{i,j}\left(Y_i, Y_j, G^{-1}\right) \bar{G} + \bar{G}^{\mathrm{T}} \mathscr{A}_{i,j}\left(Y_i, Y_j, G^{-1}\right)\right) \left(\bar{B}_{u,j}^{\perp}\right)^{\mathrm{T}} \succ 0, \\ \left(\bar{C}_y^{\mathrm{T}}\right)^{\perp} \left(\bar{G}^{-\mathrm{T}} \mathscr{A}_{i,j}\left(Y_i, Y_j, G^{-1}\right) + \mathscr{A}_{i,j}\left(Y_i, Y_j, G^{-1}\right) \bar{G}^{-1}\right) \left(\left(\bar{C}_y^{\mathrm{T}}\right)^{\perp}\right)^{\mathrm{T}} \succ 0. \end{aligned} \tag{59}$$

The first matrix inequality in (59) can be rewritten as (45). The second matrix inequality in (59) can be rewritten as

$$\left(\bar{C}_y^{\mathrm{T}}\right)^{\perp} \begin{bmatrix} \mathrm{Sym}(G^{-1}) - G^{-\mathrm{T}} Y_j G^{-1} & 0 & A_j^{\mathrm{T}} & C_q^{\mathrm{T}} \\ 0 & \mathrm{I} & -B_p^{\mathrm{T}} & 0 \\ A_j & -B_p & Y_i & 0 \\ C_q & 0 & 0 & \gamma \mathrm{I} \end{bmatrix} \left(\left(\bar{C}_y^{\mathrm{T}}\right)^{\perp}\right)^{\mathrm{T}} \succ 0. \tag{60}$$

Applying the Schur complement lemma and a congruence transformation with an invertible matrix results in the equivalences:

$$\eta_4 \begin{bmatrix} \mathrm{Sym}(G^{-1}) & 0 & A_j^{\mathrm{T}} & C_q^{\mathrm{T}} & G^{-\mathrm{T}} \\ 0 & \mathrm{I} & -B_p^{\mathrm{T}} & 0 & 0 \\ A_j & -B_p & Y_i & 0 & 0 \\ C_q & 0 & 0 & \gamma \mathrm{I} & 0 \\ G^{-1} & 0 & 0 & 0 & Y_j^{-1} \end{bmatrix} \eta_4^{\mathrm{T}} \succ 0, \tag{61}$$

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$$\Leftrightarrow \eta_4 \begin{bmatrix} \mathrm{Sym}(G^{-1}) & 0 & A_j^{\mathrm{T}} Y_i^{-1} & C_q^{\mathrm{T}} & G^{-\mathrm{T}} \\ 0 & \mathrm{I} & -B_p^{\mathrm{T}} Y_i^{-1} & 0 & 0 \\ Y_i^{-1} A_j & -Y_i^{-1} B_p & Y_i^{-1} & 0 & 0 \\ C_q & 0 & 0 & \gamma \mathrm{I} & 0 \\ G^{-1} & 0 & 0 & 0 & Y_j^{-1} \end{bmatrix} \eta_4^{\mathrm{T}} \succ 0. \tag{62}$$

From Lemma 5, feasibility of (62) is equivalent to (44) in which a dummy variable H is introduced. □

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