Technical Notes and Correspondence

On the Analysis of the Eigenvalues of Uncertain Matrices by μ and ν: Applications to Bifurcation Avoidance and Convergence Rates

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***Abstract*—Based on the structured singular value μ and the skewed structured singular value ν, this technical note presents several useful relationships between an uncertain matrix expressed in a linear fractional form and its eigenvalues. The results are used to derive 1) sufficient conditions to avoid bifurcations for systems with parametric uncertainties and 2) bounds on the convergence rate for uncertain stable matrices and uncertain Markov matrices. Illustrative examples are also provided.**

***Index Terms*—Uncertain systems, eigenvalues, bifurcation, stability analysis, robust stability.**

I. Introduction

The eigenvalues of a matrix play many important roles in science and engineering. In structural mechanics, the eigenvalues determine the vibration frequencies. In rigid body dynamics, the eigenvalues of the moment of inertia tensor are the principal moments of inertia, which determine the ease of rotation. In population ecology, the largest eigenvalue of the Leslie matrix determines the long-term growth rate. In multivariate data analysis, the eigenvalues are a measure of the data variance, which can be used in dimensional reduction. In control engineering, the eigenvalues of the state matrix for a linear time-invariant system determine the stability of the system.

The eigenvalues of a matrix can be computed in polynomial time using iterative methods, and hence properties computed from eigenvalues, such as stability, are also computable in polynomial time. Matrices for real systems typically have uncertainties, however, and the computation of even relatively simple properties for an uncertain set of matrices is NP-hard (e.g., [2], [3]).

A useful tool to analyze the effect of uncertainties is the *structured singular value* μ [4]–[6], which has been used to analyze the performance and robustness properties of linear dynamic systems. Although the computation of μ is known to be NP-hard [3], various methods have been developed to compute upper and lower bounds in polynomial time [7]–[10].

This note uses μ and its variant, the *skewed structured singular value* ν, to provide ways to obtain useful information on the eigenvalues of uncertain matrices, which is then applied to derive nonexistence conditions for bifurcations for a nonlinear dynamic system [1] and bounds on the convergence rates of uncertain stable matrices and uncertain Markov matrices.

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This note is organized as follows. Section II provides a preliminary mathematical background. Section III presents the main theorems. Based on the results in Section III, Section IV derives necessary conditions for the nonexistence of bifurcations, and Section V derives conditions on the state convergence rate. Section VI provides numerical examples, and Section VII concludes the presentation.

II. Mathematical Preliminaries

The sets of real numbers, real vectors of length n, and real matrices of size $n \times m$ are denoted by $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ respectively. The sets of complex numbers, complex vectors of length n, and complex matrices of size $n \times m$ are denoted by $\mathbb{C}$, $\mathbb{C}^n$, and $\mathbb{C}^{n\times m}$ respectively. M^T denotes the transpose of a real matrix M, M^* denotes the complex conjugate transpose of a complex matrix M, and m_{ij} denotes the i, j-element of a matrix M. The identity matrix is denoted by I, and the vector of ones is denoted by $\mathbf{1}$.

For a vector $v \in \mathbb{C}^n$, $\|v\|_\infty$ is the maximum norm and $\|v\|_2$ is the Euclidean norm. The maximum singular value of a matrix M is denoted by $\bar{\sigma}(M)$ or $\|M\|_2$. The ith eigenvalue of the matrix M is denoted by $\lambda_i(M)$ and the spectrum of a matrix M, denoted by σ_M, is the set of eigenvalues of M, $\det(M)$ is the determinant of a matrix M, and $\mathrm{tr}(M)$ is the trace of a matrix M. The Kronecker product of two matrices M and N of dimensions $l \times k$ and $p \times q$ is $M \otimes N = \begin{bmatrix} m_{11}N & \cdots & m_{1k}N \\ \vdots & \ddots & \vdots \\ m_{l1}N & \cdots & m_{lk}N \end{bmatrix}$, $\mathrm{vec}(M)$ is the vectorization of a matrix M, and $\mathrm{diag}[M_1, \cdots, M_n]$ is a block-diagonal matrix with M_i on the diagonal. For a given matrix block structure $\mathcal{K}(m_r, m_c) = (k_1, \cdots, k_{m_r}, k_{m_r+1}, \cdots, k_{m_r+m_c})$, the sets of structured uncertainties are defined as in (1)–(3), as shown at the bottom of the next page. With a given matrix block structure $\mathcal{K}(m_r, m_c)$, Δ is an m by m matrix, where $m = \sum_{i=1}^{m_r+m_c} k_i$.

For matrices $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and Δ of compatible dimensions with nonsingular $(I - M_{11}\Delta)$, the (upper) linear fractional transform (LFT) is $F_u(M, \Delta) := M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$, which is well-posed as long as $\det(I - M_{11}\Delta) \neq 0$.

Definition 1 (Structured Singular Value μ [11], [12]): For $M \in \mathbb{C}^{m\times m}$ and $\mathcal{K}(m_r, m_c)$, the structured singular value μ is defined as

$$\mu_{\mathcal{K}}(M) = \frac{1}{\min\{k \geq 0 : \Delta \in k\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}} \text{ s.t.} \det(I - M\Delta) = 0\}}$$

unless no $\Delta \in k\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$ exists that makes $I - M\Delta$ singular, in which case $\mu_{\mathcal{K}}(M) = 0$. ◇

Definition 2 (Skewed Structured Singular Value ν [13]): For $M \in \mathbb{C}^{m\times m}$, $\mathcal{K}_1$, and $\mathcal{K}_2$, the skewed structured singular value ν is defined as

$$\nu_{\mathcal{K}_1,\mathcal{K}_2}(M) = \frac{1}{\min\left\{\begin{array}{c} k \geq 0 : \Delta_1 \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}_1}, \Delta_2 \in k\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}_2}, \\ \Delta = \mathrm{diag}[\Delta_1, \Delta_2], \text{ s.t. } \det(I - M\Delta) = 0 \end{array}\right\}}$$

unless no k exists for $\Delta_2 \in k\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}_2}$ that makes $I - M\Delta$ singular, in which case $\nu_{\mathcal{K}_1,\mathcal{K}_2}(M) = 0$. ♢

III. Eigenvalues of $F_u(M,\Delta)$ and μ and ν

The next theorem determines whether an uncertain matrix has a specific eigenvalue $\lambda \in \mathbb{C}$.

Theorem 1: Consider a well-posed uncertain square matrix

$$F_u(M,\Delta) \in \mathbb{C}^{m\times m},\ \Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$$

and suppose that $\lambda \in \mathbb{C}$ is not an element of $\sigma_{M_{22}}$. Then $F_u(M,\Delta)$ does not have an eigenvalue λ for any $\Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$ if and only if

$$\mu_{\mathcal{K}}(M_\lambda) < 1$$

where

$$M_\lambda = M_{11} + M_{12}(\lambda I - M_{22})^{-1}M_{21}.$$

Proof: Under the condition that $\lambda \notin \sigma_{M_{22}}$, $\det(\lambda I - M_{22}) \neq 0$. For a well-posed LFT, $\det(I - M_{11}\Delta) \neq 0$ for all $\Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$. Under these two conditions, the following equivalences hold for any $\Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$:

$$\begin{aligned}
&\det(I - M_\lambda\Delta) \neq 0\\
&\Leftrightarrow \det\left(\begin{bmatrix} I - M_{11}\Delta & M_{12}\\ M_{21}\Delta & \lambda I - M_{22}\end{bmatrix}\right) \neq 0\\
&\Leftrightarrow \det\left(\lambda I - M_{22} - M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\right) \neq 0\\
&\Leftrightarrow \det\left(\lambda I - F_u(M,\Delta)\right) \neq 0.
\end{aligned} \tag{4}$$

By the definition of μ, $\mu_{\mathcal{K}}(M_\lambda) < 1$ holds if and only if

$$\det\left(I - M_\lambda\Delta\right) \neq 0 \tag{5}$$

for all $\Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$. Equation (5) is equivalent to

$$\det\left(\lambda I - F_u(M,\Delta)\right) \neq 0. \tag{6}$$

From the above equivalences, (4), condition (6) holds if and only if λ is not an eigenvalue of $F_u(M,\Delta)$. ■

The next theorem finds a disk that contains all the eigenvalues of an uncertain matrix.

Theorem 2: The maximum of the largest eigenvalue modulus of a well-posed uncertain square matrix, $F_u(M,\Delta) \in \mathbb{C}^{m\times m}, \Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$, is

$$|\lambda|_{\max} := \max_{\Delta\in\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}} \max_{i=1,\ldots,m} \{|\lambda_i\left(F_u(M,\Delta)\right)|\} = \nu_{\mathcal{K},\mathcal{K}_2}(M) \tag{7}$$

where $\mathcal{K}_2(m_r,m_c) = (\varnothing, m)$.

Proof: From well-posedness, $\det(I - M_{11}\Delta) \neq 0$ for all $\Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$. Then for any scalars $k \geq 0$ and $\delta^c \in \mathbb{C}$, the following equivalences hold:

$$\begin{aligned}
&\det\left(I - M\begin{bmatrix}\Delta & 0\\ 0 & k\delta^c I\end{bmatrix}\right) = 0\\
&\Leftrightarrow \det\left(\begin{bmatrix} I - M_{11}\Delta & -k\delta^c M_{12}\\ -M_{21}\Delta & I - k\delta^c M_{22}\end{bmatrix}\right) = 0\\
&\Leftrightarrow \det(I - M_{11}\Delta)\\
&\quad \times \det\left(I - k\delta^c\left(M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\right)\right) = 0\\
&\Leftrightarrow \det(I - M_{11}\Delta)\det\left(I - k\delta^c F_u(M,\Delta)\right) = 0.
\end{aligned} \tag{8}$$

Clearly, $k\delta^c = 0$ does not satisfy (8). Hence, $k\delta^c \neq 0$, and (with $\det(I - M_{11}\Delta) \neq 0$), (8) holds if and only if

$$\det\left(\frac{1}{k\delta^c}I - F_u(M,\Delta)\right) = 0 \tag{9}$$

and $1/(k\delta^c)$ is an eigenvalue of $F_u(M,\Delta)$. The minimum k that achieves (9) subject to $|\delta^c| \leq 1$ must occur with $\delta^c = \delta^{c'}$ such that $|\delta^{c'}| = 1$, because $|\delta^{c'}| < 1$ implies that there exists a smaller k for the same value of $k\delta_c$. Hence, $|1/(k\delta^{c'})| = 1/k$ is the absolute value of an eigenvalue of $F_u(M,\Delta)$. By definition, ν is the inverse of the minimum k that achieves (9), which corresponds to the maximum eigenvalue modulus of $F_u(M,\Delta)$ over $\Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$. ■

Theorem 2 naturally leads to a corollary.

Corollary 1 (from Theorem 2): The minimum of the smallest eigenvalue modulus of a well-posed uncertain square matrix $F_u(M,\Delta) \in \mathbb{C}^{m\times m}, \Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$

$$|\lambda|_{\min} := \min_{\Delta\in\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}} \min_{i=1,\ldots,m} \{|\lambda_i\left(F_u(M,\Delta)\right)|\}$$

is 0 if M_{22} is singular and

$$|\lambda|_{\min} = 1/\nu_{\mathcal{K},\mathcal{K}_2}(M_{inv})$$

otherwise, where $\mathcal{K}_2(m_r,m_c) = (\varnothing, m)$ and

$$M_{inv} = \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1}\\ M_{22}^{-1}M_{21} & M_{22}^{-1}\end{bmatrix}. \tag{10}$$

Proof: If M_{22} is singular, it is clear that $|\lambda|_{\min} = 0$ (i.e., set $\Delta = 0$). If M_{22} is not singular, then use the fact that if a nonsingular matrix M has an eigenvalue λ_i, then $1/\lambda_i$ is an eigenvalue of M^{-1}. The corollary follows by considering $F_u^{-1}(M,\Delta) = F_u(M_{inv},\Delta)$ and applying Theorem 2. ■

Proposition 1 (for Theorem 3, from Theorem 2): For any $M \in \mathbb{C}$ and well-posed uncertain square matrix $F_u(M,\Delta) \in \mathbb{C}^{m\times m},\ \Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$, the maximum of its maximum singular value is

$$\max_{\Delta\in\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}} \bar{\sigma}\left(F_u(M,\Delta)\right) = \sqrt{\nu_{\mathcal{K}_{aug},\mathcal{K}_2}(M_{aug})}$$

where M_{aug} and the structure $\mathcal{K}_{aug}$ for Δ_{aug} satisfies

$$F_u(M_{aug},\Delta_{aug}) = F_u^*(M,\Delta)F_u(M,\Delta) \tag{11}$$

and $\mathcal{K}_2(m_r,m_c) = (2m,\varnothing)$ or $(\varnothing, 2m)$.

Proof: The largest singular value of a matrix G is the square root of the largest eigenvalue of the positive-semidefinite matrix G^*G, and is a non-negative real number. The result follows from Theorem 2. M_{aug} can be obtained using a multiplication formula for LFTs and is non-unique [11]. Note the change in $\mathcal{K}_2(m_r,m_c)$, which occurs because the size of the LFT in (11) has been changed. Although the optimal second uncertainty block will be real because the eigenvalues of (11) are real, $\mathcal{K}_2(m_r,m_c) = (\varnothing, 2m)$ can be chosen to make the ν computation easier. ■

The next two theorems relate an eigenvalue of the uncertain matrix $F_u(M,\Delta)$ to an eigenvalue of the nominal matrix M_{22}.

Theorem 3: For a well-posed uncertain square matrix $F_u(M,\Delta) \in \mathbb{C}^{m\times m}, \Delta \in \mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}}$, suppose that M_{22} is diagonalizable with $M_{22} = S\Lambda_{22}S^{-1}$, where $\Lambda = \mathrm{diag}[\lambda_1,\cdots,\lambda_m]$. Then, for any eigenvalue λ

$$\boldsymbol{\Delta}_{\mathcal{K}} := \left\{\Delta = \mathrm{diag}\left[\delta_1^r I_{k_1},\cdots,\delta_{m_r}^r I_{k_{m_r}},\delta_1^c I_{k_{m_r+1}},\cdots,\delta_{m_c}^c I_{k_{m_r+m_c}}\right] : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}\right\} \tag{1}$$

$$\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}} := \left\{\Delta = \mathrm{diag}\left[\delta_1^r I_{k_1},\cdots,\delta_{m_r}^r I_{k_{m_r}},\delta_1^c I_{k_{m_r+1}},\cdots,\delta_{m_c}^c I_{k_{m_r+m_c}}\right] : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}\right\}, \bar{\sigma}(\Delta) \leq 1\} \tag{2}$$

$$k\mathbf{B}\boldsymbol{\Delta}_{\mathcal{K}} := \left\{\Delta = \mathrm{diag}\left[\delta_1^r I_{k_1},\cdots,\delta_{m_r}^r I_{k_{m_r}},\delta_1^c I_{k_{m_r+1}},\cdots,\delta_{m_c}^c I_{k_{m_r+m_c}}\right] : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}\right\}, \bar{\sigma}(\Delta) \leq k\} \tag{3}$$

of $F_u(M,\Delta)$ for any $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$, there is some eigenvalue $\lambda_i \in \sigma_{M_{22}}$ such that

$$|\lambda_i - \lambda| \le \kappa(S)\sqrt{\nu_{\mathcal{K}_{0,aug},\mathcal{K}_2}(M_{0,aug})} \tag{12}$$

where $\kappa(S)$ is the condition number of S, and $M_{0,aug}$ and the structure $\mathcal{K}_{0,aug}$ for $\Delta_{0,aug}$ satisfy

$$F_u(M_{0,aug},\Delta_{0,aug}) = F_u^*\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0\end{bmatrix},\Delta\right) F_u\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0\end{bmatrix},\Delta\right)$$

and $\mathcal{K}_2(m_r,m_c)=(2m,\varnothing)$ or $(\varnothing,2m)$.

Proof: If λ' is an eigenvalue of $F_u(M,\Delta')$ for some $\Delta' \in \mathbf{B\Delta}_{\mathcal{K}}$, then

$$\det(\lambda' I - \Lambda_{22} - S^{-1}\delta M' S) = 0$$

where

$$\delta M' = M_{21}\Delta'\left(I - M_{11}\Delta'\right)^{-1} M_{12}.$$

If $\det(\lambda' I - \Lambda_{22}) = 0$, then λ' is an eigenvalue of Λ_{22} (and M_{22}), and (12) is satisfied. If not, from

$$(\lambda' I - \Lambda_{22})^{-1}(\lambda' I - \Lambda_{22} - S^{-1}\delta M' S) = I - (\lambda' I - \Lambda_{22})^{-1} S^{-1}\delta M' S$$

$\det(I - (\lambda' I - \Lambda_{22})^{-1} S^{-1}\delta M' S) = 0$, which implies that

$$\|(\lambda' I - \Lambda_{22})^{-1} S^{-1}\delta M' S\|_2 \ge 1.$$

Hence

$$\begin{aligned} 1 \le & \|(\lambda' I - \Lambda_{22})^{-1}\|_2 \|S^{-1}\delta M' S\|_2 \\ \le & \max_{i=1,\dots,m} |\lambda' - \lambda_i|^{-1} \|S^{-1}\|_2 \|\delta M'\|_2 \|S\|_2 \\ = & \frac{1}{\min_{i=1,\dots,m}|\lambda' - \lambda_i|}\kappa(S)\bar{\sigma}(\delta M'). \end{aligned}$$

Therefore

$$\min_{i=1,\dots,m} |\lambda' - \lambda_i| \le \kappa(S)\bar{\sigma}(\delta M').$$

Considering all possible eigenvalues with $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$, take the maximum of the both sides of the inequality over the set $\mathbf{B\Delta}_{\mathcal{K}}$

$$\max_{\Delta \in \mathbf{B\Delta}_{\mathcal{K}}} \min_{i=1,\dots,m} |\lambda - \lambda_i| \le \kappa(S) \max_{\Delta \in \mathbf{B\Delta}_{\mathcal{K}}} \bar{\sigma}(\delta M)$$

where λ is an eigenvalue of $F_u(M,\Delta)$ for some $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$, and

$$\delta M = M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12} \tag{13}$$

for some $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$. From Prop. 1,

$$\begin{aligned} \max_{\Delta \in \mathbf{B\Delta}_{\mathcal{K}}} \bar{\sigma}(\delta M) &= \max_{\Delta \in \mathbf{B\Delta}_{\mathcal{K}}} \bar{\sigma}\left(M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12}\right) \\ &= \sqrt{\nu_{\mathcal{K}_{0,aug},\mathcal{K}_2}(M_{0,aug})}. \end{aligned} \tag{14}$$

Hence, (12) is obtained. ■

Theorem 4: For an uncertain square matrix

$$F_u(M,\Delta) \in \mathbb{C}^{m\times m}, \Delta \in \mathbf{B\Delta}_{\mathcal{K}}$$

suppose that $\|\delta M\|_2 = O(\epsilon)$ for δM defined as in (13). Let the right and left eigenvectors of M_{22} be x and y with corresponding simple eigenvalue λ, i.e.,

$$M_{22}x = \lambda x,\ y^* M_{22} = \lambda y^*.$$

Furthermore, let λ's individual condition number be

$$\mathrm{cond}(\lambda) := \frac{\|y\|_2\|x\|_2}{|y^* x|}.$$

Then the perturbed eigenvalue λ' of $F_u(M,\Delta)$ satisfies

$$|\lambda - \lambda'| \le \mathrm{cond}(\lambda)\sqrt{\nu_{\mathcal{K}_{0,aug},\mathcal{K}_2}(M_{0,aug})} + O(\epsilon^2) \tag{15}$$

with $\mathcal{K}_{0,aug}$, $\mathcal{K}_2$, and $M_{0,aug}$ defined in Theorem 3.

Proof: This proof is obtained by Stewart's Theorem [14]. For the perturbed eigenvector $x+\delta x$ and the eigenvalue $\lambda' = \lambda + \delta\lambda$:

$$\begin{aligned} &(M_{22} + \delta M)(x + \delta x) = (\lambda + \delta\lambda)(x + \delta x) \\ &\Leftrightarrow \delta\lambda(x+\delta x) = \delta M(x+\delta x) + (M_{22} - \lambda I)\delta x \\ &\Rightarrow \delta\lambda y^*(x+\delta x) = y^*\delta M(x+\delta x) + y^*(M_{22} - \lambda I)\delta x \\ &\Leftrightarrow \delta\lambda y^*(x+\delta x) = y^*\delta M(x+\delta x) \\ &\Rightarrow \delta\lambda y^* x = y^*\delta M x + O(\epsilon^2) \\ &\Rightarrow |\lambda - \lambda'| = |\delta\lambda| = \left|\frac{y^*\delta M x}{y^* x}\right| + O(\epsilon^2) \\ &\le \frac{\|y\|_2\|x\|_2}{|y^* x|}\bar{\sigma}(\delta M) + O(\epsilon^2). \end{aligned}$$

Hence, using Prop. 1 as in (14) gives the result. ■

IV. Applications to Bifurcation Analysis

Bifurcations appear in a wide range of systems in biology [15]–[17], fluid flow [18], [19], and quantum mechanics [20], [21], as well as in engineering applications in power systems [22], acoustics [23], chemical processes [24], and aircraft dynamics [25]. In engineering design, it is often important to avoid bifurcations because of their intimate connection to the system's stability properties and behavior. For example, bifurcations can result in voltage collapse in an electric power system [22] or loss of performance of a control system.

A continuous-time nonlinear dynamic system with parametric uncertainties can be described as

$$\dot{x} = f(x,p) \tag{16}$$

where $x \in \mathbb{R}^m$ is a state vector and $p \in \mathbb{R}^{m_r}$ is an uncertain parameter in $\mathcal{P}$ defined by

$$\mathcal{P} = \{p : p = p_c + W\delta p \in \mathbb{R}^{m_r}, \|\delta p\|_\infty \le 1\}$$

with the known nominal parameter vector p_c and the known normalizing matrix W, and the vector-valued function $f : \mathbb{R}^n \times \mathbb{R}^{m_r} \to \mathbb{R}^n$ is sufficiently differentiable with respect to x and p.

Definition 3: Any solution to $0 = f(x,p)$ is called a *steady-state parameter pair* $(x_{\mathrm{ss}}, p_{\mathrm{ss}})$, and its element x_{ss} is called a *fixed point*.

Proposition 2 ([26]): Let

$$J(x_{\mathrm{ss}}, p_{\mathrm{ss}}) = \left.\frac{\partial f}{\partial x}\right|_{x=x_{\mathrm{ss}},\, p=p_{\mathrm{ss}}} \in \mathbb{R}^{m\times m}$$

denote the Jacobian of (16) evaluated at the steady-state parameter $(x_{\mathrm{ss}}, p_{\mathrm{ss}})$. A necessary condition for the system (16) to have a local bifurcation at $(x_{\mathrm{ss}}, p_{\mathrm{ss}})$ is that $J(x_{\mathrm{ss}}, p_{\mathrm{ss}})$ has at least one eigenvalue with zero real part.

If a bifurcation occurs when the Jacobian has an eigenvalue crossing the origin, it is called a *steady-state bifurcation*, and if a bifurcation occurs when the Jacobian has a pair of complex eigenvalues that becomes a purely imaginary pair $\pm j\omega$, then it is called a *Hopf bifurcation*.

The rest of this section considers rational vector-valued functions f (i.e., both numerator and denominator of each element of the vector f are polynomial functions of p) and the case where the fixed point $x_{\mathrm{ss}}(p)$ is a rational function of p, so that $J(x_{\mathrm{ss}}, p_{\mathrm{ss}})$ can be expressed as an LFT. A reformulation of the system equations or an approximation is required if f is not rational, and [1] discusses the handling of nonrational $x_{\mathrm{ss}}(p)$.

Consider a fixed point x_{ss} that is not a bifurcation point for the nominal parameter p_c, giving a Jacobian $J(x_{\mathrm{ss}}, p_{\mathrm{ss}})$ expressed by an LFT

$$J(x_{ss}, p_{ss}) = M_{22} + M_{21}\Delta(I - \Delta M_{11})^{-1}M_{12} = F_u(M, \Delta)$$

where

$$\Delta = \text{diag}\left[\delta p_1 I_{k_1}, \delta p_2 I_{k_2}, \cdots, \delta p_{m_r} I_{k_{m_r}}\right] \in \mathbf{B\Delta}_{\mathcal{K}}$$

$$M_{22} = \left.\frac{\partial f}{\partial x}\right|_{x=x_{ss}, p=p_c}$$

and M_{11}, M_{12}, and M_{21} are minimal and found using block-diagram algebra, toolboxes [27], [28], and/or order reduction algorithms such as [29], [30]. Under the condition that x_{ss} is not a bifurcation point for p_c, M_{22} is a nonsingular square matrix.

Corollary 2 (from Theorem 1): The system (16) does not have a steady-state bifurcation at $x = x_{ss}$ if

$$\mu_{\mathcal{K}}(M_0) < 1 \tag{17}$$

where

$$M_0 = M_{11} - M_{12}M_{22}^{-1}M_{21}. \tag{18}$$

Corollary 3 (from Theorem 1): Condition (17) can be written as

$$\mu_{\mathcal{K}'}(M_{ssb,0}) < 1$$

where

$$M_{ssb,0} = M_{ssb,11} - M_{ssb,12}M_{ssb,22}^{-1}M_{ssb,12}$$

for

$$\det J(x_{ss}, p_{ss}) = \det F_u(M, \Delta) = F_u(M_{ssb}, \Delta_{ssb})$$

and $\mathcal{K}'$ specifies the structure of Δ_{sbb}.

Remark 1: The complexity of constructing M_{ssb} and Δ_{ssb} and constructing M and Δ depends on the system. Which of Corollary 2 and 3 is more convenient to use is assessed on a case-by-case basis.

The existence of pure imaginary eigenvalues needs to be checked for a Hopf bifurcation.

Theorem 5 (Application of Theorem 2): There exists no $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$ for which an uncertain square matrix,

$$F_u(M, \Delta) \in \mathbb{C}^{m\times m}$$

has nonzero pure imaginary eigenvalues if and only if

$$\nu_{\mathcal{K},\mathcal{K}_2}(M_{img}) = 0$$

where $\mathcal{K}_2(m_r, m_c) = (m, \varnothing)$ and

$$M_{img} = \begin{bmatrix} M_{11} & M_{12} \\ jM_{21} & jM_{22} \end{bmatrix}. \tag{19}$$

Proof: By restricting the second uncertainty block to be real and by replacing M_{21} and M_{22} by jM_{21} and jM_{22}, Theorem 2 gives the maximum modulus of pure imaginary eigenvalues of $F_u(M, \Delta)$. If the maximum modulus of pure imaginary eigenvalues is zero, then no nonzero pure imaginary eigenvalue exists. Note that $M_{img} = \begin{bmatrix} M_{11} & jM_{12} \\ M_{21} & jM_{22} \end{bmatrix}$ can be used instead of (19). ■

Corollary 4 (from Theorem 5): The system (16) does not have a Hopf bifurcation at $x = x_{ss}$ if

$$\nu_{\mathcal{K},\mathcal{K}_2}(M_{img}) = 0$$

where M_{img} is defined by (19).

Corollary 5: For a two-dimensional system (i.e., $m = 2$), Corollary 4 can be written as

$$\mu_{\mathcal{K}}(M_0) < 1 \text{ and } \mu_{\mathcal{K}'}(M_{tr,0}) < 1$$

where

$$M_{tr,0} = M_{tr,11} - M_{tr,12}M_{tr,22}^{-1}M_{tr,12}$$

for

$$\text{tr}J(x_{ss}, p_{ss}) = \text{tr}F_u(M, \Delta) = F_u(M_{tr}, \Delta_{tr})$$

and $\mathcal{K}'$ specifies the structure of Δ_{tr}. M_{tr} and Δ_{tr} are non-unique. One possible choice is

$$\begin{aligned} M_{tr,11} &= M_{11} \otimes I, M_{tr,12} = \text{vec}(M_{12}^T) \\ M_{tr,21} &= \text{vec}(M_{21})^T, M_{tr,22} = \text{tr}(M_{22}) \\ \Delta_{tr} &= \Delta \otimes I. \end{aligned}$$

Proof: The condition that $F_u(M, \Delta)$ has a conjugate pair of pure imaginary eigenvalues is equivalent to having $\text{tr}(F_u(M, \Delta)) = 0$. Therefore, $\text{tr}(F_u(M, \Delta)) \neq 0$ for all $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$ if and only if $\mu_{\mathcal{K}}(M_{tr,0}) < 1$. $\text{tr}(F_u(M, \Delta)) = 0$ includes the cases of two zero eigenvalues; to exclude such cases, add a condition $\mu_{\mathcal{K}}(M_0) < 1$ that guarantees that the system does not have a steady-state bifurcation. ■

Corollary 6 (from Corollaries 2 and 4): The number of stable and unstable eigenvalues of $F_u(M, \Delta)$ changes over $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$ if and only if $\mu_{\mathcal{K}}(M_0) > 1$ or $\nu_{\mathcal{K},\mathcal{K}_2}(M_{img}) \neq 0$.

Remark 2 (Discrete-Time System): For a discrete-time system, the interesting question is whether $F_u(M, \Delta)$ has an eigenvalue $\lambda = \pm 1$, or whether $|\lambda| = 1$, but $\lambda \neq \pm 1$. Whether $F_u(M, \Delta)$ has an eigenvalue $\lambda = \pm 1$ can be checked using Theorem 1. The matrix $F_u(M, \Delta)$ has an eigenvalue $|\lambda| = 1$ if and only if $F_u^*(M, \Delta)F_u(M, \Delta)$ has an eigenvalue of 1.

V. Applications to Convergence Analysis

The convergence rate and eigenvalues of a matrix are intimately related. This section derives theorems that are applications of Theorem 2. The first theorem concerns the rightmost eigenvalue for a stable matrix.

Theorem 6: Consider a well-posed uncertain square matrix

$$F_u(M, \Delta) \in \mathbb{C}^{m\times m}, \Delta \in \mathbf{B\Delta}_{\mathcal{K}}$$

with $\mathcal{K}(m_r, m_c) = (k_1, \cdots, k_{m_r}, \varnothing)$. If all the eigenvalues of M_{22} are in the left-half plane, and furthermore,

$$\mu_{\mathcal{K}}(M_0) < 1 \text{ and } \nu_{\mathcal{K},\mathcal{K}_2}(M_{img}) = 0 \tag{20}$$

where M_0 is defined in (18) and M_{img} is defined in (19), then the real parts of the eigenvalues of $F_u(M, \Delta)$ are to the left of $-1/\nu_{\mathcal{K}_{aug},\mathcal{K}_2'}(N_{inv})$ for all $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$, where $\mathcal{K}_{aug}(m_r + 1, m_c) = (k_1, \cdots, k_{m_r}, m, \varnothing)$ (pure real structure), $\mathcal{K}_2'(m_r, m_c) = (m, \varnothing)$, and N_{inv} is defined in a similar manner as in (10) for

$$N = \left[\begin{array}{cc|c} M_{11} & 0 & M_{12} \\ 0 & 0 & j|\lambda|_{\max} I \\ \hline M_{21} & I & M_{22} \end{array}\right]$$

$$\Delta_{aug} = \begin{bmatrix} \Delta & 0 \\ 0 & \delta^r I_m \end{bmatrix} \in \mathbf{B\Delta}_{\mathcal{K}_{aug}}.$$

Proof: Recall that $\mu_{\mathcal{K}}(M_0) < 1$ implies that $F_u(M, \Delta)$ does not have a zero eigenvalue (Theorem 1), and recall also that $\nu_{\mathcal{K},\mathcal{K}_2}(M_{img}) = 0$ implies that $F_u(M, \Delta)$ does not have a nonzero pure imaginary eigenvalue (Theorem 5). Therefore, the conditions that all the eigenvalues of M_{22} are in the left-half plane and (20) imply that all the eigenvalues of $F_u(M, \Delta)$ remain in the left-half plane. Hence, it suffices to find the minimum absolute value of the real part of the eigenvalues.

For any eigenvalue $\sigma + j\omega$, with $\sigma, \omega \in \mathbb{R}$, of $F_u(M, \Delta)$:

$$\det\left((\sigma + j\omega)I - F_u(M, \Delta)\right) = 0. \tag{21}$$

Noting that $|\omega| \leq |\lambda|_{\max}$ from (7), $\omega = |\lambda|_{\max}\delta^r$ with $|\delta^r| \leq 1$. Equation (21) is equivalent to

$$\det\left(\sigma I - F_u(N, \Delta_{aug})\right) = 0$$

where

$$N = \left[\begin{array}{cc|c} M_{11} & 0 & M_{12} \\ 0 & 0 & j|\lambda|_{\max} I \\ \hline M_{21} & I & M_{22} \end{array}\right].$$

$$\Delta_{aug} = \begin{bmatrix} \Delta & 0 \\ 0 & \delta^r I_m \end{bmatrix} \in \mathbf{B\Delta}_{\mathcal{K}_{aug}}$$

Using the result from Corollary 1, $|\sigma| \geq 1/\nu_{\mathcal{K}_{aug},\mathcal{K}'_2}(N_{inv})$. Because all the eigenvalues of $F_u(M, \Delta)$ remain in the left-half plane, this inequality implies that $\sigma \leq -1/\nu_{\mathcal{K}_{aug},\mathcal{K}'_2}(N_{inv})$. ■

A similar result holds for an uncertain matrix $F_u(M, \Delta)$ having $\mathcal{K}(m_r, m_c)$ with $m_c \neq \varnothing$, with reordering of terms in N and Δ_{aug}.

The next theorem gives a condition on the second largest eigenvalue modulus for an uncertain Markov matrix, which is useful for bounding the convergence rate of a Markov process.

Theorem 7: Suppose that

$$F_u(M, \Delta) \in \mathbb{R}^{m \times m},\ \Delta \in \mathbf{B\Delta}_{\mathcal{K}}$$

is a Markov matrix with all positive entries for all $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$. The maximum of the second largest eigenvalue modulus of $F_u(M, \Delta)$ over $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$ is

$$\max_{\Delta \in \mathbf{B\Delta}_{\mathcal{K}}} \max\left\{|\lambda_2|, \cdots, |\lambda_m|\right\} = \nu_{\mathcal{K},\mathcal{K}_2}(M_{\text{second}})$$

where $\mathcal{K}_2(m_r, m_c) = (\varnothing, m)$ and λ_i, $i = 1, \cdots, m$ are eigenvalues of $F_u(M, \Delta)$ for a fixed Δ and ordered such that $1 = \lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$ (recall that the modulus of every eigenvalue of a Markov matrix is less than or equal to one) and

$$M_{\text{second}} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}(I - \mathbf{1}\mathbf{1}^T) \end{bmatrix}.$$

Proof: The largest (left) eigenvalue modulus of a Markov matrix is the one with $\lambda_1 = 1$ and its corresponding (left) eigenvector $\mathbf{1}^T$, which holds for each $\Delta \in \mathbf{B\Delta}_{\mathcal{K}}$. Now apply this (left) eigenvalue and the (left) eigenvector with the largest modulus together with the known fact that the second largest eigenvalue modulus of matrix G is the largest eigenvalue modulus of $G - \lambda x x^T$, where λ is the largest eigenvalue of G and x is the corresponding eigenvector (deflation) [31]. ■

VI. Numerical Example

A. Bifurcation of a Predator-Prey Model

Consider the predator-prey model [26], [32],

$$\begin{aligned} \dot{x}_1 &= \alpha x_1(1 - x_1) - \frac{p_1 x_1 x_2}{p_2 + x_1} \\ \dot{x}_2 &= -p_3 x_2 + \frac{p_1 x_1 x_2}{p_2 + x_1} \end{aligned} \tag{22}$$

where x_1 and x_2 are scaled population numbers, and $\alpha = 0.1$ and $p = [p_1\ p_2\ p_3]^T$ are parameters that characterize the behavior of the system.

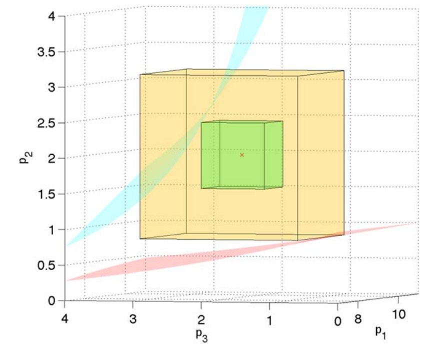

Fig. 1. Steady-state (blue) and Hopf bifurcation boundary surfaces (red) in the parameter space p_1, p_2, and p_3 with $\alpha = 1$. The green region is guaranteed to have no steady-state bifurcations, and the orange region is guaranteed to have no Hopf bifurcations around p_c (red cross).

Suppose that the parameter vector $p \in \mathcal{P}$ is uncertain, where

$$\begin{aligned} \mathcal{P} &= \left\{p = [p_1\ p_2\ p_3]^T : p = p_c + W\delta p, \|\delta p\|_\infty \leq 1\right\} \\ p_c &= [p_{1,c}\ p_{2,c}\ p_{3,c}]^T = [9\ 2\ 2]^T \\ W &= \operatorname{diag}[w_1, w_2, w_3], w_i \in \mathbb{R}, i = 1, \ldots, 3. \end{aligned}$$

The system (22) has a nontrivial fixed point at

$$x_{1,\text{ss}} = \frac{p_2 p_3}{p_1 - p_3}, \quad x_{2,\text{ss}} = \frac{\alpha p_2}{p_1 - p_3}\left(1 - \frac{p_2 p_3}{p_1 - p_3}\right) \tag{23}$$

and the Jacobian is

$$J = \begin{bmatrix} (1 - 2x_1)\alpha - \frac{p_1 x_2}{p_2 + x_1} + \frac{p_1 x_1 x_2}{(p_2 + x_1)^2} & -\frac{p_1 x_1}{p_2 + x_1} \\ \frac{p_1 x_2}{p_2 + x_1} - \frac{p_1 x_1 x_2}{(p_2 + x_1)^2} & \frac{p_1 x_1}{p_2 + x_1} - p_3 \end{bmatrix}.$$

At the fixed point (23), the Jacobian is

$$\begin{aligned} J_{\text{ss}} &= \begin{bmatrix} \frac{\alpha p_3}{p_1}\left(1 - p_2\frac{p_1 + p_3}{p_1 - p_3}\right) & -p_3 \\ \frac{\alpha}{p_1}(p_1 - p_3 - p_2 p_3) & 0 \end{bmatrix} \\ \det(J_{\text{ss}}) &= \frac{\alpha p_3}{p_1}(p_1 - p_3 - p_2 p_3) \\ \operatorname{tr}(J_{\text{ss}}) &= \frac{\alpha p_3}{p_1}\left(1 - p_2\frac{p_1 + p_3}{p_1 - p_3}\right) \end{aligned}$$

which can be easily expressed as LFTs.

Fig. 1 shows the analytically computed bifurcation boundary surfaces and the guaranteed box region of no steady-state and Hopf bifurcations for the uncertain system by applying Corollary 3 and 5 together with bisection to find the size of the uncertainty box that touches the analytical surface (computations used the MATLAB command `mussv`). The parametric uncertainty is assumed to be equal for each parameter, i.e., $W = kI$ for some k. For steady-state bifurcations, Corollary 3 gives that $k = 0.46$, and for Hopf bifurcations, Corollary 5 gives that $k = 1.15$, both of which results agree with the analytically computed values.

B. Locations of Eigenvalues for an Uncertain Matrix

A mass-spring damper system has the state matrix [11]

$$\begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \tag{24}$$

where m, c, and k represent mass, damper value, and spring stiffness respectively. Suppose that m, c, and k are uncertain, with m being within 10% of a nominal value $m_c = 1$, c within 20% of a nominal

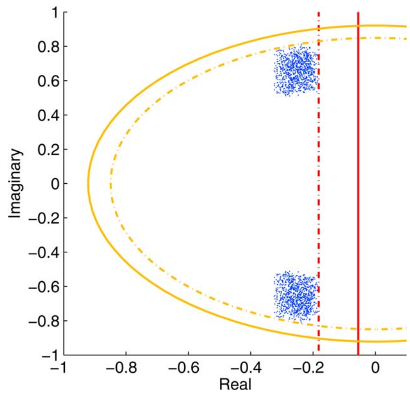

Fig. 2. Blue dots are randomly generated possible eigenvalue locations, the vertical red lines are bounds for the rightmost eigenvalue, and the orange curves are bounds for the maximum eigenvalue modulus. Solid lines represent the ν upper bounds and dashed-dotted lines the ν lower bounds.

value $c_c = 0.5$, and k within 30% of a nominal value $k_c = 0.5$. The state matrix (24) can be written as $F_u(M, \Delta)$ with

$$M = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0.3k_c & 0 \\ 0 & 0 & 0 & 0 & 0.2c_c \\ -1 & -1 & -0.1 & -k_c & -c_c \\ \hline 0 & 0 & 0 & 0 & 1 \\ -1/m_c & -1/m_c & -0.1/m_c & -k_c/m_c & -c_c/m_c \end{array}\right]$$

$$\Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}$$

with $\delta_i \in [-1,\ 1]$ for $i \in \{k, c, m\}$.

Using the polynomial-time upper and lower bounds on ν, application of Theorem 2 gives that the maximum eigenvalue modulus is between 0.85 and 0.92 (computed using the MATLAB SMAC toolbox [33]), and application of Theorem 6 gives that the eigenvalue with the smallest real part is between -0.18 and -0.06 (computed using MATLAB command `mussv` along with bisection due to the limitation on the number of uncertainties for the free version of SMAC). These results agree with Fig. 2, which illustrates the possible eigenvalue locations for randomly selected parameters in the set. The ν lower bounds for both the maximum eigenvalue modulus and the eigenvalue with the smallest real part essentially touch the cloud of eigenvalues.

VII. Conclusion

This note derives several properties on the eigenvalues of uncertain matrices using the structured singular value μ and the skewed structured singular value ν. Numerical examples illustrate possible applications of the theoretical results.

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