This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2024.3426890

Extremum-Seeking Regulator for a Class of Nonlinear Systems with Unknown Control Direction

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***Abstract*—This study proposes a design technique that solves a robust output regulation problem for a class of nonlinear systems subject to unknown control direction. Nussbaum function techniques are commonly used tools to investigate output regulation problems for various systems subject to unknown control direction. They often lead to large overshoots when the initial estimates of the control direction are wrong. In this study, an extremum-seeking control approach is proposed to overcome the need for Nussbaum functions. The approach yields control laws that can handle the robust practical output regulation problem for a class of nonlinear systems subject to a time-varying control direction whose sign or value is unknown. The stability of the design is proven via a Lie bracket averaging technique where uniform ultimate boundedness of the closed-loop signals is guaranteed. Finally, the simulation of a chaotic control problem for the generalized Lorenz system with an unknown time-varying coefficient is provided to illustrate the validity of the theoretical results.**

***Index Terms*—Output Regulation, Nonlinear Systems, Unknown Control Direction, Approximation Method**

I. Introduction

Output regulation control problems have been intensively investigated in the past decades [1–6]. The objective of output regulation control is to design a control law that achieves asymptotic tracking of reference signals while rejecting the steady-state effect of a class of disturbances. Feedback ([3, 4, 6]) and feedforward ([2]) control approaches are the two most commonly used frameworks for the solution of output regulation problems. In particular, it was shown in [2] that the output regulation of nonlinear systems can be solved by a feedforward control synthesized from a certain solvable nonlinear partial differential equation. It was also shown that both the feedforward control approach and the linear internal model principle become invalid in the presence of unknown parameters and nonlinearities as in [4, 7–9]. As a result, a feedback control framework was proposed in [10] to globally stabilize the augmented system consisting of the internal model and the plant while remaining robust in the presence of plant parameter uncertainties and nonlinearities. A comprehensive overview of the research on the output regulation problem can be found in the monographs [3, 4] and recent survey papers [11, 12].

This research was supported by the U.S. Food and Drug Administration under the FDA BAA-22-00123 program, Award Number 75F40122C00200. This research was also supported by NSERC.

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Most existing controller design techniques require knowledge of the control direction *a priori*. However, many practical situations arise, such as the autopilot design of time-varying ships in [13], in which the high-frequency gain of the control system is unknown. Moreover, knowing the signs of the virtual control gain functions also has an important role in solving the output tracking of unknown pure feedback systems with prescribed performance and bounded closed-loop signals [14]. In addition, some applications, such as the formation control of multiple high-altitude balloons, are subjected to unknown, time-varying, and unpredictable dynamics such as stratospheric wind currents [15]. In a recent study, [16], the requirement for the knowledge of the Hessian sign information for the design of an extremum-seeking controller was alleviated using a switching monitoring function-based scheme. The control direction naturally plays an essential role in the solution of output regulation problems for both nonlinear and linear systems, as pointed out in [17]. The use of controllers based on the wrong control direction can force the output regulation error of the closed-loop system to drift away from the desired control goal [18]. Therefore, the output regulation problem without a known control direction has become a research problem of interest that has attracted significant attention from the control community [19, 20]. It remains a relevant and challenging research topic as outlined in [21, 22]. Various unknown control direction problems, including traditional and cooperative output regulation problems, have been well addressed in [19] and [23], respectively. In particular, the global robust output regulation problem with unknown control direction for nonlinear systems in output feedback and lower triangular forms have been solved in [19] and [17], respectively. Recently, the output regulation problem with unknown high-frequency-gain signs has been generalized to multi-agent systems in [24, 25].

All existing results are based on the Nussbaum function technique proposed in [26]. The Nussbaum-type gain uses oscillation to degrade and reward the closed-loop system in order to identify the correct control direction adaptively [19]. For example, the approach proposed in [27] utilized the Nussbaum function to investigate the robust prescribed performance control of nth order cascade nonlinear system with partial-state feedback. The techniques developed in [28] have been applied to deal with the output regulation problem subject to unknown constant control direction in [19, 24, 29]. The application of specific choices of Nussbaum functions has also been shown to address systems with unknown time-varying control coefficients [17, 25]. Nevertheless, as pointed out in [30], existing results based on the Nussbaum function can suffer from poor transient performance. They also fail to achieve exponential stability even in the absence of uncertainties, as observed in [19, 24, 25, 28]. Furthermore, [18] used a counterexample to show that the existing Nussbaum functions are not always effective in multivariable and/or time-varying control coefficients with unknown signs. It should be noted that probing functions covering the Nussbaum functions were developed in [31, 32] in the funnel control of a class of nonlinear controlled systems. This approach constitutes an interesting and notable improvement of Nussbaum-based techniques.

Recent results [30, 33] have shown that the extremum-seeking algorithm can be applied to achieve semiglobal stabilization and tracking problem of unstable and time-varying systems with unknown time-varying control direction, respectively. Extremum-seeking control has a long history. The recent comprehensive survey [34] provides a complete account of the field over the last 100 years. Extremum-seeking control aims to steer an unknown dynamical system to the optimum of a partially or completely unknown map [35, 36] and [37]. Particularly, [38, 39] generalized an extremum-seeking control approach to solving the output regulation of a nonlinear control system to the unknown optimum of a measured objective function. It is shown that the resulting extremum-seeking regulator can achieve practical output regulation of the unknown optimum.

In this study, we propose a solution to the robust practical output regulation problem of a class of nonlinear systems with unknown control direction without using the Nussbaum-type gain technique. Using an extremum-seeking control approach as in [30, 38, 39], we construct control laws that can solve the robust practical output regulation problem for a class of nonlinear systems subject to a time-varying control direction whose sign or value is unknown. The main difference between this work and [38, 40] is that the latter results apply generally to time-varying systems. In contrast, the current study develops controllers for time-invariant systems in output feedback form but with a focus on the solution of a robust output regulation problem. The proposed extremum-seeking regulator technique is based on a Lie bracket averaging technique [41], which enhances and is different from the results in [19, 24, 29]. This article extends the result summarized in the conference paper [42] by using a more relaxed assumption and providing a detailed analysis of the properties of the closed loop not provided in [42]. We also present an alternative control structure that further improves the performance of the system.

The rest of this article is organized as follows. In Section II, we formulate the problem and introduce some standard assumptions. In Section III, we recall some existing results from [41, 43] and establish some new lemmas. The main result is presented in Section IV, followed by a numerical example in Section V.

Notation: For $X_1, \dots, X_N \in \mathbb{R}^n$, let $\mathrm{col}(X_1, \dots, X_N) = [X_1^\top, \dots, X_N^\top]^\top$. For two vector fields, $a_i(x)$ and $a_j(x)$, the Lie bracket denoted by $[a_i(x), a_j(x)]$ is given by

$$\left[a_i(x), a_j(x)\right] = \frac{\partial a_j}{\partial x} a_i(x) - \frac{\partial a_i}{\partial x} a_j(x).$$

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing. $\mathcal{K}_\infty$ is the subclass of unbounded $\mathcal{K}$ functions. For functions $f_1(\cdot)$ and $f_2(\cdot)$ with compatible dimensions, their composition $f_1(f_2(\cdot))$ is denoted by $f_1 \circ f_2(\cdot)$.

II. Problem Formulation and Assumptions

Consider the class of output feedback nonlinear systems with unity relative degree investigated in [44],

$$\begin{aligned} \dot{z} &= f(z, y, v, w), \\ \dot{y} &= g(z, y, v, w) + b(v, w)u, \\ e &= y - y_0, \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^{n_z}$, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control input, the high frequency gain $b(w, v)$ is such that $\overline{b} \geq b(w, v)^2 \geq \underline{b} > 0$ form some positive constants $\bar{b}$ and $\underline{b}$, $e \in \mathbb{R}$ is the error output, and $w \in \mathbb{R}^{n_w}$ is an uncertain constant vector. The exogenous signal $v \in \mathbb{R}^{n_v}$ represents the reference input to be tracked or disturbance to be rejected and is assumed to be generated by the exosystem:

$$\begin{aligned} \dot{v} &= Sv, \\ y_0 &= q(v, w), \end{aligned} \tag{2}$$

where $S \in \mathbb{R}^{n_v \times n_v}$ and $y_0 \in \mathbb{R}$ is the output of the exosystem. We assume the functions $f(z, y, v, w)$ and $g(z, y, v, w)$ and $q(v, w)$ are sufficiently smooth known functions with uncertainties (i.e., w) that satisfy $f(0, 0, 0, w) = 0$, $g(0, 0, 0, w) = 0$, $q(0, w) = 0$, $\forall w \in \mathbb{R}^{n_w}$.

The proposed control law is

$$\begin{aligned} u &= \mathbf{k}(\zeta, y), \\ \dot{\zeta} &= \mathbf{l}(\zeta, y), \end{aligned} \tag{3}$$

where $\zeta \in \mathbb{R}^l$ for some integer l, and $\mathbf{k}(\cdot)$ and $\mathbf{l}(\cdot)$ are some nonlinear functions of their arguments. The specific form of the functions $\mathbf{k}(\cdot)$ and $\mathbf{l}(\cdot)$ is defined later.

Problem 1 (Robust practical output regulation problem)**.** *Given system (1), (2), any compact subsets $\mathbb{W} \in \mathbb{R}^{n_w}$ and $\mathbb{V} \in \mathbb{R}^{n_v}$ with $\mathbb{W}$ and $\mathbb{V}$ contain the origin point, find a control law in the form (3) such that, for all initial conditions $v(0) \in \mathbb{V}$ and $w \in \mathbb{W}$, and any initial states* $\mathrm{col}(z(0), y(0), \zeta(0))$ *in some compact set,*

1) *the solution of the closed-loop system exists and is bounded for all $t \geq 0$,*
2) *$\lim_{t\to\infty} |e(t)| \leq \nu$, for some positive constant ν.*

We now introduce the regulator equations. We can put (1) and (2) into the compact form:

$$\begin{aligned} \dot{x}_c &= f_c(x_c, u, v, w), \\ e &= h(x_c, v, w), \end{aligned} \tag{4}$$

where $x_c = \mathrm{col}(z, y)$, and $f_c(\cdot)$ and $h(\cdot)$ are sufficiently smooth functions determined by (1). Associated with (4) are the output regulator equations:

$$\frac{\partial \mathbf{x}(v, w)}{\partial v} Sv = f_c(\mathbf{x}(v, w), \mathbf{u}(v, w), v, w),$$

$$0 = h(\mathbf{x}(v,w), v, w), \tag{5}$$

where $\mathbf{x} : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^{1+n_z}$ and $\mathbf{u} : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \mapsto \mathbb{R}$ are two smooth functions vanishing at the origin.

Now, we state the assumptions required for the solution of the output regulation problem.

Assumption 1. *All the eigenvalues of S are semi-simple with zero real part.*

Assumption 2. *There exist sufficiently smooth functions $\mathbf{z}(v,w)$ with $\mathbf{z}(0,w) = 0$ such that, for any $v \in \mathbb{R}^{n_v}$ and $w \in \mathbb{W}$,*

$$\frac{\partial \mathbf{z}(v,w)}{\partial v} Sv = f(\mathbf{z}(v,w), q(v,w), v, w). \tag{6}$$

Remark 1. *Assumption 1 guarantees that the solution of* (2) *is bounded for all $t \geq 0$. Under Assumption 2, let $\mathbf{y}(v,w) = q(v,w)$ and*

$$\mathbf{u}(v,w) = b(v,w)^{-1}\left(\frac{\partial q(v,w)}{\partial v} Sv - g(\mathbf{z}, q, v, w)\right).$$

We can verify that $\mathbf{z}(v,w)$, $\mathbf{y}(v,w)$, and $\mathbf{u}(v,w)$ are the solutions of the regulator equations associated with systems (1) and (2) *[4].*

Assumption 3. *There exists an integer n, a sufficiently smooth function $\tau : \mathbb{R}^{n_v+n_w} \mapsto \mathbb{R}^n$ vanishing at the origin, and matrices $\Phi \in \mathbb{R}^{n\times n}$, $\Gamma \in \mathbb{R}^{n\times 1}$ such that, for all trajectories $v(t)$ of the exosystem and all $w \in \mathbb{R}^{n_w}$,*

$$\frac{\partial \tau(v,w)}{\partial v} Sv = \Phi\tau(v,w), \tag{7a}$$

$$\mathbf{u}(v,w) = \Gamma\tau(v,w). \tag{7b}$$

Moreover, the pair (Φ,Γ) *is observable and all the eigenvalues of Φ are simple with zero real part.*

Assumption 3 is a standard assumption in the global robust output regulation problem of nonlinear systems subject to a time-varying control direction whose sign or value is unknown, and it can be found in [17]. System (7) is identified as the steady-state generator which could be used to generate the steady-state input $\mathbf{u}(v,w)$. We consider any given Hurwitz matrix $M \in \mathbb{R}^{n\times n}$ and vector $N \in \mathbb{R}^{n\times 1}$ such that (M,N) is controllable. Since the pair (Φ,Γ) is observable and all the eigenvalues of Φ have zero real parts, it follows that the Sylvester equation $T\Phi - MT = N\Gamma$ admits a unique nonsingular matrix solution T. We perform the transformation $\theta(v,w) = T\tau(v,w)$ and $\Psi = \Gamma T^{-1}$ satisfying

$$\dot{\theta}(v,w) = (M + N\Psi)\theta(v,w),$$
$$\mathbf{u} = \Psi\theta(v,w).$$

Then, we define a dynamic compensator,

$$\dot{\eta} = M\eta + Nu, \tag{8}$$

which is called *an internal model* of system (1) (see [4]). Following the framework in [10], we convert the output regulation problem of the given plant via the internal model approach to a stabilization problem of well-defined augmented systems. To achieve this, we consider the coordinate transformation,

$$\bar{z} = z - \mathbf{z}(v,w),$$
$$\bar{\eta} = \eta - \theta(v,w) - b(v,w)^{-1}Ne,$$
$$\bar{u} = u - \Psi\eta.$$

This transformation yields the augmented system

$$\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu), \tag{9a}$$

$$\dot{\bar{\eta}} = M\bar{\eta} + \bar{\varphi}(\bar{z}, e, \mu), \tag{9b}$$

$$\dot{e} = \bar{g}(\bar{z}, \bar{\eta}, e, \mu) + b(v,w)\bar{u}, \tag{9c}$$

where $\mu = \mathrm{col}(v,w)$,

$$\begin{aligned}
\bar{f}(\bar{z}, e, \mu) &= f(\bar{z} + \mathbf{z}(v,w), e + q(v,w), v, w) \\
&\quad - f(\mathbf{z}(v,w), q(v,w), v, w), \\
\bar{\varphi}(\bar{z}, e, \mu) &= b(v,w)^{-1}(MNe - N\varpi(\bar{z}, e, v, w)) \\
&\quad - N\frac{\partial b(v,w)^{-1}}{\partial v} Sve, \\
\bar{g}(\bar{z}, \bar{\eta}, e, \mu) &= \varpi(\bar{z}, e, \mu) + b\Psi\bar{\eta} + \Psi Ne, \\
\varpi(\bar{z}, e, \mu) &= g(\bar{z} + \mathbf{z}(v,w), e + q(v,w), v, w) \\
&\quad - g(\mathbf{z}(v,w), q(v,w), v, w).
\end{aligned}$$

The augmented system (9) is composed of the original plant (1) and the internal model (8). In [10], it was shown that the output regulation problem of (1) and (2) can be converted to the stabilization problem of the augmented system (9) under certain conditions. We list the standard assumptions on the first equation of (9).

Assumption 4. *Given any compact subset $\Omega \subset \mathbb{R}^{n_v} \times \mathbb{W}$, there exists C^1 function $V_{\bar{z}}(\bar{z})$ satisfying*

$$\underline{\alpha}_1(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \overline{\alpha}_1(\|\bar{z}\|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1(\cdot)$ and $\overline{\alpha}_1(\cdot)$ such that, for any $\mu \in \Omega$, along the trajectories of the $\bar{z}$ subsystem,

$$\dot{V}_{\bar{z}}(\bar{z}) \leq -\alpha_1(\|\bar{z}\|) + \delta\gamma(e),$$

where δ is some positive number, $\alpha_1(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying $\limsup_{s\to 0^+}(\alpha_1^{-1}(s^2)/s) < +\infty$, and $\gamma(\cdot)$ is some known smooth positive definite function.

III. Background and Lemmas

A. Lie bracket approximations

Before presenting our main results, we first review some content related to the Lie bracket averaging approach. Let us consider a control affine nonlinear system of the form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\sqrt{\omega}u_i(\omega t), \tag{10}$$

where $x \in \mathbb{R}^n$, $x(0) \in \mathbb{R}^n$, $\omega > 0$, $t \in [0,\infty)$, and $f(x)$ and $g_i(x)$ are twice continuously differentiable. For $i = 1,\ldots,m$, the input functions $u_i(\omega t)$ are assumed to be uniformly bounded and periodic with period T such that $\int_0^T u_i(\omega\tau)d\tau = 0$.

Following the approach proposed in [41, 45], the Lie bracket average of a nonlinear system (10) can be calculated in the form:

$$\dot{\tilde{x}} = f(\tilde{x})$$

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$$+\frac{1}{T}\sum_{i<j}[g_i, g_j](\tilde{x})\int_0^T\int_0^\theta u_j(\omega\theta)u_i(\omega\tau)d\tau d\theta. \quad (11)$$

Consider the nonlinear parameterized dynamical system

$$\dot{x}^\epsilon = F^\epsilon(t, x^\epsilon) \quad (12)$$

with a small positive parameter ϵ. The solution of (12) is denoted by $x^\epsilon(t) = \phi_\epsilon(t, t_0, x_0)$, where ϕ_ϵ is the flow of the system for $t > 0$ with initial conditions t_0, $x^\epsilon(t_0) = x_0^\epsilon$. The averaged dynamics are defined as

$$\dot{x} = F(t, x) \quad (13)$$

whose solution of (13) is denoted by $x(t) = \phi(t, t_0, x_0)$, where ϕ is the flow of the system for $t > 0$ with initial conditions t_0, $x(t_0) = x_0$. The definition of the convergence property is listed below.

Definition 1. *[46] The systems* (12) *and* (13) *are said to satisfy the convergence property if, for every $T \in (0, \infty)$ and compact set $\mathbb{K} \in \mathbb{R}^n$ satisfying $\{(t, t_0, x_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n : t \in [t_0, t_0 + T], x_0 \in \mathbb{K}\} \subset \mathrm{Dom}\ \phi$, for every $\delta \in (0, \infty)$ there exists ϵ^* such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in \mathbb{K}$, and for all $\epsilon \in (0, \epsilon^*)$,*

$$\|\phi^\epsilon(t, t_0, x_0) - \phi(t, t_0, x_0)\| < \delta, \qquad \forall t \in [t_0, t_0 + T].$$

Then, we define ϵ-semiglobal practical uniform asymptotic stability (ϵ-SPUAS).

Definition 2 (ϵ-SPUAS)**.** *An equilibrium point of* (12) *is said to be ϵ-SPUAS if it satisfies uniform stability, uniform boundedness, and global uniform attractivity.*

Lemma 1. *[46] If systems* (12) *and* (13) *satisfy the converging trajectories property and if the origin of system* (13) *is a global uniform asymptotical stable equilibrium point, then the origin of system* (12) *is ϵ-SPUAS.*

Then, the systems (10) and (11) satisfy the below lemma.

Lemma 2. *[38] If a compact set $\mathbb{S} \in \mathbb{R}^n$ is locally (globally) uniformly asymptotically stable for system* (11)*, then it is locally (semi-globally) practically uniformly asymptotically stable for* (10)*.*

IV. Main Results

A. Extremum-seeking control design

In this section, we propose the use of an extremum-seeking control approach to handle the unknown control direction.

Motivated by the extremum-seeking control approach proposed in [30, 38–40], we propose the control laws (14) and (21) to solve the robust practical output regulation problem for a class of nonlinear systems subject to a time-varying control direction whose sign or value is unknown. It is noted that the oscillatory part of the proposed controllers (14) and (21) uses the bounded ES expressions utilized in [30, 40]. In contrast to [30, 40], the controllers (14) and (21) modify the amplitude and phase by introducing an output-dependent function $\rho(e)$.

Theorem 1. *Under Assumptions 1–4, for sufficiently large enough smooth positive functions $\rho^2(\cdot) \geq 1$ and some sufficiently large enough positive constants k and α, the dynamic output feedback controller*

$$u = \sqrt{\alpha\omega}\cos(\omega t + ke^2)\rho(e) + \Psi\eta, \quad (14a)$$

$$\dot{\eta} = M\eta + Nu, \quad (14b)$$

solves the robust practical output regulation problem 1 for the closed-loop system composed of (9) *and* (14)*.*

Proof. The error dynamics (9) with the extremum-seeking control (14) are given by

$$\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu), \quad (15a)$$

$$\dot{\bar{\eta}} = M\bar{\eta} + \bar{\varphi}(\bar{z}, e, \mu), \quad (15b)$$

$$\dot{e} = \bar{g}(\bar{z}, \bar{\eta}, e, \mu) + b(w, v)\sqrt{\alpha\omega}\cos(\omega t + ke^2)\rho(e). \quad (15c)$$

Let $X = \mathrm{col}(\bar{z}, \bar{\eta}, e)$ and

$$P(X, \mu) = \begin{bmatrix} \bar{f}(\bar{z}, e, \mu) \\ M\bar{\eta} + \bar{\varphi}(\bar{z}, e, \mu) \\ \bar{g}(\bar{z}, \bar{\eta}, e, \mu) \end{bmatrix}.$$

The closed-loop system (15) can be expanded as

$$\dot{X} = P(X, \mu) + \left(\underbrace{\begin{bmatrix} 0_{2\times1} \\ b(w, v)\sqrt{\alpha\omega}\cos(ke^2)\rho(e) \end{bmatrix}}_{a_1(X)} \cos(\omega t) - \underbrace{\begin{bmatrix} 0_{2\times1} \\ b(w, v)\sqrt{\alpha\omega}\sin(ke^2)\rho(e) \end{bmatrix}}_{a_2(X)} \sin(\omega t) \right).$$

Then, the corresponding Lie-bracket averaged system can be calculated as

$$\dot{\tilde{X}} = P(\tilde{X}, \mu) + \frac{1}{T}[a_1(\tilde{X}), a_2(\tilde{X})]\int_0^T\int_0^\theta \cos(\omega\theta)\sin(\omega\tau)d\tau d\theta.$$

where

$$\left[a_1(\tilde{X}), a_2(\tilde{X})\right] = 2\omega\begin{bmatrix} 0_{2\times1} \\ kb(w, v)^2\rho(\tilde{e})^2\tilde{e}\alpha \end{bmatrix},$$

$$\frac{1}{T}\int_0^T\int_0^\theta \cos(\omega\theta)\sin(\omega\tau)d\tau d\theta = -\frac{1}{2\omega}.$$

Then we have the averaged system

$$\dot{\tilde{z}} = \bar{f}(\tilde{z}, \tilde{e}, \mu), \quad (16a)$$

$$\dot{\tilde{\eta}} = M\tilde{\eta} + \bar{\varphi}(\tilde{z}, \tilde{e}, \mu), \quad (16b)$$

$$\dot{\tilde{e}} = \bar{g}(\tilde{z}, \tilde{\eta}, \tilde{e}, \mu) - k\alpha b(w, v)^2\rho(\tilde{e})^2\tilde{e}. \quad (16c)$$

Given any initial condition $\mathrm{col}(v_0, w) \in \mathbb{R}^{n_v}\times\mathbb{W}$, under Assumption 1, the signal generated by the system (2) is bounded for all $t > 0$. Then, we can always find a compact subset $\Omega \subseteq \mathbb{R}^{n_v}\times\mathbb{W}$ containing μ for all $t \geq 0$. Define the Lyapunov function candidate $V_{\tilde{\eta}}(\tilde{\eta}) = \tilde{\eta}^\top P\tilde{\eta}$, where P satisfies $PM + M^\top P = -2I$. Let λ_p and λ_P be the minimum and

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maximum eigenvalues of P. The time derivative of $V_{\tilde{\eta}}(\tilde{\eta})$ long (16b) can be evaluated as

$$\dot{V}_{\tilde{\eta}}(\tilde{\eta}) \leq -\|\tilde{\eta}\|^2 + \|P\bar{\varphi}(\tilde{z}, \tilde{e}, \mu)\|^2.$$

Since $\bar{\varphi}(0, 0, \mu) = 0$, for all $\mu \in \Sigma$, by Lemma 7.8 in [4],

$$\|P\bar{\varphi}(\tilde{z}, \tilde{e}, \mu)\|^2 \leq \pi_1(\tilde{z})\|\tilde{z}\|^2 + \phi_1(\tilde{e})\tilde{e}^2$$

for some known smooth functions $\pi_1(\cdot) \geq 1$ and $\phi_1(\cdot) \geq 1$. Then we have

$$\dot{V}_{\tilde{\eta}}(\tilde{\eta}) \leq -\|\tilde{\eta}\|^2 + \pi_1(\tilde{z})\|\tilde{z}\|^2 + \phi_1(\tilde{e})\tilde{e}^2.$$

Define the Lyapunov function $U_{\tilde{z}}(\tilde{z}) = \int_0^{V_{\tilde{z}}(\tilde{z})} \kappa(s)ds$, where the positive function $\kappa(\cdot)$ will be specified later. Under Assumption 4, the time derivative of $U_{\tilde{z}}(\tilde{z})$ along (16) can be evaluated as

$$\dot{U}_{\tilde{z}}(\tilde{z}) \leq -\kappa \circ V_{\tilde{z}}(\tilde{z}) \left[\alpha_1(\|\tilde{z}\|) - \delta\gamma(\tilde{e})\right].$$

By the changing supply rate technique [47], under Assumption 4, we have that

$$\dot{U}_{\tilde{z}}(\tilde{z}) \leq -\frac{1}{2}\kappa \circ \underline{\alpha}_1(\|\tilde{z}\|)\alpha_1(\|\tilde{z}\|) + \kappa \circ \theta(\tilde{e})\delta\gamma(\tilde{e}), \quad (17)$$

where $\theta \in \mathcal{K}_\infty$ is defined as $\theta := \bar{\alpha}_1 \circ \alpha_1^{-1} \circ (2\delta\gamma)$. Under Assumption 4, $\alpha_1(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying $\limsup_{s\to 0^+} (\alpha_1^{-1}(s^2)/s) < +\infty$. Then, there exists a smooth function $\alpha_0(\|\tilde{z}\|)$ such that

$$\alpha_0(\|\tilde{z}\|)\alpha_1(\|\tilde{z}\|) \geq \|\tilde{z}\|^2.$$

As $\limsup_{s\to 0^+} (\alpha_1^{-1}(s^2)/s) < +\infty$, and there exists constant $l_1 \geq 1$ such that $\alpha_1(\|\tilde{z}\|) \geq \|\tilde{z}\|^2/l_1^2$ for all $\|\tilde{z}\| \leq 1$. Besides, $\alpha_1(\|\tilde{z}\|)$ is of class $\mathcal{K}_\infty$, there exists a constant $l_2 > 0$ such that $\alpha_1(\|\tilde{z}\|) \geq l_2$ for all $s \geq 1$. Hence, we have that

$$\alpha_0(\|\tilde{z}\|)\alpha_1(\|\tilde{z}\|) \geq \|\tilde{z}\|^2$$

for any $\alpha_0(\|\tilde{z}\|) \geq l_1^2 + l_2\|\tilde{z}\|^2$. We can choose a positive smooth nondecreasing function $\kappa(\cdot)$ such that

$$\frac{1}{2}\kappa \circ \underline{\alpha}_1(\|\tilde{z}\|) \geq \alpha_0(\|\tilde{z}\|) \times (\pi_1(\tilde{z}) + 1).$$

Let $\tilde{Z} = \mathrm{col}(\tilde{z}, \tilde{\eta})$ and consider the $\tilde{Z}$-subsystem of (16). Define the Lyapunov function candidate $V_1(\tilde{Z}) = U_{\tilde{z}}(\tilde{z}) + V_{\tilde{\eta}}(\tilde{\eta})$. The time derivative of $V_1(\tilde{Z})$ along (16) can be evaluated as

$$\begin{aligned}\dot{V}_1(\tilde{Z}) \leq& -\frac{1}{2}\kappa \circ \underline{\alpha}_1(\|\tilde{z}\|)\alpha_1(\|\tilde{z}\|) + \kappa \circ \theta(\tilde{e})\delta\gamma(\tilde{e}) \\ &- \|\tilde{\eta}\|^2 + \pi_1(\tilde{z})\|\tilde{z}\|^2 + \phi_1(\tilde{e})\tilde{e}^2 \\ \leq& -\|\tilde{Z}\|^2 + \bar{\gamma}(\tilde{e}),\end{aligned} \quad (18)$$

where $\bar{\gamma}(\tilde{e}) = \kappa \circ \theta(\tilde{e})\delta\gamma(\tilde{e}) + \phi_1(\tilde{e})\tilde{e}^2$. Using Lemma 11.3 in [43], we can choose the class $\mathcal{K}_\infty$ functions $\underline{\beta}_1(\cdot)$ and $\overline{\beta}_1(\cdot)$ such that

$$\underline{\beta}_1(\|\tilde{Z}\|) \leq \int_0^{\underline{\alpha}_1(\|\tilde{z}\|)} \kappa(s)ds + \lambda_p\|\tilde{\eta}\|^2,$$
$$\overline{\beta}_1(\|\tilde{Z}\|) \geq \int_0^{\overline{\alpha}_1(\|\tilde{z}\|)} \kappa(s)ds + \lambda_P\|\tilde{\eta}\|^2.$$

From (18), with the same development and by using the changing supply rate technique [47] again, given any smooth function $\Theta(\tilde{Z}) > 0$, there exists a C^1 function $V_2(\tilde{Z})$ satisfying

$$\underline{\alpha}_2(\|\tilde{Z}\|^2) \leq V_2(\tilde{Z}) \leq \overline{\alpha}_2(\|\tilde{Z}\|^2)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_2(\cdot)$ and $\overline{\alpha}_2(\cdot)$, such that, for all $\mu \in \Sigma$, along the trajectory of the Z subsystem,

$$\dot{V}_2 \leq -\Theta(\tilde{Z})\|\tilde{Z}\|^2 + \hat{\delta}\hat{\gamma}(\tilde{e})\tilde{e}^2,$$

where $\hat{\delta}$ is some known positive numbers and $\hat{\gamma}(\cdot) \geq 1$ is some known smooth positive function. Next, consider the augmented system (16). Since $\bar{g}(0, 0, \mu) = 0$, for all $\mu \in \Sigma$, by Lemma 7.8 in [4],

$$\|\bar{g}(\tilde{Z}, \tilde{e}, \mu)\|^2 \leq c_0\big(\pi(\tilde{Z})\|\tilde{Z}\|^2 + \phi(\tilde{e})\tilde{e}^2\big)$$

for some known positive constant c_0 and some known smooth functions $\pi(\cdot) \geq 1$ and $\phi(\cdot) \geq 1$. Then, considering the Lyapunov function

$$V(\tilde{Z}, \tilde{e}) = V_2(\tilde{Z}) + \frac{1}{2}\tilde{e}^2. \quad (19)$$

Then V is globally positive definite and radially unbounded, and the derivative of V along the trajectory of the system (16) under the controller (14) satisfies

$$\begin{aligned}\dot{V} &= \dot{V}_2 + \tilde{e}^\top\dot{\tilde{e}} \\ &= \dot{V}_2 + \tilde{e}^\top\bar{g}(\tilde{Z}, \tilde{e}, \mu) - k\alpha b(w, v)^2\tilde{e}^2\rho(\tilde{e})^2 \\ &\leq -(\Theta(\tilde{Z}) - \pi(\tilde{Z}))\|\tilde{Z}\|^2 \\ &\quad - (b(w, v)^2 k\alpha\rho(\tilde{e})^2 - \hat{\delta}\hat{\gamma}(\tilde{e}) - \frac{c_0}{4}\tilde{e}^2 - \phi(\tilde{e}))\tilde{e}^2.\end{aligned} \quad (20)$$

Let the smooth functions $\Theta(\cdot)$, $\rho(\cdot)$, and the positive number k be such that $\Theta(\tilde{Z}) \geq \pi(\tilde{Z}) + 1$ and $\rho(\tilde{e})^2 \geq \{\hat{\gamma}(\tilde{e}), \phi(\tilde{e}), 1\}$, and $k\alpha \geq \frac{4\hat{\delta}+c_0}{4b(w,v)^2}$, the equation (20) gives

$$\dot{V} \leq -\|\tilde{Z}\|^2 - \tilde{e}^2.$$

Therefore, system (16) is globally uniformly asymptotically stable for all $\mathrm{col}(v, w) \in \mathbb{V} \times \mathbb{W}$. From the convergence property, Lemma 1, and Lemma 2, we have that system (15) is $\frac{1}{\omega}$-semi-globally uniformly asymptotically stable, which further implies that there exists a constant $\nu(\frac{1}{\omega})$ and a ω^* such that, for all initial conditions $\mathrm{col}(z(0), y(0), \eta(0))$ in some compact set and $v(0) \in \mathbb{V}$ and $\omega > \omega^*$, the nominal trajectories are such that $\|\mathrm{col}(\bar{Z}, \bar{e}) - \mathrm{col}(\tilde{Z}, \tilde{e})\| < \nu(\frac{1}{\omega})$. This completes the proof. □

Remark 2. *The controller* (14) *does not require a dynamic gain as needed by the Nussbaum function-based technique [19, 24, 29]. Moreover, one of the improvements of the control law in* (14) *is to allow the function* $\rho(\cdot)$ *to be negative. As a result,* $\frac{1}{\omega}$*-semi-global uniform asymptotic convergence can be guaranteed, and the poor transients resulting from the wrong initial sign estimate can be removed. The nonlinear part in the controller* (14) *may lead to large inputs outside the input range of the actuator, resulting in high-gain feedback. In the following, we propose a more suitable controller that exploits the characteristics of the trigonometric functions. The proposed controller removes the need for dynamic gain,* *reduces the onset of high-gain impact, and provides a bounded control action.*

Theorem 2. *Under Assumptions 1–4, for sufficiently large enough smooth positive functions $\rho(\cdot) \geq 1$ and some sufficiently large enough positive constants k and α, the dynamic output feedback controller*

$$u = \sqrt{\alpha\omega}\cos\left(\omega t + k\int_0^{e^2}\rho(s)ds\right) + \Psi\eta, \tag{21a}$$
$$\dot{\eta} = M\eta + Nu, \tag{21b}$$

solves the robust practical out regulation problem 1 for the closed-loop system composed of (9) *and* (21)*.*

Proof. The error dynamics with the extremum-seeking control (21) are given by

$$\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu), \tag{22a}$$
$$\dot{\bar{\eta}} = M\bar{\eta} + \bar{\varphi}(\bar{z}, e, \mu), \tag{22b}$$
$$\dot{e} = \bar{g}(\bar{z}, \bar{\eta}, e, \mu) + b(w,v)\sqrt{\alpha\omega}\cos\left(\omega t + k\int_0^{e^2}\rho(s)ds\right). \tag{22c}$$

Let $X = \mathrm{col}(\bar{z}, \bar{\eta}, e)$ and

$$P(X,\mu) = \begin{bmatrix} \bar{f}(\bar{z}, e, \mu) \\ M\bar{\eta} + \bar{\varphi}(\bar{z}, e, \mu) \\ \bar{g}(\bar{z}, \bar{\eta}, e, \mu) \end{bmatrix}.$$

The closed-loop system (22) can be expanded as

$$\dot{X} = P(X,\mu) + \left(\underbrace{\begin{bmatrix} 0_{2\times 1} \\ b(w,v)\sqrt{\alpha\omega}\cos(k\int_0^{e^2}\rho(s)ds) \end{bmatrix}}_{b_1(X)} \cos(\omega t) - \underbrace{\begin{bmatrix} 0_{2\times 1} \\ b(w,v)\sqrt{\alpha\omega}\sin(k\int_0^{e^2}\rho(s)ds) \end{bmatrix}}_{b_2(X)} \sin(\omega t) \right).$$

Then, the corresponding Lie-bracket averaged system can be calculated as

$$\dot{\tilde{X}} = P(\tilde{X},\mu) + \frac{1}{T}[b_1(\tilde{X}), b_2(\tilde{X})]\int_0^T\int_0^\theta \cos(\omega\theta)\sin(\omega\tau)d\tau d\theta.$$

where

$$\left[b_1(\tilde{X}), b_2(\tilde{X})\right] = 2\omega\begin{bmatrix} 0_{2\times 1} \\ kb(w,v)^2\rho(\tilde{e}^2)\tilde{e}\alpha \end{bmatrix},$$

$$\frac{1}{T}\int_0^T\int_0^\theta \cos(\omega\theta)\sin(\omega\tau)d\tau d\theta = -\frac{1}{2\omega}.$$

Then, we have the averaged system

$$\dot{\tilde{z}} = \bar{f}(\tilde{z}, \tilde{e}, \mu), \tag{23a}$$
$$\dot{\tilde{\eta}} = M\tilde{\eta} + \bar{\varphi}(\tilde{z}, \tilde{e}, \mu), \tag{23b}$$
$$\dot{\tilde{e}} = \bar{g}(\tilde{z}, \tilde{\eta}, \tilde{e}, \mu) - k\alpha b(w,v)^2\rho(\tilde{e}^2)\tilde{e}. \tag{23c}$$

The rest of the proof follows the same development as the proof of Theorem 1. □

Remark 3. *Motivated by the techniques proposed in [30, 40], we apply an extremum-seeking control approach to solving robust practical output regulation problems with unknown control directions. This work uses different techniques and considers a different problem than [30, 40]. The techniques in [30, 40] solve stabilization and tracking problems for a class of unknown and time-varying nonlinear systems in the full-state feedback form. In contrast, the current study investigates the output regulation problem for a class of nonlinear systems where $f(\cdot)$, $g(\cdot)$ are known with some uncertainties for systems in the output feedback form.*

V. Numerical Example

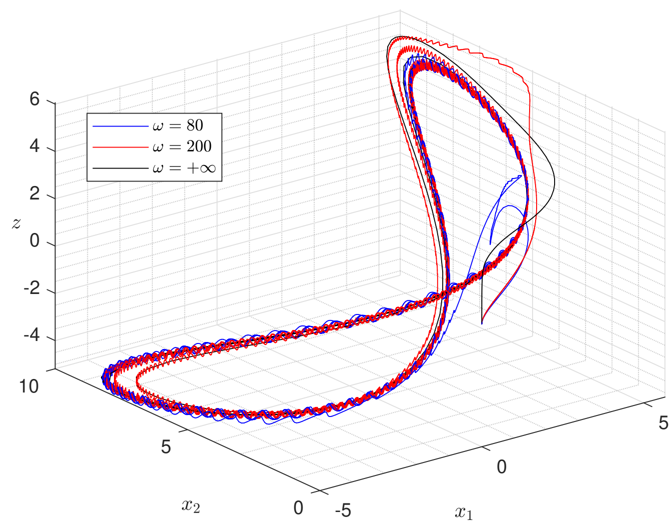

Figure 1. 3D plot of the controlled Lorenz system's trajectory $\mathrm{col}(x, y)$

This example considers a generalized Lorenz system taken from [44],

$$\dot{x} = \begin{bmatrix} -a_1 & 0 \\ z & -a_2 \end{bmatrix} x + \begin{bmatrix} a_1 z \\ 0 \end{bmatrix},$$
$$\dot{z} = [1,0]\,x(a_3 - [0,1]\,x) - z + b(v,w)u,$$
$$y = z, \tag{24}$$

where $x = \mathrm{col}(x_1, x_2)$ and z are the state, $a = \mathrm{col}(a_1, a_2, a_3)$ is a constant parameter vector that satisfies $a_1 > 0$, $a_3 < 0$ and $b(v,w)$ is nonzero with an unknown sign. For convenience, let $a = \bar{a} + w$, where $\bar{a} = \mathrm{col}(\bar{a}_1, a_2, \bar{a}_3)$ is the true value of a and $w = \mathrm{col}(w_1, w_2, w_3)$ is the uncertain parameter of a. We assume that the uncertainty $w \in \mathbb{W} \subseteq \mathbb{R}^3$. The signal v is produced by the ecosystem:

$$\dot{v} = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix} v.$$

The regulated error is defined as $e = y - v_1$. Assume $\bar{a} = \mathrm{col}(10, 28, -\frac{8}{3})$, $b = -1$, and $\sigma = 2$. It has been verified that

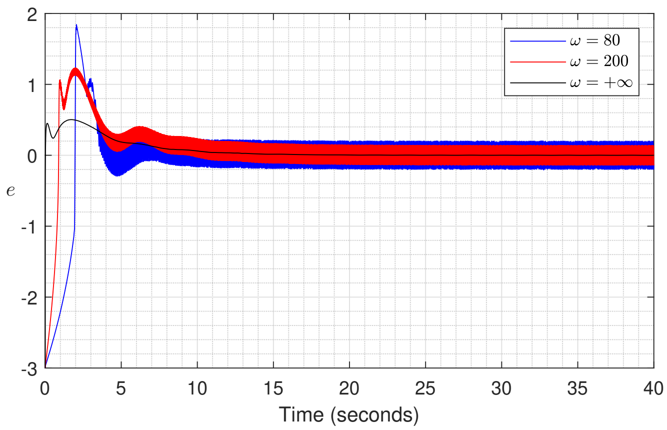

Figure 2. Tracking error over different frequency subject to the controller (14)

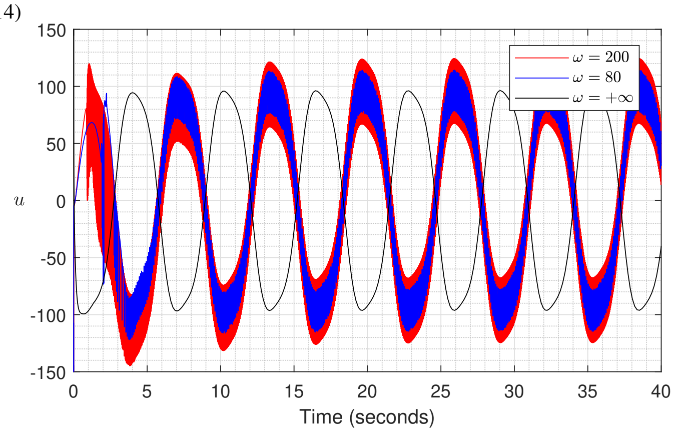

Figure 3. Trajectories of the controller (14) over different frequencies

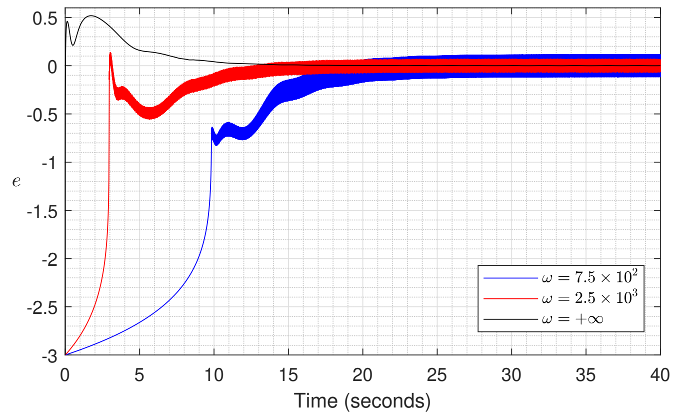

Figure 4. Tracking error over different frequency subject to the controller (21)

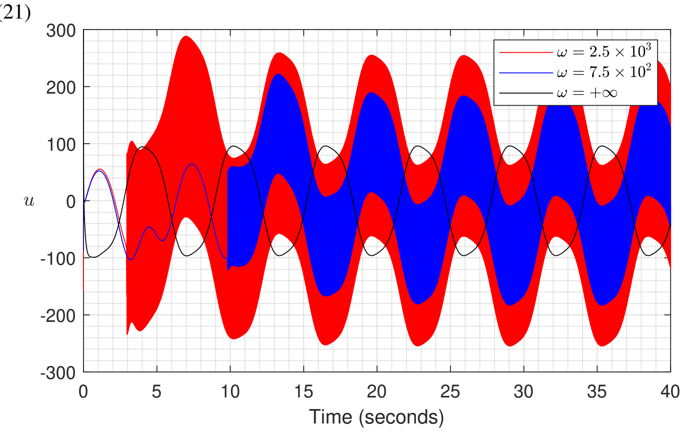

Figure 5. Trajectories of the controller (21) over different frequencies

the system satisfies all the assumptions in [44]. The controller (14) is designed with

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -12 & -13 & -6 \end{bmatrix}, \; N = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$
$$\Psi_i^\sigma = \begin{bmatrix} 4 - 9\sigma^4 & 12 & 13 - 10\sigma^2 & 6 \end{bmatrix}.$$

The numerical simulations are conducted for initial states of v and $\text{col}(x_1, x_2, y)$ randomly chosen in $(0, 2)$ and all initial conditions in the controller set to zero. The uncertain parameter $w = \text{col}(-5, 0.15, -3)$.

The parameters k and α in (14) are chosen as $k = 20$ and $\alpha = 4$. Figure 1 shows the trajectory of y over different frequencies with $\rho(s) = (s^2 + 1)^2$. Figure 2 shows the trajectories of $e = y - v_1$ over different frequencies subject to the controller (14). Figure 3 shows trajectories of the controller (14) over different frequencies. The parameters k and α in (21) are chosen as $k = 10$ and $\alpha = 10$. Figure 4 shows the trajectories of $e = y - v_1$ over different frequencies subject to the controller (21) with $\rho(s) = 1 + 3s^2$. Figure 5 shows trajectories of the controller (21) over different frequencies. For the proposed control structure, it is possible to offset the impact of a larger frequency on the control action by reducing the value of α.

VI. Conclusion

This article studies the robust practical output regulation problem for a class of nonlinear systems in the absence of knowledge of the control direction. Based on a Lie bracket averaging technique, the proposed extremum-seeking controller ensures that all signals in the closed-loop system are bounded and converge to a compact set. The large overshoots resulting from the wrong initial sign estimate in Nussbaum-type gain techniques can be mitigated by implementing an extremum-seeking controller, and $\frac{1}{\omega}$-semi-global uniform asymptotic convergence can be guaranteed, thus enhancing the results in [19, 24, 29]. In contrast to existing works [19, 24, 29], the proposed method is the first design method that can solve output regulation problems with unknown control direction without the need for Nussbaum-type gain techniques.

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This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2024.3426890

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