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Switched model predictive control of switched linear systems: Feasibility, stability and robustness[☆]

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ABSTRACT

This paper is concerned with the issues of feasibility, stability and robustness on the switched model predictive control (MPC) of a class of discrete-time switched linear systems with mode-dependent dwell time (MDT). The concept of conventional MDT in the literature of switched systems is extended to the stage MDT of lengths that vary with the stages of the switching. By computing the steps over which all the reachable sets of a starting region are contained into a targeting region, the minimum admissible MDT is offline determined so as to guarantee the persistent feasibility of MPC design. Then, conditions stronger than the criteria for persistent feasibility are explored to ensure the asymptotic stability. A concept of the extended controllable set is further proposed, by which the complete feasible region for given constant MDT can be determined such that the switched MPC law can be persistently solved and the resulting closed-loop system is asymptotically stable. The techniques developed for nominal systems lay a foundation for the same issues on systems with bounded additive disturbance, and the switched tube-based MPC methodology is established. A required "switched" tube in the form of mode-dependent cross section is determined by computing a mode-dependent generalized robust positive invariant set for each error subsystem between nominal subsystem and disturbed subsystem. The theoretical results are testified via an illustrative example of a population ecological system.

1. Introduction

1.1. Background and related work

Switched systems are commonly used to model multiple-mode plants or hybrid control systems that use a family of controllers. Switching signals can be categorized as state-dependent or time-dependent in terms of how a switching is generated (Liberzon, 2003). Control problems that have been studied for switched systems with both classes of switching include stability analysis, control synthesis, and the design of switching signals to ensure the stability of the resulting closed-loop system, see, e.g., Branicky (1998), De Persis, De Santis, and Morse (2003), Ferrari-Trecate, Cuzzola, Mignone, and Morari (2002), Hespanha (2004), Liberzon (2003) and Zhao and Hill (2008). In particular, an important subject on properties of switching signals is to determine minimum admissible dwell time such that the resulting switched system is (asymptotically) stable, see, e.g., Chesi, Colaneri, Geromel, Middleton, and Shorten (2012) and Geromel and Colaneri (2006).

The model predictive control (MPC) of switched systems is of interest, as constraints are frequently encountered in practice. For state-dependent switching, MPC of a class of piecewise affine systems—which is also termed *hybrid MPC* in the literature—has been extensively investigated, e.g., Borrelli, Mato Baotic, Bemporad, and Morari (2005) and Lazar (2006). In this type of systems, switching among different system modes occurs when the system state hits a certain switching surface; consequently, the mode variations during the prediction horizon are known *a priori* for the MPC optimization at each step. As for time-dependent switching, if the switching sequence consisting of both

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switching instants and switching indices is exactly prescribed, then the results of the hybrid MPC literature can be directly used. In many practical systems, the switching instants are unknown *a priori*, such as occurs in systems that encounter faults during their operation (Chiang, Russell, & Braatz, 2001). In this case, an MPC algorithm would not have *a priori* knowledge of the mode changes over the prediction horizon, and so there would not be any single known predicted trajectory, even in the case of no model uncertainties, unlike what is commonly assumed in the hybrid MPC literature. This lack of knowledge motivates the design of the MPC algorithm to be individually configured for each subsystem. In this scenario, some works have considered system stability (Colaneri & Scattolini, 2007; Franco, Sacone, & Parisini, 2004; Gorges, Izak, & Liu, 2011) by applying the classical multiple Lyapunov-like functions (MLFs) approach (Branicky, 1998) commonly used for switched systems.

In addition to the stability requirement, a more predominated issue is the persistent feasibility of the MPC synthesis for the switched systems with unknown switching instants. In Muller, Martius, and Allgower (2012), persistent feasibility has been explicitly addressed for a class of continuous-time switched nonlinear systems with average dwell time (ADT) switching. Once the MPC algorithm for each subsystem is designed, an admissible ADT ensuring the system stability can be specified and the feasible region estimated. Although ADT switching is flexible, as the running time sometimes can be shorter than the required ADT, a resulting drawback is that the feasible region will shrink, as a state may not be steered to the feasible region of another system mode if the running time is short. A more fundamental problem in MPC for switched systems with unknown switching instants, i.e., determining the admissible dwell time to guarantee both persistent feasibility and system stability, has even not been fully solved. In addition, for given dwell time, how to determine the complete feasible region for the switched systems is also largely open, even when the subsystems are linear.

Turning to the robust MPC synthesis for the switched systems with uncertainties, the subject is more significant as uncertainties are unavoidable in practice; but it has also almost not been investigated unlike in the area of robust MPC of non-switched systems (Rawlings & Mayne, 2009) or piecewise affine systems (Lazar, 2006). Recent years have witnessed rapid progress in robust MPC based on diverse methodologies including open-loop and closed-loop min–max MPC (Alamo, Munoz de la Pena, Limon, & Camacho, 2005; Limon, Alamo, Salas, & Camacho, 2006) and tube-based MPC (Mayne, Seron, & Raković, 2005; Raković, Kouvaritakis, Cannon, Panos, & Findeisen, 2011). As a method that employs finite-dimensional optimization in robust MPC synthesis, the tube-based MPC law presents a concise separate control policy, which consists of a conventional MPC for the nominal systems and a local feedback control law that steers the states of the uncertain systems to be, for all time, within a tube centered on the nominal trajectory. The tube can be determined allowing for the cross sections to be constant or time-varying, computed online or offline. In the presence of switching dynamics, however, it is challenging to determine the tube (even with constant cross section) such that the persistent feasibility within each subsystem and at switching instants can be both ensured. The aforesaid two issues, i.e., determining the admissible dwell time and feasible region for given dwell time, for uncertain switched systems are more difficult and so far, not addressed up to the authors' knowledge.

1.2. Objectives and contributions

This paper investigates the switched MPC of a class of discrete-time switched linear systems with mode-dependent dwell time (MDT). Three key issues including feasibility, stability and robustness will be addressed. The nominal systems are first studied, upon which is built a basis for the scenario of systems involved with bounded additive disturbance. The detailed objectives are as below.

A. Nominal systems

(i) *The first objective* of the paper is to determine the minimum admissible dwell time for nominal switched systems, in which the switching is autonomous with switching times unknown *a priori*, in ensuring both the system (asymptotic) stability and the persistent feasibility of MPC design. (ii) With the problem in the first objective being solved, the feasible region of the switched systems for given dwell time (less than the minimum admissible ones computed in (i)) needs to be determined. The developments of corresponding algorithms to obtain a complete feasible region for the underlying systems form *the second objective*.

B. Systems with bounded additive disturbance

The above two objectives involve much more difficulties for practical switched systems in the presence of uncertainties. Allowing for the bounded additive disturbance to the systems in the paper, the specific objectives in this part are twofold paralleling the ones for nominal systems. (i) The determination of the minimum admissible dwell time for disturbed systems, as *the third objective,* entails the computation of a "switched" tube in order to establish the switched tube-based MPC methodology, in addition to the techniques developed for nominal systems. (ii) *The final objective* of the paper is to further the algorithms explored for the second objective to determine a complete feasible region of the disturbed switched systems on the basis of the solution to the problem in the third objective.

Towards these goals, the contributions of the paper are highlighted as follows.

(i) The *stage* MDT of lengths varying with the stages of the switching is proposed. By computing the steps over which all the reachable sets of a starting region are contained into a targeting region, the minimum admissible MDT is offline determined so as to guarantee the persistent feasibility of MPC design. Then, stronger conditions are also developed to ensure asymptotic stability. (ii) A concept of the extended controllable set (ECS) is proposed. For a targeting region, by determining its ECS that can cover the states at the switching instants, the obtained stage MDT can be shortened allowing for them to be further state-dependent. The pros and cons of non-state-dependent and state-dependent MDTs are also presented analytically and testified via an illustrative example of a population ecological system. (iii) Further, via the ECS approach, the complete feasible region for given constant MDT is determined such that the switched MPC law can be persistently solved and the resulting closed-loop system is asymptotically stable. (iv) As for the systems with bounded additive disturbance, the switched tube-based MPC methodology is established to address the two issues of determination of minimum admissible stage MDT and the feasible region for given constant MDT. A required "switched" tube in the form of mode-dependent cross section is determined by computing a generalized robust positive invariant set for each error subsystem between nominal subsystem and disturbed subsystem.

Note that compared with (Zhang & Braatz, 2013) (the conference version of the paper), the paper not only further addresses the case of the disturbed systems, but also generalizes the MDT to the stage ones, obtains the complete feasible region based on the ECS proposed in this paper, and develops the criteria ensuring the asymptotic stability.

1.3. Notation

Notation: The superscript "T" stands for matrix transpose; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; the $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the sets of non-negative real numbers integers, respectively; and $\mathbb{Z}_{\geq s_1}$ and $\mathbb{Z}_{[s_1,s_2]}$ denote the sets $\{k \in \mathbb{Z}_+ | k \geq s_1\}$ and $\{k \in \mathbb{Z}_+ | s_1 \leq k \leq s_2\}$, respectively, for some $s_1, s_2 \in \mathbb{Z}_+$. The spectral radius of matrix A is denoted by $\rho(A)$. A real-valued scalar function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ that is continuous, strictly increasing, and has $\alpha(0) = 0$ is said to be of *class* $\mathcal{K}$. For any real $\lambda \geq 0$ and set $\mathcal{P} \subset \mathbb{R}^n$, the set $\lambda\mathcal{P}$ is defined as $\lambda\mathcal{P} \triangleq \{x \in \mathbb{R}^n | x = \lambda y \text{for some } y \in \mathcal{P}\}$. The Pontryagin difference, Minkowski sum of two arbitrary sets $\mathcal{P}_1 \subset \mathbb{R}^n$, $\mathcal{P}_2 \subset \mathbb{R}^n$, are denoted as $\mathcal{P}_1 \ominus \mathcal{P}_2$ and $\mathcal{P}_1 \oplus \mathcal{P}_2$, respectively; $co\{\mathcal{P}\}$ denotes the convex-hull of $\mathcal{P}$. The cardinality of set $\mathbf{\Delta}$ is denoted as card$\{\mathbf{\Delta}\}$. For two sets $\mathbf{\Delta} \triangleq \{\Delta_1, \Delta_2, \ldots, \Delta_M\}$ and $\boldsymbol{\eta} \triangleq \{\eta_1, \eta_2, \ldots, \eta_M\}$, $\Delta_i, \eta_i \in \mathbb{Z}_+, i \in \mathbb{Z}_{[1,M]}$, $\mathbf{\Delta}^{[\pm z]}(\boldsymbol{\eta})$ stands for the set $\{\Delta_1^{[\pm z]}, \Delta_2^{[\pm z]}, \ldots, \Delta_M^{[\pm z]} | \Delta_i^{[\pm z]} \geq \eta_i, \sum_{i=1}^M \Delta_i^{[\pm z]} = (\sum_{i=1}^M \Delta_i) \pm z\}$, $z \in \mathbb{Z}_+$.

2. Nominal systems

2.1. Preliminaries and problem formulation

Consider the class of discrete-time switched linear systems:

$$(\Omega_{\sigma(k)}) : x_{k+1} = A_{\sigma(k)}x_k + B_{\sigma(k)}u_k \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, and $u_k \in \mathbb{R}^{n_u}$ is the control input; $\sigma(k)$ is a switching signal that is a piecewise constant function of time k, continuous from the right everywhere, and takes values at the sampling times in a finite set $\mathcal{I} = \{1, \ldots, M\}$, where $M > 1$ is the number of subsystems. The switching is supposed to be autonomous, and the switching sequences $\mathbf{S} \triangleq \{k_0, k_1, \ldots, k_l, \ldots\}$ are unknown *a priori*, but are known instantly, where the switching instant is denoted as k_{l-1}, $l \in \mathbb{Z}_{\geq 1}$. When $k \in [k_{l-1}, k_l)$, the $\sigma(k_{l-1})$th subsystem (or system mode) is said to be *activated* and the length of the current running time of the subsystem is $k_l - k_{l-1}$.

As commonly considered in the literature of switched linear systems (e.g., Branicky, 1998, Chesi et al., 2012, Geromel & Colaneri, 2006, Hespanha, 2004 and Liberzon, 2003), the individual subsystems are assumed to have the origin as the common equilibrium. In addition, generally the switching in a discrete-time switched system does not necessarily take place exactly at the sampling instants. This work is based on the following assumption in the context of discrete-time switched systems.

Assumption 1. The switching instants are assumed to exactly be the sampling instants of system (1).

Both the system state and control input are subject to mode-dependent constraints, i.e., $\forall\sigma(k) = m \in \mathcal{I}$,

$$x_k \in \mathbb{X}_m \subseteq \mathbb{R}^{n_x}, \qquad u_k \in \mathbb{U}_m \subseteq \mathbb{R}^{n_u} \tag{2}$$

where both $\mathbb{X}_m$ and $\mathbb{U}_m$ are compact polyhedral sets that contain the origin in their interior.

The switching signals are considered to have *mode-dependent dwell time* property that is defined as below.

Definition 1. Consider system (1) and switching instants k_0, $k_1, \ldots, k_l, \ldots$ with $k_0 = 0$. A positive constant τ_m is said to be *mode-dependent dwell time*(MDT) associated with subsystem Ω_m, if $k_l - k_{l-1} \geq \tau_m$ when $\sigma(k) = m$ for $k \in [k_{l-1}, k_l)$, $l \in \mathbb{Z}_{\geq 1}$.

Note that the MDT in Definition 1 is constant, regardless of the lth switching of the switched system, $l \in \mathbb{Z}_{\geq 1}$. The concept can be generalized to the following *stage* MDT that is of variable lengths, in the sense of the lth *stage of switching* (the stage begins with the lth switching at k_{l-1}, and ends with the $(l+1)$th switching of the switched systems at k_l).

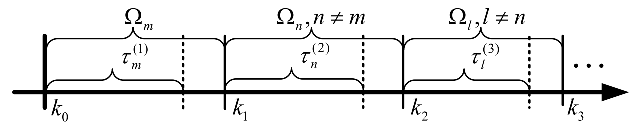


Fig. 1. Stage MDT $\tau_m^{(l)}$, $l \in \mathbb{Z}_{\geq 1}$, $m \in \mathcal{I}$.

Definition 2. Consider system (1) and switching instants $k_0, k_1, \ldots, k_l, \ldots$ with $k_0 = 0$ and the lth switching occurring at $k_{l-1}, l \in \mathbb{Z}_{\geq 1}$. A positive constant $\tau_m^{(l)}$ is said to be the lth stage MDT associated with subsystem Ω_m ($x_{k+1} = A_m x_k + B_m u_k$), if $k_l - k_{l-1} \geq \tau_m^{(l)}$ when $\sigma(k) = m$ for $k \in [k_{l-1}, k_l)$, $l \in \mathbb{Z}_{\geq 1}$.

An illustration about the stage MDT[1] is given in Fig. 1.

The control input u_k is designed based on the current subsystem model, allowing for the switching times to be unknown *a priori*. Once a switching is detected, the model for the control design is switched. A regular model predictive control (MPC) strategy for each subsystem is adopted, and accordingly a mode-dependent MPC optimization is solved at each sampling time k. More specifically, $\forall\sigma(k) = m \in \mathcal{I}$, let N_m denote the prediction horizon for subsystem Ω_m, for the given system state x_k, the following optimization problem

$$\min_{\mathbf{u}_k} J_m(x_k, \mathbf{u}_k) \triangleq T_m(x_{N_m/k}) + \sum_{i=0}^{N_m-1} L_m(x_{i/k}, u_{i/k})$$
$$\text{subject to } x_{i+1/k} = A_m x_{i/k} + B_m u_{i/k}, \quad x_{0/k} = x_k,$$
$$x_{i/k} \in \mathbb{X}_m, \; u_{i/k} \in \mathbb{U}_m, \; \forall i \in \mathbb{Z}_{[0,N_m-1]}, \; x_{N_m/k} \in \mathcal{T}_m \tag{3}$$

is solved at time k, where $x_{i/k}$, $i = 1, \ldots, N_m$ denotes the state predicted through subsystem Ω_m by applying the input sequence $\mathbf{u}_k \triangleq (u_{0/k}, u_{1/k}, \ldots, u_{N_m-1/k})$, the terminal set $\mathcal{T}_m$ is mode-dependent, and the cost function $J_m(x_k, \mathbf{u}_k)$ consists of stage cost $L_m(x_{i/k}, u_{i/k})$ and terminal cost $T_m(x_{N_m/k})$ that are also both mode-dependent mappings with $L_m : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}_+$, $L_m(0, 0) = 0$, and $T_m : \mathbb{R}^{n_x} \to \mathbb{R}_+$, $T_m(0) = 0$.

A state $x_k \in \mathbb{X}_m$ is said to be *feasible* for the optimization problem (3) if

$$\mathcal{U}_{N_m}^m(x_k) \triangleq \Big\{u_{i/k} \in \mathbb{U}_m \,\Big|\, x_{i/k} \in \mathbb{X}_m, x_{N_m/k} \in \mathcal{T}_m,$$
$$x_{i+1/k} = A_m x_{i/k} + B_m u_{i/k}, x_{0/k} = x_k,$$
$$\forall i \in \mathbb{Z}_{[0,N_m-1]}\Big\} \neq \varnothing.$$

Denoting the optimal sequence of controls solving the above optimization as $\mathbf{u}_k^* \triangleq (u_{0/k}^*, u_{1/k}^*, \ldots, u_{N_m-1/k}^*)$, the MPC law is defined as

$$u^{MPC}(\Omega_m, x_k) \triangleq u_{0/k}^*. \tag{4}$$

Let $\mathcal{X}_{N_m}^m$ denote the set of all of the feasible states for subsystem Ω_m, and $V_{N_m}^m : \mathcal{X}_{N_m}^m \to \mathbb{R}_+$ denote the value function, i.e., the infimum of the cost function, i.e., $V_{N_m}^m(x_k) = J_m(x_k, \mathbf{u}_k^*)$.

The control input at each $k \in [k_{l-1}, k_l)$, $l \in \mathbb{Z}_{\geq 1}$, is $u_k = u^{MPC}(\Omega_{\sigma(k)}, x_k)$, which can be obtained by solving (3) that implicitly gives (4). Also, u_k can be explicitly solved in certain scenarios particularly if the cost function is quadratic, each subsystem is linear, and both control and state constraints are polyhedra (cf. Page 484 in Rawlings & Mayne, 2009).

The following definitions of the positive invariant set and control invariant set are needed.

[1] Unless specifically stated, the term MDT will be slightly abused in later developments to serve both Definitions 1 and 2, and when necessary, the superscript "(l)" will be used to indicate the MDT be a stage one.

Definition 3 (Rawlings & Mayne, 2009). A set $\mathcal{O} \subseteq \mathbb{X} \subseteq \mathbb{R}^{n_x}$ is said to be, respectively, a positive invariant set for autonomous system $x_{k+1} = f(x_k),\ x_k \in \mathbb{X}$, if $x_k \in \mathcal{O}$ implies $x_t \in \mathcal{O},\ t \in \mathbb{Z}_{\geq k+1}$; a control invariant set for controlled system $x_{k+1} = f(x_k, u_k),\ x_k \in \mathbb{X}, u_k \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$, if $x_k \in \mathcal{O}$ implies there exists $u_k \in \mathbb{U}$ such that $x_t \in \mathcal{O},\ t \in \mathbb{Z}_{\geq k+1}$.

Let $\mathcal{B}$ denote the closed unit ball in $\mathbb{R}^{n_x}$. The asymptotic stability in a region of attraction for a constrained system is defined as below.

Definition 4 (*Rawlings & Mayne, 2009, Definition B.9*). Suppose $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ is positive invariant set for $x_{k+1} = f(x_k)$. The system origin is (i) stable in $\mathbb{X}$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ *such that for all* $x_0 \in \mathbb{X} \cap \delta\mathcal{B}$ *implies that* $\|x_k\| \leq \varepsilon$ *for all* $k \geq 0$. (ii) attractive in $\mathbb{X}$ if $\|x_k\| \to 0$ as $k \to \infty$ *for all* $x_0 \in \mathbb{X}$. (iii) asymptotically stable with a region of attraction $\mathbb{X}$ if it is stable in $\mathbb{X}$ and attractive in $\mathbb{X}$.

The objectives of this section include: (i) Determine the admissible MDT such that the resulting switched system (1)–(4) is asymptotically stable in the sense of Definition 4, with such MDT concisely denoted as $\mathcal{AS}$-MDT; (ii) Given MDT, find the feasible region (the set of all feasible states) such that (4) can be persistently solved and the resulting switched system (1)–(4) is asymptotically stable, likewise, the feasible region is denoted as $\mathcal{AS}$-FR.

Since one special switching is that the system stays at one of subsystems, the above objectives require that each subsystem $\Omega_m,\ m \in \mathcal{I}$, in closed-loop with MPC law (4) is asymptotically stable, which is ensured by the following assumption that is standard in the MPC literature, see, e.g., Lazar (2006), Muller et al. (2012) and Rawlings and Mayne (2009).

Assumption 2. There exist $\alpha_{m,i} \in \mathcal{K},\ i = 1, 2,\ \forall\sigma(k) = m \in \mathcal{I}$ and a feedback control law $K_m(\cdot)$ such that $\mathcal{T}_m$ is a control invariant set for subsystem Ω_m in closed-loop with $u_k = K_m(x_k)$ and (i) $L_m(x, u) \geq \alpha_{m,1}(\|x\|),\ \forall x \in \mathcal{X}^m_{N_m},\ \forall u \in \mathbb{U}_m$. (ii) $T_m(x) \leq \alpha_{m,2}(\|x\|),\ \forall x \in \mathcal{T}_m$ (iii) $\Delta T_m(x) + L_m(x, K_m(x)) \leq 0,\ \forall x \in \mathcal{T}_m$, where $\Delta T_m(x) \triangleq T_m(A_m x + B_m K_m(x)) - T_m(x)$.

Remark 1. Note that Assumption 2 is comparable to the typical assumption in the area of switched systems that each subsystem in closed-loop via a stabilizing linear state feedback is asymptotically stable. With Assumption 2, it can be concluded from Rawlings and Mayne (2009, see Page 119) that $\mathcal{X}^m_{N_m}$ is the region of attraction for subsystem Ω_m in closed-loop with (4) and is positively invariant, as $\mathcal{T}_m$ is a control invariant set for subsystem Ω_m.

2.2. Determination of $\mathcal{AS}$-MDT

This subsection first determines the admissible MDT such that the MPC design for switched system (1)–(4) is persistently feasible (denoted as $\mathcal{F}$-MDT), the underlying system is attractive besides the persistent feasibility ($\mathcal{FA}$-MDT), and furthers the results to $\mathcal{AS}$-MDT case. The logic relation among the criteria to be developed in this subsection and later (sub)sections can be seen in Appendix B.

To determine the admissible $\mathcal{AS}$-MDT, before ensuring the stability and attractivity of closed-loop switched system (1)–(4), persistent feasibility (aka recursive feasibility, cf. Page 111 in Rawlings & Mayne, 2009) of solving the controller (4) needs to be first guaranteed. The persistent feasibility means that the MPC optimization is persistently feasible at all sampling times, i.e., both within each subsystem and at switching instants. By Assumption 2, $\mathcal{T}_m$ is a control invariant set for each subsystem Ω_m; then the persistent feasibility within each subsystem is ensured (cf. Page 111 in Rawlings & Mayne, 2009).

To guarantee persistent feasibility at the switching instants, it intuitively suffices that, for $\sigma(k_{l-1}) = m$ and $\sigma(k_l) = n, \forall n \neq m \in \mathcal{I}$, the running time of Ω_m, $H_m \triangleq k_l - k_{l-1}, l \in \mathbb{Z}_{\geq 1}$, belongs to

$$\mathcal{H}_{m,n} \triangleq \left\{ H \in \mathbb{Z}_+ \left| x_{k_s+H} \in \mathcal{X}^n_{N_n},\ \forall x_{k_s} \in \mathcal{X}^m_{N_m} \right. \right\}. \tag{5}$$

However, since the feasible region $\mathcal{X}^n_{N_n}$ of the next subsystem Ω_n is generally not an invariant set for the current subsystem Ω_m, a state entering $\mathcal{X}^n_{N_n}$ at some sampling time may leave the set at a later sampling time; this means that $\mathcal{H}_{m,n}$ may be a set consisting of dispersed values of admissible running time.

Therefore, even if there are no extra requirements in ensuring (asymptotic) stability, to obtain the set $\mathcal{H}_{m,n}$ in (5) for a state will not offer an explicit criterion for each subsystem how long the admissible MDT should be. To overcome such a problem, in what follows, the positive invariance of reachable sets of a system controlled by MPC will be utilized to determine the $\mathcal{F}$-MDT.

For subsystem $\Omega_m : x_{k+1} = A_m x_k + B_m u_k$, where u_k is the control input designed by the MPC law, the one-step reachable set from $\mathcal{X} \subseteq \mathcal{X}^m_{N_m}$ along subsystem Ω_m is denoted as

$$\begin{aligned} &\mathcal{R}each(\mathcal{X}, \Omega_m) \triangleq \big\{ x \in \mathbb{R}^n \,|x_0 \in \mathcal{X}, \\ &x = A_m x_0 + B_m u^{MPC}(\Omega_m, x_0) \big\} \end{aligned} \tag{6}$$

and the H-step reachable set, $\mathcal{R}^m_H(\mathcal{X})$ is defined as

$$\begin{aligned} &\mathcal{R}^m_{y+1}(\mathcal{X}) \triangleq \mathcal{R}each\left(\mathcal{R}^m_y(\mathcal{X}), \Omega_m\right), \quad y \in \mathbb{Z}_{[0,H-1]}, \\ &\mathcal{R}^m_0(\mathcal{X}) = \mathcal{X}. \end{aligned} \tag{7}$$

Theorem 1. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}^m_{N_m}$, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold, the MPC design for system (1)–(4) is persistently feasible with admissible MDT $\tau^{(l)}_m$, $l \in \mathbb{Z}_{\geq 1}$, if $\tau^{(1)}_m$ satisfies*

$$\mathcal{R}^m_{\tau^{(1)}_m}(\mathcal{X}^m_{N_m}) \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{X}^n_{N_n} \tag{8}$$

and $\forall\sigma(k_{l-1}) = m \in \mathcal{I},\ l \in \mathbb{Z}_{\geq 2},\ \tau^{(l)}_{\sigma(k_{l-1})}$ satisfies

$$\Upsilon(\tau^{(l)}_{\sigma(k_{l-1})}) \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{X}^n_{N_n} \tag{9}$$

where $\Upsilon(\tau^{(l)}_{\sigma(k_{l-1})}) \triangleq \bigcup_{i=0}^{\tau^{(1)}_{\sigma(k_{l-1})} - \tau^{(l)}_{\sigma(k_{l-1})}} \bigcup_{\sigma(k_{l-2}) \in \mathcal{I}} \mathcal{R}^{\sigma(k_{l-1})}_{\tau^{(l)}_{\sigma(k_{l-1})}+i} (\hat{\mathcal{R}}^{\cup}(\tau^{(l-1)}_{\sigma(k_{l-2})}))$ *with* $\hat{\mathcal{R}}^{\cup}(\tau^{(l-1)}_{\sigma(k_{l-2})}) \triangleq \mathcal{R}^{\sigma(k_0)}_{\tau^{(1)}_{\sigma(k_0)}} (\mathcal{X}^{\sigma(k_0)}_{N_{\sigma(k_0)}}), l = 2$ *and* $\hat{\mathcal{R}}^{\cup}(\tau^{(l-1)}_{\sigma(k_{l-2})}) \triangleq \bigcup_{\sigma(k_{l-3}) \in \mathcal{I}} \mathcal{R}^{\sigma(k_{l-2})}_{\tau^{(l-1)}_{\sigma(k_{l-2})}} (\hat{\mathcal{R}}^{\cup}(\tau^{(l-2)}_{\sigma(k_{l-3})})),\ l \in \mathbb{Z}_{\geq 3}$.

Proof. If Assumption 2 holds, the feasible region $\mathcal{X}^m_{N_m}$ is positively invariant, which implies that $\mathcal{R}^m_1(\mathcal{X}^m_{N_m}) \subseteq \mathcal{X}^m_{N_m}$. By the geometric condition for invariance (cf. Dorea & Hennet, 1999), $\mathcal{R}^m_1(\mathcal{X}^m_{N_m})$ is also positively invariant for Ω_m, and so is $\mathcal{R}^m_H(\mathcal{X}^m_{N_m})$, $\forall H \in \mathbb{Z}_{\geq 2}$. Consider $l = 1$ (the 1st stage of switching) and $\sigma(k_0) = m$, due to $\hat{\mathcal{R}}^{\cup}(\tau^{(1)}_{\sigma(k_0)}) = \mathcal{R}^m_{\tau^{(1)}_m}(\mathcal{X}^m_{N_m}) \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{X}^n_{N_n}$ ((8)) and the fact that the running time $k_1 - k_0$ is not less than $\tau^{(1)}_m$, then $\mathcal{R}^m_{k_1-k_0}(\mathcal{X}^m_{N_m}) \subseteq \mathcal{R}^m_{\tau^{(1)}_m}(\mathcal{X}^m_{N_m}) \subseteq \mathcal{X}^n_{N_n}$, $\forall n \in \mathcal{I},\ n \neq m$, that is, the system trajectory in $\mathcal{R}^m_{k_1-k_0}(\mathcal{X}^m_{N_m})$ will stay inside $\mathcal{R}^m_{\tau^{(1)}_m}(\mathcal{X}^m_{N_m}) \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{X}^n_{N_n}$ until the second switching occurs. Suppose at $l = \upsilon$ (the υth stage of switching), $\tilde{\mathcal{R}}_\upsilon \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{X}^n_{N_n}$ where $\tilde{\mathcal{R}}_\upsilon$ denotes the region evolved from $\mathcal{X}^{\sigma(k_0)}_{N_{\sigma(k_0)}}$ after $\upsilon - 1$ stages according to the concrete switching sequence of the system, i.e., the system trajectory falls into $\bigcap_{n \in \mathcal{I}} \mathcal{X}^n_{N_n}$ after running time $k_{\upsilon-1} - k_{\upsilon-2}$. Bear in mind that

$\tau^{(v)}_{\sigma(k_{v-1})} \le k_v - k_{v-1}$. Then, at $l = v+1$ (the $(v+1)$th stage of switching), it follows from the requirement in (9) on $\tau^{(l)}_{\sigma(k_{l-1})}$, $l \in \mathbb{Z}_{\ge 2}$ that $\tilde{\mathcal{R}}_{v+1} = \mathcal{R}^{\sigma(k_{v-1})}_{k_v-k_{v-1}}(\tilde{\mathcal{R}}_v) \subseteq \bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}$. Therefore, the MDT $\tau^{(l)}_m$ satisfying (8) and (9) is admissible for the system in terms of persistent feasibility. □

In Theorem 1, $\tau^{(l)}_m$ needs to be computed at each stage of switching, thus for a switching signal with infinite switching sequences, it is not practical to determine the admissible $\mathcal{F}$-MDT based on Theorem 1. The following two corollaries avoid the problem.

Corollary 1. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}^m_{N_m}$, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold, the MPC design for system (1)–(4) is persistently feasible with admissible MDT $\tau^{(l)}_m$, $l \in \mathbb{Z}_{\ge 1}$, if $\tau^{(1)}_m$ satisfies (8) and $\tau^{(l)}_m \equiv \tau^{(2)}_m$, $l \in \mathbb{Z}_{\ge 2}$ where $\tau^{(2)}_m$ satisfies*

$$\bigcup_{i=0}^{\tau^{(1)}_m-\tau^{(2)}_m} \mathcal{R}^m_{\tau^{(2)}_m+i}\left(\bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}\right) \subseteq \bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}. \tag{10}$$

Proof. By Theorem 1, $\tau^{(1)}_m$ is admissible. For any $l \in \mathbb{Z}_{\ge 2}$, $\tau^{(l)}_m$ satisfying (10) will be no less than the one satisfying (9) due to $\hat{\mathcal{R}}^{\cup}(\tau^{(l-1)}_{\sigma(k_{l-2})}) \subseteq \bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}$. □

Corollary 2. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}^m_{N_m}$, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold, the MPC design for system (1)–(4) is persistently feasible with admissible MDT $\tau^{(l)}_m$, $l \in \mathbb{Z}_{\ge 1}$, if $\tau^{(l)}_m \equiv \tau_m$ where τ_m satisfies*

$$\mathcal{R}^m_{\tau_m}(\mathcal{X}^m_{N_m}) \subseteq \bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}. \tag{11}$$

Proof. The proof is straightforward from Corollary 1 as $\bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n} \subseteq \mathcal{X}^m_{N_m}$, $m \in \mathcal{I}$. □

Based on Theorem 1, Corollary 1, and Corollary 2, the corresponding minimum admissible $\mathcal{F}$-MDT can be obtained by solving the minimization procedures

$$\underline{\tau}^{(l)}_m \triangleq \min\{\tau^{(l)}_m\}, \quad \text{for } \tau^{(l)}_m \in \mathbb{Z}_+ \text{ subject to (8)–(9) (or (8) and (10), or (11))}. \tag{12}$$

The algorithms for solving for $\underline{\tau}^{(l)}_m$ are trivial, which can be developed by iterations of increasing $\tau^{(l)}_m$ by one in the left-hand side of (8)–(9) (or (8) and (10), or (11)).

Note that in Theorem 1, Corollary 1, and Corollary 2, the persistent feasibility is ensured, but the system trajectory may not converge, i.e., the system state at the switching instants may just stay close to the margin of $\bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}$. Stronger conditions are needed to determine the $\mathcal{F}\mathcal{A}$-MDT, i.e., the admissible MDT such that the underlying system is attractive besides the persistent feasibility.

Theorem 2. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}^m_{N_m}$, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold, system (1)–(4) is attractive in $\bigcup_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}$ with admissible MDT $\tau^{(l)}_m$, $l \in \mathbb{Z}_{\ge 1}$, if $\tau^{(1)}_m$ satisfies (8) and $\forall \sigma(k_{l-1}) = m \in \mathcal{I}$, $l \in \mathbb{Z}_{\ge 2}$, $\tau^{(l)}_{\sigma(k_{l-1})}$ satisfies*

$$\Upsilon(\tau^{(l)}_{\sigma(k_{l-1})}) \subseteq \bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-1}(\mathcal{X}^n_{N_n}) \tag{13}$$

where $\Upsilon(\tau^{(l)}_{\sigma(k_{l-1})})$ is denoted in (9).

Proof. (i) Persistent feasibility. It suffices to consider the cases when $l \in \mathbb{Z}_{\ge 2}$, as $\tau^{(1)}_m$ satisfies (8). Since $\bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-1}(\mathcal{X}^n_{N_n}) \subseteq \bigcap_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}$, $\forall l \in \mathbb{Z}_{\ge 2}$, then the MDT satisfying (13) is no less than the one satisfying (9) and thus admissible in terms of persistent feasibility. (ii) Attractivity. Since $\bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-1}(\mathcal{X}^n_{N_n}) \to \{\mathbf{0}\}$ as $l \to \infty$, $\|x_{k_l}\| \to 0$ and $V^n_{N_n}(x_{k_l}) \to 0$ as $l \to \infty$ (Note that $\bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-1}(\mathcal{X}^n_{N_n}) \nrightarrow \{\mathbf{0}\}$ will contradict with the fact, ensured by Assumption 2, that each state can be steered to the origin within each subsystem). By Assumption 2 and Lem. 2.19 in Rawlings and Mayne (2009), $V^m_{N_m}(x_{k+1}) - V^m_{N_m}(x_k) \le -\alpha_m(\|x_k\|)$. Since $\mathcal{I}$ is a finite set, then $\alpha \in \mathcal{K}$ can be chosen independent of m such that $V^m_{N_m}(x_{k+1}) - V^m_{N_m}(x_k) \le -\alpha(\|x_k\|)$. Hence, $\forall k \in [k_{l-1}, k_l)$, $\|x_k\| \to 0$ due to $\|x_k\| \le \alpha^{-1}(V^m_{N_m}(x_k)) \le \alpha^{-1}(V^m_{N_m}(x_{k_{l-1}}))$. □

A direct result extended from Theorem 2 is given as below, paralleling the extension from Theorem 1 to Corollary 1. The proof can be done by referring to the proof for Corollary 1 and omitted.

Corollary 3. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}^m_{N_m}$, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold, system (1)–(4) is attractive in $\bigcup_{n\in\mathcal{I}} \mathcal{X}^n_{N_n}$ with admissible MDT $\tau^{(l)}_m$, if $\tau^{(1)}_m$ satisfies (8) and $\tau^{(l)}_m$, $l \in \mathbb{Z}_{\ge 2}$ satisfies*

$$\bigcup_{i=0}^{\tau^{(1)}_m-\tau^{(2)}_m} \mathcal{R}^m_{\tau^{(2)}_m+i}\left(\bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-2}(\mathcal{X}^n_{N_n})\right) \subseteq \bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-1}(\mathcal{X}^n_{N_n}). \tag{14}$$

Remark 2. The admissible $\tau^{(l)}_m, l \in \mathbb{Z}_{\ge 1}$ in Theorem 2 and Corollary 3 is required to steer a starting region to targeting region $\bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-1}(\mathcal{X}^n_{N_n})$. As $l \to l+1$, $\bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l-1}(\mathcal{X}^n_{N_n})$ becomes $\bigcap_{n\in\mathcal{I}} \mathcal{R}^n_{l}(\mathcal{X}^n_{N_n})$, i.e., it contracts innerly "every one step" to ensure the convergence of system trajectory. In fact, the convergence pace can be changed to any others that can ensure convergence, e.g., "every two steps" or even contract innerly but sometimes expand outerly, but contract in a whole trend.

Though the $\mathcal{F}\mathcal{A}$-MDT can be determined by Theorem 2 or Corollary 3, it is unknown that the system stability can be ensured in terms of Definition 4 (the system trajectory can always approach to zero but may break through a ball with an arbitrarily-given radius ε). In addition, in both Theorem 2 and Corollary 3, $\tau^{(l)}_m$ needs to be computed at each stage of switching. Therefore, if a switching sequence is infinite, the infinite times of computation will make Theorem 2 and Corollary 3 impractical for determining $\mathcal{F}\mathcal{A}$-MDT. However, since each subsystem is linear, then if the stage cost is quadratic and the terminal cost $T_m(x)$ is set to be the value function of the unconstrained infinite horizon optimal quadratic control problem (denoted as $V^{UC}_{\infty}(\Omega_m, x)$ and the associated controller gain is denoted as K_m), which is common in linear MPC field, then an interesting property (cf. Chapter 2.5 in Rawlings & Mayne, 2009), i.e., $u^{MPC}(\Omega_m, x) \equiv K_m x$, $\forall x \in \mathcal{T}_m$, where $\mathcal{T}_m$ is a control invariant set for subsystem $x_{k+1} = (A_m + B_m K_m)x_k$, can be utilized to determine admissible $\mathcal{F}\mathcal{A}$-MDT by finite times of computation. The stability of underlying system will be also ensured in Theorem 3 as shown below; the resulting admissible $\mathcal{F}\mathcal{A}$-MDT will be therefore the required $\mathcal{A}\mathcal{S}$-MDT.

To proceed, let the set of MDT τ_m's be denoted by $\boldsymbol{\tau} \triangleq \{\tau_1, \tau_2, \dots, \tau_M\}$, and the admissible MDT switching sequence with set $\boldsymbol{\tau}$ until time k by $\mathbf{S}_{\boldsymbol{\tau}}(k) \triangleq \{k_0, k_1, \dots, k_s, \dots, k-1\}$ (the switching instants in $\mathbf{S}_{\boldsymbol{\tau}}(k)$ are required to satisfy Definition 1), the definition of contractive set for the underlying switched linear systems is needed.

Definition 5 (Dehghan & Ong, 2012a). Consider the constrained unforced switched linear system

$$x_{k+1} = \hat{A}_{\sigma(k)} x_k, \quad x_k \in \mathbb{X} \tag{15}$$

where $\mathbb{X}$ is a compact polytope and $\rho(\hat{A}_m) < 1$, $\forall \sigma(k) = m \in \mathcal{I}$. A set $\Omega \subseteq \mathbb{X}$ is said to be a $[\lambda, \boldsymbol{\tau}]$-contractive set of system (15) with MDT τ_m, if $x_0 \in \Omega$ implies $x_t \in \mathbb{X}$, $\forall t \in \mathbb{Z}_{[1,k]}$ and $\hat{A}_{\mathbf{S}_{\boldsymbol{\tau}}(k)} x_0 \in \lambda \Omega$, where the contraction factor $\lambda \in (0, 1)$ and $\hat{A}_{\mathbf{S}_{\boldsymbol{\tau}}(k)} \triangleq \prod_{s=0}^{k-1} \hat{A}_{\sigma(s)}$.

Under a certain MDT set $\boldsymbol{\Delta}$, the existence of the contractive set is ensured since each subsystem in (15) is required to be asymptotically stable, cf., Dehghan and Ong (2012a) and Dehghan and Ong (2013). Then, the two following criteria present sufficient conditions to determine $\mathcal{AS}$-MDT based on the property $u^{MPC}(\Omega_m, x) \equiv K_m x$, $\forall x \in \mathcal{T}_m$ when setting $T_m(x) = V_\infty^{UC}(\Omega_m, x)$.

Theorem 3. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}_{N_m}^m$, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold, the stage cost is quadratic, $T_m(x) = V_\infty^{UC}(\Omega_m, x)$ and the associated controller gain within $\mathcal{T}_m$ is K_m. If under a MDT set $\boldsymbol{\Delta} \triangleq \{\Delta_1, \Delta_2, \ldots, \Delta_M\}$, a $[\lambda, \boldsymbol{\Delta}]$-contractive set $\mathcal{O}_\infty^\lambda$ exists for system*

$$x_{k+1} = \hat{A}_{\sigma(k)} x_k, \quad x_k \in \bigcap_{m \in \mathcal{I}} \mathcal{T}_m \tag{16}$$

where $\hat{A}_{\sigma(k)} \triangleq A_{\sigma(k)} + B_{\sigma(k)} K_{\sigma(k)}$, then system (1)–(4) is asymptotically stable with a region of attraction $\bigcup_{n \in \mathcal{I}} \mathcal{X}_{N_n}^n$ under admissible $\tau_m^{(l)}$, if $\tau_m^{(1)}$ satisfies (8), $\tau_m^{(l)}$ satisfies (14), $l \in \mathbb{Z}_{[2,v]}$, and $\tau_m^{(l)} = \Delta_m$, $l \in \mathbb{Z}_{\geq v+1}$, where v satisfies

$$\bigcap_{n \in \mathcal{I}} \mathcal{R}_{v-1}^n(\mathcal{X}_{N_n}^n) \subseteq \mathcal{O}_\infty^\lambda. \tag{17}$$

Proof. Suppose under $\boldsymbol{\Delta} \triangleq \{\Delta_1, \Delta_2, \ldots, \Delta_M\}$, the $[\lambda, \boldsymbol{\Delta}]$-contractive set $\mathcal{O}_\infty^\lambda \subseteq \bigcap_{m \in \mathcal{I}} \mathcal{T}_m \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{X}_{N_n}^n$ exists for system (16). As demonstrated in Theorem 2, $\bigcap_{n \in \mathcal{I}} \mathcal{R}_{l-1}^n(\mathcal{X}_{N_n}^n) \rightarrow \{\mathbf{0}\}$ as $l \rightarrow \infty$, then there must exist a $v \in \mathbb{Z}_{\geq 2}$ such that $\bigcap_{n \in \mathcal{I}} \mathcal{R}_{v-1}^n(\mathcal{X}_{N_n}^n) \subseteq \mathcal{O}_\infty^\lambda$. (i) Persistent feasibility. It follows from Theorem 1 and Corollary 3 that $\tau_m^{(l)}$, $l \in \mathbb{Z}_{[2,v]}$ is admissible in terms of persistent feasibility. Also, $\tau_m^{(l)} = \Delta_m$, $l \in \mathbb{Z}_{\geq v+1}$ will be admissible since system trajectory will remain inside $\bigcap_{m \in \mathcal{I}} \mathcal{T}_m \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{X}_{N_n}^n$. (ii) Stability. Due to $T_m(x) = V_\infty^{UC}(\Omega_m, x)$, the system (1)–(4) reduces to (16) within $\bigcap_{m \in \mathcal{I}} \mathcal{T}_m$. Since $\bigcap_{m \in \mathcal{I}} \mathcal{T}_m$ contains the origin in its interior, then there exist ε_1 and δ_1 such that the following fact holds

$$\delta_1 \mathcal{B} \subseteq \mathcal{O}_\infty^\lambda \subseteq \bigcap_{m \in \mathcal{I}} \mathcal{T}_m \subseteq \varepsilon_1 \mathcal{B}. \tag{18}$$

Then if a given $\varepsilon \geq \varepsilon_1$, let $\delta \in (0, \delta_1]$, then since $\mathcal{O}_\infty^\lambda$ is MDT contractive within the constraint $\bigcap_{m \in \mathcal{I}} \mathcal{T}_m$, it follows that $\|x_0\| \leq \delta$ implies that $x_k \in \bigcap_{m \in \mathcal{I}} \mathcal{T}_m \subseteq \varepsilon_1 \mathcal{B} \subseteq \varepsilon \mathcal{B}$, which is $\|x_k\| \leq \varepsilon$. On the other hand, if $\varepsilon < \varepsilon_1$, then there must exist a $N \in \mathbb{Z}_{\geq 1}$ such that $\lambda^N \varepsilon_1 \leq \varepsilon$. Due to (18), for a $\eta > 0$, we have $\eta \delta_1 \mathcal{B} \subseteq \eta \mathcal{O}_\infty^\lambda \subseteq \eta \bigcap_{m \in \mathcal{I}} \mathcal{T}_m \subseteq \eta \varepsilon_1 \mathcal{B}$. Then, consider constraint $\lambda^N \bigcap_{m \in \mathcal{I}} \mathcal{T}_m$, it holds that $\lambda^N \mathcal{O}_\infty^\lambda$ will be a contractive set within $\lambda^N \bigcap_{m \in \mathcal{I}} \mathcal{T}_m$ under the same $\boldsymbol{\Delta}$, i.e., all the $x_k \in \lambda^N \bigcap_{m \in \mathcal{I}} \mathcal{T}_m$. Let $\delta \in (0, \lambda^N \delta_1]$, it follows that $\|x_0\| \leq \delta$ implies that $x_k \in \lambda^N \bigcap_{m \in \mathcal{I}} \mathcal{T}_m \subseteq \lambda^N \varepsilon_1 \mathcal{B} \subseteq \varepsilon \mathcal{B}$, i.e., $\|x_k\| \leq \varepsilon$. In sum, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x_0\| \leq \delta \Rightarrow \|x_k\| \leq \varepsilon$. (iii) Attractivity. First, under $\tau_m^{(l)}$, $l \in \mathbb{Z}_{[1,v]}$, the system trajectory converges to $\mathcal{O}_\infty^\lambda$ at switching instant k_v by Theorem 1 and Corollary 3. After k_v, with $\tau_m^{(l)} = \Delta_m$, $l \in \mathbb{Z}_{\geq v+1}$, it follows that, for any $x_{k_v} \in \mathcal{O}_\infty^\lambda$, $x_k = \hat{A}_{\mathbf{S}_{\boldsymbol{\Delta}}(k)} x_{k_v} \in \lambda^M \mathcal{O}_\infty^\lambda$ holds, where M is the number of switching after k_v till k. Since $\lambda \in (0, 1)$, then $M \rightarrow \infty$ as $k \rightarrow \infty$, which ensures $\|x_k\| \rightarrow 0$. □

Corollary 4. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}_{N_m}^m$, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold, the stage cost is quadratic, $T_m(x) = V_\infty^{UC}(\Omega_m, x)$ and the associated controller gain within $\mathcal{T}_m$ is K_m. If under a MDT set $\boldsymbol{\Delta} \triangleq \{\Delta_1, \Delta_2, \ldots, \Delta_M\}$, a $[\lambda, \boldsymbol{\Delta}]$-contractive set $\mathcal{O}_\infty^\lambda$ exists for system (16), then system (1)–(4) is asymptotically stable with a region of attraction $\bigcup_{n \in \mathcal{I}} \mathcal{X}_{N_n}^n$ under admissible MDT $\tau_m^{(l)}$, $l \in \mathbb{Z}_{\geq 1}$, if $\tau_m^{(l)} \equiv \Delta_m$, $\forall l \in \mathbb{Z}_{\geq 2}$ and $\tau_m^{(1)}$ satisfies*

$$\mathcal{R}_{\tau_m^{(1)}}^m(\mathcal{X}_{N_m}^m) \subseteq \mathcal{O}_\infty^\lambda. \tag{19}$$

In (19), a longer $\tau_m^{(1)}$ is required such that the system trajectory can enter $\mathcal{O}_\infty^\lambda$ in one stage. The proof of Corollary 4 is similar to the one for Theorem 3 and omitted here.

Remark 3. The determination of $\mathcal{AS}$-MDT by Theorem 3 and Corollary 4 relies on the computation of $\mathcal{O}_\infty^\lambda$. An algorithm to determine $\mathcal{O}_\infty^\lambda$ for given λ and $\boldsymbol{\Delta}$ is given in the Appendix (Algorithm A1) by combining the Algorithm 1 in Dehghan and Ong (2012a) and Algorithm 1 in Dehghan and Ong (2013), where it is noted that the constrained switched system is (16). In addition, the minimum MDT can be found to ensure the existence of $\mathcal{O}_\infty^\lambda$ for a given λ, and the minimum MDT set can be many (Dehghan & Ong, 2013). For the purpose that the system trajectory converges in the shortest time, the minimum MDT with both the smallest $\|\boldsymbol{\Delta}\|_1$ and the smallest variance of $\boldsymbol{\Delta}$.

The minimum $\mathcal{FA}$-MDT and $\mathcal{AS}$-MDT can be also determined by a minimization procedure similar to (12).

2.3. Discussions on conservatism and computability, and testification

So far, the admissible $\mathcal{F}$-MDT, $\mathcal{FA}$-MDT and $\mathcal{AS}$-MDT have been obtained by using the concept of reachable sets. Two direct questions can be raised as follows.

(i) The first question would be the conservatism of the criteria derived above, since it can be seen that $\tau_m^{(l)}$, $l \in \mathbb{Z}_{\geq 1}$, is computed despite where the state at switching instant $x_{k_{l-1}}$ is. For instance, if $x_{k_{l-1}}$ is close to $\bigcap_{n \in \mathcal{I}} \mathcal{X}_{N_n}^n$, it will be probably conservative for the currently active subsystem to dwell for $\tau_m^{(l)}$ to ensure persistent feasibility. One can even argue that in the framework of MPC design (if state-feedback), the state is measurable, then it can be online checked if $x_k \in \bigcap_{n \in \mathcal{I}} \mathcal{X}_{N_n}^n$. If yes, the switching can occur, then it seems that the determination of admissible $\mathcal{F}$-MDT, $\mathcal{FA}$-MDT and $\mathcal{AS}$-MDT are not needed and $k - k_{l-1}$ in such cases should be the shortest required stage MDT.

To answer this question, attention shall be first turned to the determination of admissible state-dependent MDT (denoted as $\tau_m^{(l)}(x_{k_{l-1}})$, cf. the state-dependent dwell time concept defined in De Persis et al., 2003), where the "state" is meant to the state at a switching instant. To this end, a concept of extended controllable set (ECS) for a system controlled by MPC is needed, for which the definition of controllable sets is first given as follows.

Similar to (6) and (7), for subsystem $\Omega_m : x_{k+1} = A_m x_k + B_m u_k$, where u_k is the control input designed by the MPC law, the one-step controllable set from $\mathcal{X} \subseteq \mathcal{X}_{N_m}^m$ along subsystem Ω_m is denoted as

$$\mathcal{P}re(\mathcal{X}, \Omega_m) \triangleq \left\{ x \in \mathcal{X}_{N_m}^m \,\middle|\, A_m x + B_m u^{MPC}(\Omega_m, x) \subseteq \mathcal{X} \right\} \tag{20}$$

and the H-step controllable set, $\mathcal{P}_H^m(\mathcal{X})$ is defined as

$$\begin{aligned} &\mathcal{P}_{y+1}^m(\mathcal{X}) \triangleq \mathcal{P}re\left(\mathcal{P}_y^m(\mathcal{X}), \Omega_m\right), \quad y \in \mathbb{Z}_{[0,H-1]}, \\ &\mathcal{P}_0^m(\mathcal{X}) = \mathcal{X}. \end{aligned} \tag{21}$$

Table 1
Π in (23) for determining state-dependent MDT.

	Π ($\mathcal{F}$-MDT)
Theorem 1	$\mathcal{X}_{N_m}^m \cap \mathcal{C}_{g+}^m(\hat{\mathcal{R}}^{\cup}(\tau_m^{(l)})), l = 1$ $\hat{\mathcal{R}}^{\cup}(\tau_{\sigma(k_{l-2})}^{(l-1)}) \cap \mathcal{C}_{g+}^m(\hat{\mathcal{R}}^{\cup}(\tau_m^{(l)})),\ l \in Z_{\geq 2},\ \tau_m^{(l)}$ s.t. (9)
Corollary 1	$\mathcal{X}_{N_m}^m \cap \mathcal{C}_{g+}^m(\bigcap_{n\in\mathcal{I}} \mathcal{X}_{N_n}^n),\ l = 1$ $(\bigcap_{n\in\mathcal{I}} \mathcal{X}_{N_n}^n) \cap \mathcal{C}_{g+}^m(\bigcap_{n\in\mathcal{I}} \mathcal{X}_{N_n}^n),\ l \in \mathbb{Z}_{\geq 2}$
Corollary 2	$\mathcal{X}_{N_m}^m \cap \mathcal{C}_{g+}^m(\bigcap_{n\in\mathcal{I}} \mathcal{X}_{N_n}^n), l \in \mathbb{Z}_{\geq 1}$
	Π ($\mathcal{F}\mathcal{A}$-MDT)
Thm.2	$\mathcal{X}_{N_m}^m \cap \mathcal{C}_{g+}^m(\hat{\mathcal{R}}^{\cup}(\tau_m^{(l)})),\ l = 1$ $\hat{\mathcal{R}}^{\cup}(\tau_{\sigma(k_{l-2})}^{(l-1)}) \cap \mathcal{C}_{g+}^m(\hat{\mathcal{R}}^{\cup}(\tau_m^{(l)})),\ l \in Z_{\geq 2},\ \tau_m^{(l)}$ s.t. (13)
Corollary 3	$\mathcal{X}_{N_m}^m \cap \mathcal{C}_{g+}^m(\bigcap_{n\in\mathcal{I}} \mathcal{R}_{l-1}^n(\mathcal{X}_{N_n}^n)), l = 1$ $\bigcap_{n\in\mathcal{I}} \mathcal{R}_{l-2}^n(\mathcal{X}_{N_n}^n) \cap \mathcal{C}_{g+}^m(\bigcap_{n\in\mathcal{I}} \mathcal{R}_{l-1}^n(\mathcal{X}_{N_n}^n)),\ l \in \mathbb{Z}_{\geq 2}$
	Π ($\mathcal{A}\mathcal{S}$-MDT)
Theorem 3	$\mathcal{X}_{N_m}^m \cap \mathcal{C}_{g+}^m(\bigcap_{n\in\mathcal{I}} \mathcal{R}_{l-1}^n(\mathcal{X}_{N_n}^n)), l = 1$ $\bigcap_{n\in\mathcal{I}} \mathcal{R}_{l-2}^n(\mathcal{X}_{N_n}^n) \cap \mathcal{C}_{g+}^m(\bigcap_{n\in\mathcal{I}} \mathcal{R}_{l-1}^n(\mathcal{X}_{N_n}^n)),\ l \in \mathbb{Z}_{[2,v]}$ $\lambda^{l-v-1}\mathcal{O}_\infty^\lambda \cap \mathcal{C}_{g+}^m(\lambda^{l-v}\mathcal{O}_\infty^\lambda),\ l \in \mathbb{Z}_{\geq v+1}$
Corollary 4	$\mathcal{X}_{N_m}^m \cap \mathcal{C}_{g+}^m(\mathcal{O}_\infty^\lambda), l = 1$ $\lambda^{l-2}\mathcal{O}_\infty^\lambda \cap \mathcal{C}_{g+}^m(\lambda^{l-1}\mathcal{O}_\infty^\lambda),\ l \in \mathbb{Z}_{\geq 2}$

Then by (20) and (21), for a given H and a set $\mathcal{X} \subseteq \mathcal{X}_{N_m}^m$, let

$$\mathcal{C}_{H+}^m(\mathcal{X}) \triangleq \bigcap_{t\in\mathbb{Z}_{[H,H_m^{\sup}]}} \mathcal{P}_t^m(\mathcal{X}) \tag{22}$$

denote the extended controllable set (ECS) from which the system state can be steered to $\mathcal{X}$ in H steps and will not leave $\mathcal{X}$ any more, where $H_m^{\sup} \triangleq \max\left\{H, \min\left\{t \in \mathbb{Z}_+ \left| \mathcal{R}_t^m(\mathcal{X}_{N_m}^m) \subseteq \mathcal{X}\right.\right\}\right\}$.

Based on the definition of $\mathcal{C}_{H+}^m(\mathcal{X})$, the state-dependent $\mathcal{F}$-MDT, $\mathcal{F}\mathcal{A}$-MDT and $\mathcal{A}\mathcal{S}$-MDT can be determined as

$$\underline{\tau}_m^{(l)}(x_{k_{l-1}}) \triangleq \min\left\{g \in \mathbb{Z}_+ \left| x_{k_{l-1}} \in \Pi\right.\right\} \tag{23}$$

where Π is denoted in Table 1 for different criteria obtained above.

The state-dependent MDT indicates how long it will take for the state $x_{k_{l-1}}$ at switching instants to ensure the feasibility, attractivity and asymptotic stability. It can be seen from (23) that state-dependent $\mathcal{F}$-MDT requires online judgement for the states at switching instants and offline computation of ECSs of either $\hat{\mathcal{R}}^{\cup}(\tau_m^{(l)})$ or $\bigcap_{n\in\mathcal{I}} \mathcal{X}_{N_n}^n$ as a targeting region. For state-dependent $\mathcal{F}\mathcal{A}$-MDT or $\mathcal{A}\mathcal{S}$-MDT, the targeting region should be a set contracting at each stage of switching. An additional online record on switching stages is therefore also required. These requirements will cause additional operation burden and may not be desirable for some practical applications with shorter sampling interval. Besides, even though the conservatism of non-state-dependent MDT are reduced, i.e., the required dwell time for each stage of switching can be shorter, the resulting system behavior may be accordingly deteriorated, such as oscillations of the state response.

The shortcomings of online judgements in the determination of state-dependent MDT will also exist in the case that the switching occurs immediately after judging that x_{k_l} belongs to the intersection of $\mathcal{X}_{N_n}^n$ (or a tighter one for convergence) as described in the question.

(ii) The second question could be the testability of the explored criteria, since all the conditions are based on the computability of either the reachable sets of feasible region or the ECSs of inner sets of feasible region.

The admissible MDT determined above, either stage or constant, either non-state-dependent or state-dependent, are theoretically applicable to switched systems that can be with high state dimensions, non-polytopic constraints on control input and system state. The testification of such conditions can be prohibitively difficult in general. However, for those systems for which the *explicit* MPC can be employed (explicit affine control laws are offline determined within their respective *critical regions* partitioning the feasible region, cf. Borrelli, Baotic, Pekar, & Stewart, 2010), the testification can be relatively tractable. The computation of reachable sets or ECSs in these conditions will involve manipulations on these critical regions including the intersection and union among them, and certain sets addition and multiplication.

The theoretical results in Section 2.2, together with the above discussions, are demonstrated in the following illustrative example of a population ecological system.

Example 1. Consider an ecological system consisting of two types of population P_1 and P_2, and suppose the situated environment be subject to autonomous switching between two scenarios $\mathbb{E}_1$ and $\mathbb{E}_2$. Let the number of individuals in population P_i be denoted by $N_i,\ i = 1, 2$. The basic mathematical principle of growth of N_i is the logistic equation described by (cf. Vandermeer & Goldberg, 2003)

$$\dot{N}_i = a_i^{(\sigma)} N_i(1 - N_i/K_i^{(\sigma)}) \tag{24}$$

where $a_i^{(\sigma)}$ is the maximum per-capita rate of change of P_i, and $K_i^{(\sigma)}$ is the carrying capacity of the population in environment $\mathbb{E}_\sigma$, $\sigma = 1, 2$. Namely, the rate of change in the population P_i (i.e., $\dot{N}_i$) is equal to growth $a_i^{(\sigma)}N_i$ that is limited by carrying capacity $(1 - N_i/K_i^{(\sigma)})$. Assume $K_i^{(\sigma)}$ be approximately proportional to N_i within $[\underline{M}_i^{(\sigma)},\ \overline{M}_i^{(\sigma)}]$, the range of N_i in $\mathbb{E}_\sigma$, i.e., $K_i^{(\sigma)} = (1/\rho_i^{(\sigma)})N_i$. Then include the mutual influence between P_1 and P_2, and the effect of immigration and emigration (denoted by $I_{g,i}$ and $E_{g,i}$, respectively) on N_i (Turchin, 2001), the Eq. (24) is extended here as, $\forall i \neq j$

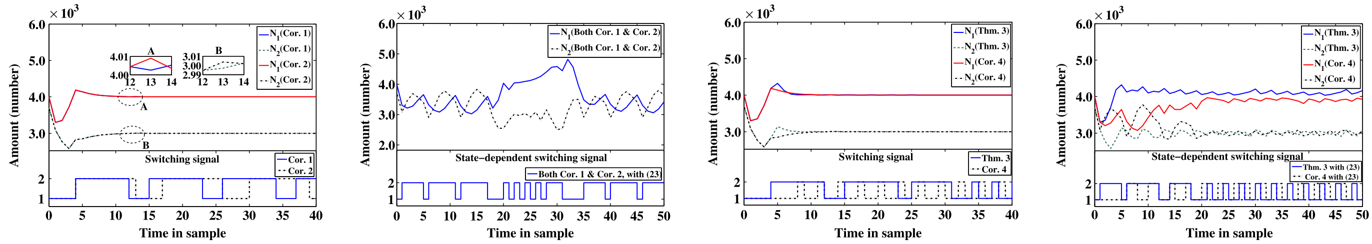
$$\dot{N}_i = a_i^{(\sigma)} N_i(1 - \rho_i^{(\sigma)}) + b_{ij}^{(\sigma)} N_j + c_i^{(\sigma)}(I_{g,i} - E_{g,i}) \tag{25}$$

where $b_{ij}^{(\sigma)}$ is a transfer coefficient modeling the mutual influence of N_1 and N_2, and $c_i^{(\sigma)}$ the effect of immigration and emigration on N_i in different $\mathbb{E}_\sigma$. Set the quaternary $(a_i^{(\sigma)}, \rho_i^{(\sigma)}, b_{ij}^{(\sigma)}, c_i^{(\sigma)})$ as $(0.2, 2, -1, 0),\ i = 1, \sigma = 1,\ (0.3, 2, 1, 1),\ i = 2, \sigma = 1,\ (0.4, 0.5, 0.5, 0),\ i = 1, \sigma = 2$, and $(0.4, 0.5, 0, 0.5),\ i = 2, \sigma = 2$, and it is supposed that $[\underline{M}_1^{(1)},\ \overline{M}_1^{(1)}] = [2000, 6000]$, $[\underline{M}_2^{(1)},\ \overline{M}_2^{(1)}] = [1000, 5000]$, $[\underline{M}_1^{(2)},\ \overline{M}_1^{(2)}] = [2400, 5600]$ and $[\underline{M}_2^{(2)},\ \overline{M}_2^{(2)}] = [1400, 4600]$. Let $x = [N_1 \quad N_2]^{\mathrm{T}}$ denote the system states, and regard the difference between immigration and emigration of P_2 as control input to system, i.e., $u = I_{g,2} - E_{g,2}$, the discrete-time state-space expression for the model can be obtained as (1) with $B_1 = [0 \quad 1]^T$, $B_2 = [0 \quad 0.5]^T$ and

$$A_1 = \begin{bmatrix} 0.8 & -1 \\ 1 & 0.7 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.2 & 0.5 \\ 0 & 1.2 \end{bmatrix}$$

by the first-order Euler approximation with the sampling period $T_s = 1$ (time unit). Suppose the control constraint be $\mathbb{U}_m = \left\{u \in \mathbb{R}^1 \left| -400 \leq u \leq 400\right.\right\},\ m = 1, 2$.

The purpose of the example is to design a switched MPC to regulate N_1 and N_2 to the equilibrium (4000, 3000) against the autonomous variation of $\mathbb{E}_\sigma$. The demanded minimum admissible MDT will be determined as well. Note that if the practical MDTs are less than the ones solved, the feasible region will shrink and the corresponding derivations and testification will be given in next subsection.



(a) $\mathcal{F}$-MDT case. (b) SD $\mathcal{F}$-MDT case. (c) $\mathcal{AS}$-MDT case. (d) SD $\mathcal{AS}$-MDT case.

Fig. 2. Amount of two individuals versus time in the population ecological system with different MDT switching, where the MDT (computed by different criteria) needed for the switching signals in the subfigures are shown in Table 2.

Table 2
Minimum admissible MDT determined by different criteria.

Criteria	$\mathcal{F}$-MDT
Corollary 1	$\underline{\tau}_1^{(1)} = 4, \underline{\tau}_2^{(1)} = 9; \underline{\tau}_1^{(l)} = 3, \underline{\tau}_2^{(l)} = 8,\ \forall l \geq 2$
Corollary 2	$\underline{\tau}_1 = 4, \underline{\tau}_2 = 9$
Criteria	$\mathcal{AS}$-MDT
Theorem 3	$\underline{\tau}_1^{(1)} = 4, \underline{\tau}_2^{(1)} = 9; \underline{\tau}_1^{(2)} = 3, \underline{\tau}_2^{(2)} = 8$ $\underline{\tau}_1^{(3)} = 3, \underline{\tau}_2^{(3)} = 8; \underline{\tau}_1^{(4)} = 3, \underline{\tau}_2^{(4)} = 8$ $\underline{\tau}_1^{(5)} = 2, \underline{\tau}_2^{(5)} = 7; \underline{\tau}_1^{(6)} = 3, \underline{\tau}_2^{(6)} = 6$ $\underline{\tau}_1^{(7)} = 3, \underline{\tau}_2^{(7)} = 1; \underline{\tau}_1^{(l)} = 2, \underline{\tau}_2^{(l)} = 1,\ \forall l \geq 8$
Corollary 4	$\underline{\tau}_1^{(1)} = 8, \underline{\tau}_2^{(1)} = 11; \underline{\tau}_1^{(l)} = 2, \underline{\tau}_2^{(l)} = 1,\ l \geq 2$

For both subsystems, consider the quadratic stage and terminal cost functions $L_m(x, u) \triangleq x^{\mathrm{T}}Q_m x + u^{\mathrm{T}}R_m u$, $T_m(x) \triangleq x^{\mathrm{T}}P_m x$ with weights Q_m, R_m and prediction horizon $Q_1 = 10I$, $Q_2 = 5I$, $R_m = 1$, $N_m = 5$, $m = 1, 2$. The terminal weight matrices P_m and the feedback control laws $K_m(x) \triangleq K_m^{LQR}x$, $\forall x \in \mathcal{T}_m$ are obtained for each subsystem as described in Rawlings and Mayne (2009, Chapter 2.5) such that Assumption 2 holds. The terminal set $\mathcal{T}_m$ is considered to be the maximal constraint admissible set for each subsystem in closed-loop with $u_k = K_m^{LQR}x_k$.

Since the admissible MDT satisfying either Theorem 1, Theorem 2, or Corollary 3 needs to be computed for an infinite number of stages of switching, the verification here only considers the other four criteria that are more practical. The computation of the feasible region $\mathcal{X}_{N_m}^m$ of each subsystem Ω_m, reachable or controllable sets, and ECSs are based on MPT (Kvasnica, Grieder, & Baotić, 2006).

Table 2 lists the computation data of $\mathcal{F}$-MDT by Corollaries 1 and 2 and $\mathcal{AS}$-MDT by Theorem 3 and Corollary 4. respectively, where the minimum MDT such that $\mathcal{O}_\infty^\lambda$ exists for system (16) in Theorem 3 and Corollary 4 can be found as $\Delta_1 = 2$ and $\Delta_2 = 1$ based on Algorithm A1 for $\lambda = 0.9$. The solved $\mathcal{F}$-MDT or $\mathcal{AS}$-MDT for each subsystem (independent of the system state, but dependent on the stage of switching) only need finite times of computation. Further, given initial condition $x_0^{\mathrm{T}} = [4000 \quad 3700]$, the state-dependent $\mathcal{F}$-MDT and $\mathcal{AS}$-MDT can be obtained by (23). By considering the running time equivalent to dwell time at each stage of switching, the resulting switching signals and the corresponding state response in different cases are given in Fig. 2. In Fig. 2(a), the system behaves very well (comparable to Fig. 2(c)), which actually results from the fact that all the used $\mathcal{F}$-MDT are no less than the required $\mathcal{AS}$-MDT, see Table 2. It can be also observed from Table 2 and Fig. 2 that the state-dependent $\mathcal{F}$-MDT or $\mathcal{AS}$-MDT reduces the conservatism (make the admissible MDT shorter), but they lead to a greater oscillation of the state response. In addition, the oscillation in Fig. 2(b) is more serious than the one in Fig. 2(d), which shows that the state-dependent $\mathcal{F}$-MDT does not necessarily lead to a convergence of the system trajectory.

2.4. Determination of $\mathcal{AS}$-FR

In this subsection, for a given constant MDT set $\boldsymbol{\tau}^{giv} \triangleq \{\tau_1^{giv}, \tau_2^{giv}, \ldots, \tau_M^{giv}\}$, the feasible region in which the MPC design for system (1)–(4) is persistently feasible, i.e., the $\mathcal{F}$-FR is first determined, then the results are further extended to $\mathcal{AS}$-FR (in which the underlying system is asymptotically stable for $\boldsymbol{\tau}^{giv}$).

Since for the case $\tau_m^{giv} \geq \underline{\tau}_m^{(1)}$, $\forall m \in \mathcal{I}$, where $\underline{\tau}_m^{(1)}$ can be obtained from Section 2.2 (e.g., by (8) and (12)), the feasible region of the switched system can be directly obtained as $\bigcup_{m \in \mathcal{I}} \mathcal{X}_{N_m}^m$. Therefore, in what follows, attention is only placed on a given MDT that satisfies $\tau_m^{giv} < \underline{\tau}_m^{(1)}$.

Proposition 1. *Consider system (1)–(4) with feasible region of subsystem Ω_m being $\mathcal{X}_{N_m}^m$, $\forall m \in \mathcal{I}$, and MDT τ_m^{giv} being given, $\forall m \in \mathcal{I}$. Suppose that Assumptions 1–2 hold and there exists a set of $W_m \in \mathbb{Z}_+$, $m \in \mathcal{I}$ such that*

$$\mathcal{R}^m_{W_m+\tau_m^{giv}}(\mathcal{X}^m_{N_m}) \subseteq \bigcap_{n \in \mathcal{I}} \mathcal{R}^n_{W_n}(\mathcal{X}^n_{N_n}). \tag{26}$$

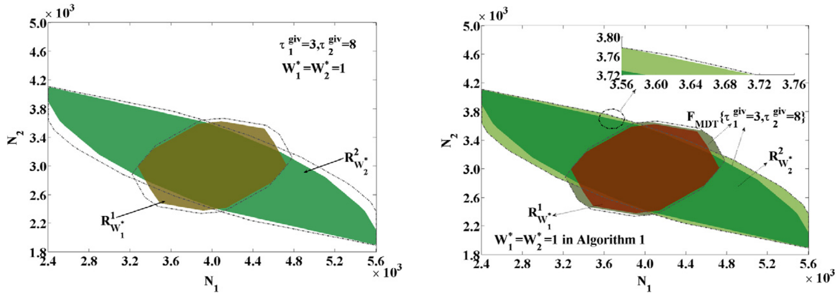
Then the MPC design for system (1)–(4) is persistently feasible within the feasible region $\mathcal{F}_{MDT}\{\boldsymbol{\tau}^{giv}\} = \bigcup_{m \in \mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\}$ with $\mathcal{F}_{MDT}^m\{\tau_m^{giv}\} = \mathcal{R}^m_{W_m}(\mathcal{X}^m_{N_m})$.

Proof. Suppose that the current subsystem is Ω_m ($\sigma(k_l) = m$, $l \in \mathbb{Z}_+$). By Assumption 2, the positive invariance of reachable sets of $\mathcal{X}_{N_m}^m$ implies that $\mathcal{R}^m_{W_m}(\mathcal{X}^m_{N_m}) \subseteq \mathcal{X}^m_{N_m}$ for a $W_m \in \mathbb{Z}_+$. From (26), $\forall n \in \mathcal{I}$, $n \neq m$, $\mathcal{R}^m_{W_m+k_{l+1}-k_l}(\mathcal{X}^m_{N_m}) \subseteq \mathcal{R}^m_{W_m+\tau_m^{giv}}(\mathcal{X}^m_{N_m}) \subseteq \mathcal{R}^n_{W_n}(\mathcal{X}^n_{N_n}) \subseteq \mathcal{X}^n_{N_n}$ holds, which means that the switching after τ_m^{giv} is admissible for all of the states within the region $\mathcal{R}^m_{W_m}(\mathcal{X}^m_{N_m})$. Therefore, in terms of persistent feasibility, the whole feasible region for the switched system (1)–(4) will be $\mathcal{F}_{MDT}\{\boldsymbol{\tau}^{giv}\} = \bigcup_{m \in \mathcal{I}} \mathcal{R}^m_{W_m}(\mathcal{X}^m_{N_m})$. $\square$

If the running time is exactly the given τ_m^{giv}, then the maximal feasible region $\mathcal{F}_{MDT}\{\boldsymbol{\tau}^{giv}\}$ based on Proposition 1 can be obtained by solving the minimization procedure

$$W_m^* \triangleq \min W_m, \quad \text{for } W_m \in \mathbb{Z}_+, \text{ subject to (26)}. \tag{27}$$

As the running time is generally greater than τ_m, the corresponding feasible region will contain $\mathcal{F}_{MDT}\{\boldsymbol{\tau}^{giv}\}$. Unlike (12), both the left-hand side and right-hand side in (26) contain the optimizer. An algorithm is then needed as below to find W_m^* by increasing W_m by one.



(a) By Algorithm 1. (b) By Algorithms 1 and 2.

Fig. 3. $\mathcal{F}$-FR for given MDT.

Table 3
Different pairs of (W_1^*, W_2^*) for given different τ_1 and τ_2.

	$\tau_2 \leq 4$	$\tau_2 = 5$	$\tau_2 = 6$	$\tau_2 = 7$	$\tau_2 = 8$
$\tau_1 = 1$	$(-, -)$	$(-, -)$	$(5, 4)$	$(4, 3)$	$(4, 2)$
$\tau_1 = 2$	$(-, -)$	$(5, 5)$	$(4, 4)$	$(3, 3)$	$(2, 1)$
$\tau_1 = 3$	$(-, -)$	$(4, 5)$	$(2, 3)$	$(2, 2)$	$(1, 1)$

Algorithm 1. Determination of $\mathcal{F}$-FR based on Proposition 1 (Input: τ_m^{giv}, $\underline{\tau}_m^{(1)}$, $\mathcal{X}_{N_m}^m$, $\forall m \in \mathcal{I}$)

(i) Initialization: Compute the minimum admissible MDT $\underline{\tau}_m^{(1)}$ (can be obtained by (8) and (12)), and set $W_m = \underline{\tau}_m^{(1)} - \tau_m^{giv}$ for each m.
(ii) Compute $\mathcal{R}_{W_m}^m(\mathcal{X}_{N_m}^m)$, $\mathcal{R}_{W_m+\tau_m^{giv}}^m(\mathcal{X}_{N_m}^m)$, and permutate the judgement conditions given by (26) (m conditions).
(iii) Check if (26) holds for each m. If yes, **exit** and **output** $\mathcal{F}_{MDT}\{\tau^{giv}\} = \bigcup_{m\in\mathcal{I}} \mathcal{R}_{W_m}^m(\mathcal{X}_{N_m}^m)$. Otherwise, among the remaining $2^m - 1$ cases (mutually exclusive), pick the matched case out and denote $\mathcal{I}_f \triangleq \{m \in \mathcal{I} | \mathcal{R}_{W_m+\tau_m^{giv}}^m(\mathcal{X}_{N_m}^m) \nsubseteq \bigcap_{n\in\mathcal{I}} \mathcal{R}_{W_n}^n(\mathcal{X}_{N_n}^n)\}$ in the case. Set $W_m = W_m + 1$ for each $m \in \mathcal{I}_f$, and goto Step (ii).

Example 2. Consider Example 1 for given constant MDT, and with different constraints on control input as $\mathbb{U}_1 = \{u \in \mathbb{R}^1 | -340 \leq u \leq 340\}$, $\mathbb{U}_2 = \{u \in \mathbb{R}^1 | -600 \leq u \leq 600\}$. Other parameters used for the switched MPC design are the same as in Example 1.

Based on Corollary 1, the minimum admissible $\mathcal{F}$-MDT can be solved as $\underline{\tau}_1^{(1)} = 4$, $\underline{\tau}_2^{(1)} = 9$, and $\underline{\tau}_1^{(l)} = 3$, $\underline{\tau}_2^{(l)} = 8$, $l \in \mathbb{Z}_{\geq 2}$. Then for given different MDT $\tau_1^{giv} \leq 3$, $\tau_2^{giv} \leq 8$, by Algorithm 1, the different pairs of W_1^*, W_2^* can be obtained by (27) as shown in Table 3, where "−" indicates that W_m^* does not have a feasible solution (i.e., the algorithm is not convergent in the cases). Fig. 3(a) shows the feasible region $\mathcal{F}_{MDT}\{\tau_1^{giv} = 3, \tau_2^{giv} = 8\}$.

In Fig. 3(a), since $\tau_1^{giv} = 3 = \underline{\tau}_1^{(2)}$ and $\tau_2^{giv} = 8 = \underline{\tau}_2^{(2)}$, the whole $\bigcap_{n=1,2}(\mathcal{X}_{N_n}^n)$ should be $\mathcal{F}$-FR. However, it can be seen that some of $\bigcap_{n=1,2}(\mathcal{X}_{N_n}^n)$ does not belong to $\mathcal{F}$-FR computed by Algorithm 1. The contradiction means that it is actually incomplete to determine $\mathcal{F}$-FR only using Algorithm 1 developed by Proposition 1.

By the definition of $\mathcal{C}_{H+}^m(\mathcal{X})$ in (22), the following theorem presents a criterion for development of an algorithm capable of determining a complete $\mathcal{F}$-FR.

Lemma 1. *Consider system (1)–(4) with feasible region of subsystem Ω_m being $\mathcal{X}_{N_m}^m$, $\forall m \in \mathcal{I}$, and the MDT τ_m^{giv} being given, $\forall m \in \mathcal{I}$. Suppose $\mathcal{F}_{MDT}^m\{\tau_m^{giv}\}$ is a $\mathcal{F}$-FR of subsystem Ω_m, so is $\mathcal{C}_{(\tau_m^{giv})+}^m(\bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\})$.*

Proof. To show a $\Psi_m \subseteq \mathcal{X}_{N_m}^m$ is a $\mathcal{F}$-FR for subsystem Ω_m for given MDT τ_m^{giv}, it suffices to show that $\mathcal{R}_t^m(\Psi_m) \subseteq \bigcap_{m\in\mathcal{I}} \Psi_m$, $\forall t \in \mathbb{Z}_{\geq \tau_m^{giv}}$ (within $\bigcap_{m\in\mathcal{I}} \Psi_m$, the stability is ensured as shown in part (ii) of the proof for Theorem 1). By definition of $\mathcal{C}_{H+}^m(\mathcal{X})$, it follows that, $\forall t \in \mathbb{Z}_{\geq \tau_m^{giv}}$

$$\mathcal{R}_t^m\left(\mathcal{C}_{(\tau_m^{giv})+}^m\left(\bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\}\right)\right) \subseteq \bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\}$$
$$\subseteq \bigcap_{m\in\mathcal{I}} \mathcal{C}_{(\tau_m^{giv})+}^m\left(\bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\}\right) \tag{28}$$

i.e., $\mathcal{C}_{(\tau_m^{giv})+}^m(\bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\})$ is $\mathcal{F}$-FR of system (1)–(4). The 2nd '$\subseteq$' of (28) holds due to $\mathcal{F}_{MDT}^m\{\tau_m^{giv}\} \subseteq \mathcal{C}_{(\tau_m^{giv})+}^m(\bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\})$, which holds because $\mathcal{F}_{MDT}^m\{\tau_m^{giv}\}$ is a $\mathcal{F}$-FR for given τ_m^{giv}, i.e., $\forall t \in \mathbb{Z}_{\geq \tau_m^{giv}}$, $\mathcal{R}_t^m(\mathcal{F}_{MDT}^m\{\tau_m^{giv}\}) \subseteq \bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\tau_m^{giv}\}$. □

Then, based on Lemma 1, by iterating the starting $\mathcal{F}$-FR in $\mathcal{C}_{(\tau_m^{giv})+}^m(\cdot)$, the algorithm to determine the complete $\mathcal{F}$-FR is as follows:

Algorithm 2. Determination of a complete $\mathcal{F}$-FR (Input: τ_m^{giv}, $\mathcal{X}_{N_m}^m$, an initial $\mathcal{F}$-FR Ψ_m, $\forall m \in \mathcal{I}$)

(i) Initialization: Set $k = 1$ and $\mathcal{F}_{MDT}^{m,(k)}\{\tau_m^{giv}\} = \Psi_m$.
(ii) Set $k = k + 1$. Update $\mathcal{F}_{MDT}^{m,(k)}\{\tau_m^{giv}\}$, $\forall m \in \mathcal{I}$ by $\mathcal{F}_{MDT}^{m,(k)}\{\tau_m^{giv}\} = \mathcal{C}_{(\tau_m^{giv})+}^m(\bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^{m,(k-1)}\{\tau_m^{giv}\})$.
(iii) If $\mathcal{F}_{MDT}^{m,(k)}\{\tau_m^{giv}\} \equiv \mathcal{F}_{MDT}^{m,(k-1)}\{\tau_m^{giv}\}$, **exit** and **output** $\mathcal{F}_{MDT}\{\tau_m^{giv}\} = \bigcup_{m\in\mathcal{I}} \mathcal{F}_{MDT}^{m,(k)}\{\tau^{giv}\}$; else goto Step (ii).

Example 3. Consider Example 2 with $\boldsymbol{\tau}^{giv} = \{3, 8\}$.

Set the initial $\mathcal{F}$-FR Ψ_m to be that yielded by Algorithm 1. By Algorithm 2, the complete $\mathcal{F}$-FR can be obtained as shown in Fig. 3(b).

Remark 4. Algorithm 2 is convergent as the set sequence $\{\bigcap_{m\in\mathcal{I}} \mathcal{F}_{MDT}^{m,(k)}\{\tau_m^{giv}\}, k \in \mathbb{Z}_+\}$ is non-contractive and bounded above from $\bigcap_{n\in\mathcal{I}} \mathcal{X}_{N_n}^n$, but it lies on the existence of the initial $\mathcal{F}$-FR Ψ_m that can be determined by Algorithm 1. However, as demonstrated in Example 2, for some small τ_m^{giv}, Algorithm 1 may not be convergent (e.g., $\tau_1^{giv} = \tau_2^{giv} = 3$), due to both iterations of increasing W_m and W_n by one simultaneously in (26), and accordingly cannot give an initial $\mathcal{F}$-FR Ψ_m to Algorithm 2. Note that such a case cannot yet conclude that the $\mathcal{F}$-FR of system (1)–(4) does not exist. Specifically, since in Example 2 a $[0.9, \{\tau_1 = 2, \tau_2 = 1\}]$-contractive set $\mathcal{O}_\infty^\lambda$ can be found for system (16), then at least the $\mathcal{O}_\infty^\lambda$ is a $\mathcal{F}$-FR (actually a $\mathcal{AS}$-FR). Therefore, for $\tau_m^{giv} \geq \Delta_m$, the initial $\mathcal{F}$-FR Ψ_m can be changed to $\mathcal{O}_\infty^\lambda$ as a start if using quadratic stage cost and setting $T_m(x) = V_\infty^{UC}(\Omega_m, x)$ for the system. The setup can give rise to the determination of $\mathcal{AS}$-FR as proposed below.

Proposition 2. *Consider system (1)–(4) with the feasible region of subsystem Ω_m being $\mathcal{X}^m_{N_m}$, $\forall m \in \mathcal{I}$ and MDT τ^{giv}_m being given. Suppose that Assumptions 1–2 hold, the stage cost is quadratic, $T_m(x) = V^{UC}_\infty(\Omega_m, x)$ and the associated controller gain within $\mathcal{T}_m$ is K_m. If under a MDT set $\mathbf{\Delta} \triangleq \{\Delta_1, \Delta_2, \ldots, \Delta_M\}$ with $\tau^{giv}_m \geq \Delta_m$, a $[\lambda, \mathbf{\Delta}]$-contractive set $\mathcal{O}^\lambda_\infty$ exists for system (16), then system (1)–(4) is asymptotically stable within a $\mathcal{AS}$-FR that is formed by $\mathcal{F}_{MDT}\{\boldsymbol{\tau}^{giv}\}$ outputted from Algorithm 2 in which $\Psi_m = \mathcal{O}^\lambda_\infty$.*

By referring to Theorem 3, the proof of Proposition 2 is rather straightforward since $\mathcal{F}_{MDT}\{\boldsymbol{\tau}^{giv}\}$ contracts to $\mathcal{O}^\lambda_\infty$ under MDT τ^{giv}_m, which is implied by Algorithm 2. In addition, by definition of ECSs in (22), it can be concluded that $\mathcal{F}_{MDT}\{\boldsymbol{\tau}^{giv}\}$ will be enlarged if increasing τ^{giv}_m. Setting $\Psi_m = \mathcal{O}^\lambda_\infty$ in Algorithm 2, The verifications can be readily obtained and omitted here.

3. The case of systems with bounded disturbance

This section aims to establish switched tube-based MPC methodology. More specifically, when the system is involved with disturbance, a switched tube (its cross section is of different size for different subsystem) will be determined, by which the two issues of determining admissible MDT and admissible feasible region for given MDT can be coped with. Consider system (1) with additive disturbance as

$$(\Phi_{\sigma(k)}) : \tilde{x}_{k+1} = A_{\sigma(k)}\tilde{x}_k + B_{\sigma(k)}\tilde{u}_k + w_k \tag{29}$$

where $w_k \in \mathbb{W} \subseteq \mathbb{R}^{n_x}$; $\mathbb{W}$ is a compact polyhedral set containing the origin in its interior. The system state and control constraints in (2) are rewritten as $\tilde{\mathbb{X}}_m$ and $\tilde{\mathbb{U}}_m$, respectively. The initial state $\tilde{x}_0 = x_0$ is considered. For system (29), the tube-based MPC strategy (Mayne et al., 2005) is adopted with

$$\tilde{u}_k = u_k + K^u_{\sigma(k)}(\tilde{x}_k - x_k) \tag{30}$$

where K^u_m is a controller gain such that $\bar{A}_m = A_m + B_m K^u_m$, $\forall \sigma(k) = m \in \mathcal{I}$ satisfies $\rho(\bar{A}_m) < 1$, and u_k is the control input to nominal system (1). The error system between (1) and (29) satisfies

$$(\Xi_{\sigma(k)}) : e_{k+1} = \bar{A}_{\sigma(k)} e_k + w_k \tag{31}$$

where $e_k \triangleq \tilde{x}_k - x_k$. Suppose a robust positive invariant (RPI) set[2] E_m exists for each subsystem Ξ_m, so that u_k can be obtained by solving the regular MPC optimization procedure (3) with the constraints tightly updated by

$$\mathbb{X}_m = \tilde{\mathbb{X}}_m \ominus E_m, \qquad \mathbb{U}_m = \tilde{\mathbb{U}}_m \ominus K^u_m E_m. \tag{32}$$

The feasible region of system (29) $\mathcal{X}^m_{N_m}$ will shrink compared to the nominal one without the disturbance.

In tube-based MPC design for non-switched systems (Mayne et al., 2005), the E_m is adopted as the minimum RPI set as $S_m(\infty) \triangleq \sum_{i=0}^{\infty} \bar{A}^i_m \mathbb{W}$. Due to the switching dynamics, however, $S_m(\infty)$ is not competent any more since generally $S_m(\infty)$ is not the robust invariant set for subsystem Ξ_n, $\forall n \in \mathcal{I}$, $n \neq m$. In this paper, a generalized RPI set for each Ξ_m is used, which requires the concepts of the MDT RPI set and the reachable set for switched system (31) as below.

Definition 6 (*Dehghan & Ong, 2012b*)**.** A set $\Theta(\boldsymbol{\eta}) \subset \mathbb{R}^n$ is said to be a MDT RPI set for system (31) with MDT set $\boldsymbol{\eta} \triangleq \{\eta_1, \eta_2, \ldots, \eta_M\}$, if $e_0 \in \Theta(\boldsymbol{\eta})$ implies $e_k \in \Theta(\boldsymbol{\eta})$ for every admissible switching $\mathbf{S}_{\boldsymbol{\eta}}(k)$ and for $w_t \in \mathbb{W}$, $t \in \mathbb{Z}_{[1,k-1]}$.

[2] A set $\mathcal{O} \subseteq \mathbb{X} \subseteq \mathbb{R}^{n_x}$ is said to be a robust positive invariant (RPI) set for autonomous system $x_{k+1} = f(x_k, w_k)$, $x_k \in \mathbb{X}$, $w_k \in \mathbb{W}$ if $x_k \in \mathcal{O}$ implies $x_t \in \mathcal{O}$ for any $w_t \in \mathbb{W}$, $t \in \mathbb{Z}_{\geq k+1}$.

Remark 5. An algorithm (Algorithm A2 in the Appendix) to determine the MDT RPI set $\Theta(\boldsymbol{\eta})$ for a switched linear system can be developed by extending Algorithm 1 in Dehghan and Ong (2012b) to the context of MDT switching (Dehghan & Ong, 2013). It can be concluded from both (Dehghan & Ong, 2012b, 2013) that the minimum MDT such that $\Theta(\boldsymbol{\eta})$ exists for system (31) are those such that system (31) is asymptotically stable. Such minimum MDT can be many, among which this paper considers the minimality in the sense that the MDT are with both the smallest $\|\boldsymbol{\eta}\|_1$ and the smallest variance of $\boldsymbol{\eta}$, similar to Remark 3.

Let one-step reachable set of $\mathcal{X}$ along subsystem Ξ_m, $\mathcal{LR}^m_1(\mathcal{X})$ be denoted as $\mathcal{LR}^m_1(\mathcal{X}) \triangleq \{\bar{A}_m x + w : x \in \mathcal{X},\ w \in \mathbb{W}\} = \bar{A}_m\mathcal{X} \oplus \mathbb{W}$, the H-step reachable set $\mathcal{LR}^m_H(\mathcal{X})$ is defined as $\mathcal{LR}^m_{y+1}(\mathcal{X}) \triangleq \mathcal{LR}^m_1\left(\mathcal{LR}^m_y(\mathcal{X})\right), y \in \mathbb{Z}_{[0,H-1]}$, where $\mathcal{LR}^m_0(\mathcal{X}) \triangleq \mathcal{X}$. Thus $\mathcal{LR}^m_H(\mathcal{X}) = \bar{A}^H_m\mathcal{X} \oplus \bar{A}^{H-1}_m\mathbb{W} \oplus \cdots \bar{A}_m\mathbb{W} \oplus \mathbb{W}$.

Since the MDT RPI set $\Theta(\boldsymbol{\eta})$ is characterized with $\mathcal{LR}^m_s(\Theta(\boldsymbol{\eta})) \subseteq \Theta(\boldsymbol{\eta})$, $\forall s \in \mathbb{Z}_{\geq \eta_m}$ (cf. Dehghan and Ong (2012b) for more details), the generalized RPI set E_m used in tube-based MPC for each Ξ_m is defined as

$$E_m \triangleq co\{\mathcal{LR}^m_{\eta_m-1}(\Theta(\boldsymbol{\eta})), \mathcal{LR}^m_{\eta_m-2}(\Theta(\boldsymbol{\eta})), \ldots, \Theta(\boldsymbol{\eta})\} \tag{33}$$

from which it holds that, for any $e_k \in \Theta(\boldsymbol{\eta}) \subseteq E_m$, $e_t \in E_m$, $t \in \mathbb{Z}_{\geq k+1}$.

Let the distance of vector x to set E, $x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$, be denoted by $\|x\|_E \triangleq \inf_{y \in E}\|x - y\|$. The following definition is needed by extending the origin in Definition 4 to a positive invariant set, cf., Rawlings and Mayne (2009).

Definition 7. Consider system (29)–(30), (32)–(33), (3)–(4) with each the feasible region of subsystem Φ_m being $\mathcal{X}^m_{N_m}$, a RPI set $E \subset \mathbb{R}^n$ is said to be asymptotically stable in $\bigcup_{m\in\mathcal{I}} \mathcal{X}^m_{N_m}$, if E is stable, i.e., $\forall k \in \mathbb{Z}_+$, $\left\|\tilde{x}_k\right\|_E \leq \alpha\left(\left\|\tilde{x}_0\right\|_E\right)$, where $\alpha \in \mathcal{K}$ and attractive, i.e., $\left\|\tilde{x}_k\right\|_E \to 0$ as $k \to \infty$, in $\bigcup_{m\in\mathcal{I}} \mathcal{X}^m_{N_m}$.

The objectives of this section include: (i) Determine the admissible MDT such that the uncertain closed-loop switched system (29)–(30), (32)–(33), (3)–(4) is asymptotically stable in the sense of Definition 7. Such MDT are denoted concisely as $\mathcal{RAS}$-MDT; (ii) Given MDT, find the feasible region such that (4) can be persistently solved and the underlying uncertain switched system is asymptotically stable, denoted as $\mathcal{RAS}$-FR.

Also, this subsection first determines the admissible MDT such that the MPC design for the uncertain switched system is persistently feasible (denoted as $\mathcal{RF}$-MDT), and the underlying uncertain system is attractive besides the persistent feasibility ($\mathcal{RFA}$-MDT). Let $\boldsymbol{\eta} \triangleq \{\eta_1, \eta_2, \ldots, \eta_M\}$ and $E^\cup \triangleq \bigcup_{m\in\mathcal{I}} E_m$.

Proposition 3. *Consider systems (1) and (29). Suppose that Assumptions 1–2 hold, a MDT RPI set $\Theta(\boldsymbol{\eta})$ exists for error system (31). Then, $\forall m \in \mathcal{I}$, if $\tau_m \in \mathbb{Z}_+$ satisfies (11) in which $\mathcal{X}^m_{N_m}$ is determined by (3), (32) and (33), then the MPC design for system (29)–(30), (32)–(33), (3)–(4) is persistently feasible with admissible MDT $\theta_m \triangleq \max(\tau_m, \eta_m)$.*

Proof. By Assumption 2, within each subsystem Φ_m, the persistent feasibility holds due to the fact $\mathcal{X}^m_{N_m} \oplus E_m \subseteq \tilde{\mathbb{X}}_m$. According to Corollary 2, $\theta_m \geq \tau_m$ ensuring (11) is admissible for the switching of the nominal system. As η_m guarantees the existence of $\Theta(\boldsymbol{\eta})$ that belongs to $\bigcap_{n\in\mathcal{I}} E_n$ owing to (33), it is also ensured that for any $e_{k_{l-1}} \in \Theta(\boldsymbol{\eta}) \subseteq E_m$, $e_{k_l} \in \Theta(\boldsymbol{\eta}) \subseteq E_n$ holds, $l \in \mathbb{Z}_{\geq 1}$, $\forall m \times n \in \mathcal{I} \times \mathcal{I}$, $m \neq n$, as long as $k_l - k_{l-1} \geq \theta_m \geq \eta_m$. Therefore, since $e_{k_0} = \tilde{x}_0 - x_0 = 0 \in \Theta(\boldsymbol{\eta})$, the persistent feasibility remains at all k_l, $l \in \mathbb{Z}_+$. □

Note that the obtained θ_m is not necessarily minimal although it is admissible. An algorithm to determine the minimum $\mathcal{RF}$-MDT can be developed as follows.

Algorithm 3. Determination of $\mathcal{RF}$-MDT based on Proposition 3 (Input: $\eta_m, \bar{A}_m, \forall m \in \mathcal{I}, \mathbb{W}$)

(i) Define $\eta_{(k)} := \{\eta^{[+(k-1)]}(\eta)\}$ with $\eta_{(k),r} := \{\eta_{1,(k),r}, \eta_{2,(k),r}, \ldots, \eta_{M,(k),r}\}$ denoting the r^{th} element of $\eta_{(k)}$, and define $U_k := \mathbb{Z}_{[1,\text{card}\{\eta_{(k)}\}]}$. Set $k = 1$, $r = 1$, and $\eta_{(k),r} = \eta$. Compute $\Theta(\eta_{(k),r})$ based on Algorithm A2 and $E^{(k)}_{m,r}$ by (33), $\forall m \in \mathcal{I}$.
(ii) Update the constraints $\mathbb{X}_m = \tilde{\mathbb{X}}_m \ominus E^{(k)}_{m,r}$ and the feasible region $\mathcal{X}^m_{N_m}$ by solving (3), and compute the τ_m based on (11) and (12); set $\theta_{m,(k),r} := \max(\tau_m, \eta_{m,(k),r}), \forall m \in \mathcal{I}$.
(iii) If $\eta_{m,(k),r} = \min_{i\in\mathbb{Z}_{[1,k]}, v\in U_i} \theta_{m,(i),v}$, then **exit** and **output** $\theta_m = \min_{i\in\mathbb{Z}_{[1,k]}, v\in U_i} \theta_{m,(i),v}$ and $\eta_m = \eta_{m,(i),v} | (i, v) = \arg(\min_{i\in\mathbb{Z}_{[1,k]}, v\in U_i} \theta_{m,(i),v})$, $\forall m \in \mathcal{I}$; else check if $r = 1$, set $k = k + 1$, $r = \text{card}\{\eta_{(k)}\}$; else set $r = r - 1$. Update $\Theta(\eta_{(k),r})$ and $E^{(k)}_{m,r}$, $\forall m \in \mathcal{I}$ and goto step (ii).

Though Proposition 3 and Algorithm 3 are proposed not allowing for the MDT to be a stage one, the obtained MDT can be considered as $\theta^{(1)}_m$ which is same in all the criteria given in Section 2.2 (except Corollary 4). Meanwhile, the feasible region $\mathcal{X}^m_{N_m}$ is therefore fixed, upon which the stage $\mathcal{RF}$-MDT, $\mathcal{RFA}$-MDT and $\mathcal{RAS}$-MDT $\theta^{(l)}_m$, $l \in \mathbb{Z}_{\geq 2}$ can be derived while using different criteria in Section 2.2. The following proposition presents a case for $\mathcal{RFA}$-MDT.

Proposition 4. *Consider systems (1) and (29). Suppose that Assumptions 1–2 hold, the MDT RPI set $\Theta(\boldsymbol{\eta})$ for error system (31) be determined by the η_m outputted from Algorithm 3 . If $\theta^{(1)}_m$ is determined by Algorithm 3 and $\tau^{(l)}_m \in \mathbb{Z}_+$, $l \in \mathbb{Z}_{\geq 2}$ satisfies (13), in which $\mathcal{X}^m_{N_m}$ is determined by (3) and (32)–(33), then set $E^{\cup}$ is attractive in $\bigcup_{m\in\mathcal{I}} \mathcal{X}^m_{N_m}$, for system (29)–(30), (32)–(33), (3)–(4) with admissible MDT $\theta^{(l)}_m \triangleq \max(\tau^{(l)}_m, \eta_m)$, $l \in \mathbb{Z}_+$.*

The proof of Proposition 4 can be done by combining the proofs for Proposition 3 and Theorem 2 (note that $\|\tilde{x}_k\|_{E^{\cup}} \triangleq d(\tilde{x}_k, E^{\cup}) = d(x_k + e_k, E^{\cup}) \leq d(x_k + e_k, e_k) = \|x_k\|$, thus $\|\tilde{x}_k\|_{E^{\cup}} \to 0$ if $\|x_k\| \to 0$). Replacing (13) by (9), (10) and (14) can give different corollaries for determination of $\mathcal{RF}$-MDT or $\mathcal{RFA}$-MDT similar to the counterpart in Section 2.2. Likewise, the $\mathcal{RAS}$-MDT can be determined as follows.

Theorem 4. *Consider systems (1) and (29). Suppose that Assumptions 1–2 hold, the MDT RPI set $\Theta(\boldsymbol{\eta})$ for error system (31) be determined by the η_m outputted from Algorithm 3 , the stage cost is quadratic, $T_m(x) = V^{UC}_\infty(\Omega_m, x)$, the associated controller gain within $\mathcal{T}_m$ be K_m, and a $[\lambda, \boldsymbol{\Delta}]$-contractive set $\mathcal{O}^\lambda_\infty$ exists for system (16), where $\boldsymbol{\Delta} \triangleq \{\Delta_1, \Delta_2, \ldots, \Delta_M\}$. If $\theta^{(1)}_m$ is determined by Algorithm 3 and $\tau^{(l)}_m \in \mathbb{Z}_+$, $l \in \mathbb{Z}_{[2,v]}$ satisfies (17) in which $\mathcal{X}^m_{N_m}$ is determined by (3) and (32)–(33), then set $\bar{E} \triangleq E^{\cup} \times \{0\}$ is asymptotically stable in $\bigcup_{m\in\mathcal{I}} \mathcal{X}^m_{N_m}$, for the composite system (1), (29)–(30), (32)–(33), (3)–(4) with admissible MDT $\theta^{(l)}_m = \max(\tau^{(l)}_m, \eta_m)$, $l \in \mathbb{Z}_{[1,v]}$ and $\theta^{(l)}_m = \max(\eta_m, \Delta_m)$, $l \in \mathbb{Z}_{\geq v+1}$, where v satisfies (17).*

Proof. Both persistent feasibility and attractivity can be proved by combining the proofs for Proposition 3, Theorems 2 and 3. The same line of the proof for Proposition 3.15 in Rawlings and Mayne (2009) is used here to demonstrate the stability of the composite switched system. With the requirement on $\theta^{(l)}_m$, by Theorem 3, the nominal switched system is stable, i.e., $\|x_k\| \leq \alpha(\|x_{k_0}\|)$, where $\alpha \in \mathcal{K}$. Since $\tilde{x}_k = x_k + e_k$, where $e_k \in E^{\cup} = \bigcup_{m\in\mathcal{I}} E_m$, it holds that $\|\tilde{x}_k\|_{E^{\cup}} = d(x_k + e_k, E^{\cup}) \leq d(x_k + e_k, e_k) = \|x_k\| \leq \alpha(\|x_{k_0}\|)$. Denote $\|(\tilde{x}_k, x_k)\| \triangleq \|\tilde{x}_k\| + \|x_k\|$, it follows that the extended state $(\tilde{x}_k, x_k)$ of the composite system satisfies $\|(\tilde{x}_k, x_k)\|_{\bar{E}} = \inf_{\tilde{y}\in E^{\cup}} \|(\tilde{x}_k, x_k) - (\tilde{y}, 0)\| = \inf_{\tilde{y}\in E^{\cup}} \|(\tilde{x}_k - \tilde{y}, x_k)\| = \inf_{\tilde{y}\in E^{\cup}} \|\tilde{x}_k - \tilde{y}\| + \|x_k\| = \|\tilde{x}_k\|_{E^{\cup}} + \|x_k\| \leq 2\alpha(\|x_{k_0}\|) \leq 2\alpha(\|\tilde{x}_{k_0}\|_{E^{\cup}} + \|x_{k_0}\|) = 2\alpha(\|(\tilde{x}_{k_0}, x_{k_0})\|_{\bar{E}})$, which means that set $\bar{E}$ is stable for the composite system in the sense of Definition 7. □

A similar extension from Theorem 3 to Corollary 4 is also applicable to Theorem 4 to realize entering $\mathcal{O}^\lambda_\infty$ within one stage, i.e., $\mathcal{R}^n_{\tau^{(1)}_m}(\mathcal{X}^m_{N_m}) \subseteq \mathcal{O}^\lambda_\infty$; the corresponding corollary is omitted.

Moreover, the state-dependent $\mathcal{RF}$-MDT, $\mathcal{RFA}$-MDT and $\mathcal{RAS}$-MDT can be further determined as $\theta^{(1)}_m(x_{k_0}) = \underline{\tau}^{(1)}_m(x_{k_0})$ and $\theta^{(l)}_m(x_{k_{l-1}}) = \max(\underline{\tau}^{(l)}_m(x_{k_{l-1}}), \eta_m)$, $l \in \mathbb{Z}_{\geq 2}$, where $\underline{\tau}^{(l)}_m(x_{k_{l-1}})$ is determined in (23). Note that $\theta^{(1)}_m(x_{k_0})$ can be less than η_m (outputted from Algorithm 3), since $\forall t \in \mathbb{Z}_{[1,\eta_m]}$, $e_t \in S_m(\eta_m) \triangleq \sum_{t=1}^{\eta_m} \bar{A}^{t-1}_m \mathbb{W} \subseteq \Theta(\boldsymbol{\eta}) \subseteq E_m$ due to $e_0 = 0$.

Remark 6. Compared with the existing tube-based MPC methodology for non-switched systems, it can be seen that the generalized RPI set E_m can be regarded as a mode-dependent cross section of a "switched" tube within Φ_m, and the cross section of this switched tube at switching instants reduces to $\Theta(\boldsymbol{\eta})$.

Remark 7. In the above results, the feasible region is fixed at the first stage of switching while minimizing $\theta^{(1)}_m$ by Algorithm 3 such that $\theta^{(l)}_m$, $l \in \mathbb{Z}_{\geq 2}$ can be further determined. Note that further increasing η_m corresponding to the minimum of $\theta^{(1)}_m$ with $\boldsymbol{\eta}^{[+z]}$ likely decrease $\theta^{(2)}_m$ and so on, based on the idea of Algorithm 3. However, such a way may not ensure the persistent feasibility, i.e., for a $e_{k_{l-1}} \in \Theta(\boldsymbol{\eta}) \subseteq E_m$ (derived from $\boldsymbol{\eta}$), it is possible that $e_{k_{l-1}} \notin \Theta(\boldsymbol{\eta}^{[+z]})$ which may give rise to $e_k \notin E_n$ (derived from $\boldsymbol{\eta}^{[+z]}$), $k \in [k_{l-1}, k_l]$, $l \in \mathbb{Z}_{\geq 1}$, $\forall m \times n \in \mathcal{I} \times \mathcal{I}$, $m \neq n$, even though $k_l - k_{l-1} \geq \theta^{(l)}_m \geq \eta^{[+z]}_m$, cause that $e_{k_l} + x_{k_l} \in \tilde{\mathbb{X}}_n$ fails at k_l.

Example 4. Consider the system in Example 2 with bounded additive disturbance. The original model (25) becomes $\dot{N}_i = a^{(\sigma)}_i N_i(1-\rho^{(\sigma)}_i) + b^{(\sigma)}_{ij} N_j + c^{(\sigma)}_i (I_{g,i} - E_{g,i}) + w$. Suppose $\|w\|_\infty = 1$.

The purpose of this example is to illustrate the switched tube-based MPC methodology proposed above via Theorem 4. Other criteria to determine $\mathcal{RAS}$-MDT and their state-dependent versions can be shown in a similar vein.

Let $K^u_m = K^{LQR}_m$, based on Algorithm A2, the MDT set such that $\Theta(\boldsymbol{\eta})$ exists can be solved as $\boldsymbol{\eta} = \{2, 1\}$. Then by Algorithm 3, the minimum of $\theta^{(1)}_m$ is obtained as $\theta^{(1)}_1 = 4$, $\theta^{(1)}_2 = 8$; meanwhile, three different MDT sets of $\boldsymbol{\eta}$ corresponding to the minimum of $\theta^{(1)}_m$ are outputted, $\{2, 1\}$, $\{2, 2\}$, $\{2, 3\}$, respectively. Consider the minimality of admissible MDT, the set $\boldsymbol{\eta} = \{2, 1\}$ is continuously used to fix the feasible region $\mathcal{X}^m_{N_m}$ (updated by (3), (32) and (33)), such that the $\mathcal{RAS}$-MDT $\theta^{(l)}_m$, $l \geq 2$ can be computed by Theorem 4. Note that $\mathcal{O}^\lambda_\infty$ in Theorem 4 is also updated since $\mathcal{T}_m$ (the maximal constraint admissible set for each subsystem in closed-loop with $u_k = K^{LQR}_m x_k$) is updated, and the minimum MDT set $\boldsymbol{\Delta}$ such that $\mathcal{O}^\lambda_\infty$ exists can be found as $\boldsymbol{\Delta} = \{2, 1\}$ by Algorithm A1.

Consider initial condition $x^{\mathrm{T}}_0 = [4400\ 3500]$, and the running time being equivalent to dwell time at each stage of switching, Fig. 4 shows the state trajectory of the nominal system and tube evolution from $E_1 = co\{\mathcal{LR}^1_1(\Theta(\boldsymbol{\eta})), \Theta(\boldsymbol{\eta})\}$ at the first stage of switching to $E_2 = co\{\Theta(\boldsymbol{\eta})\} = \Theta(\boldsymbol{\eta})$ at the second stage. The later evolutions are omitted for clarity of illustration. Generating disturbance with $\|w\|_\infty = 1$ randomly until $k = 10^4$, Case I in Fig. 5 shows the position of the practical states at switching

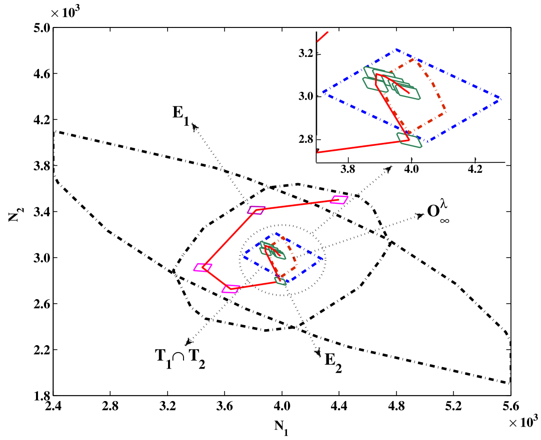


Fig. 4. State trajectory of the nominal system and tube evolution from E_1 to E_2 under $\mathcal{RAS}$-MDT.

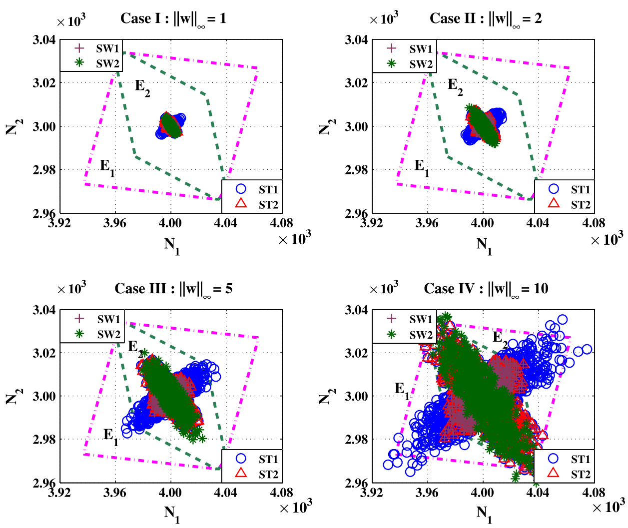


Fig. 5. The projection of state trajectories into a same coordinate where the origin is the center of E_1 and E_2, for given random disturbance with different bounds. "SW1" and "ST1" (or "SW2" and "ST2") stand for the state of subsystem 1 (or subsystem 2) at switching instants and during the stage of switching, respectively.

instants and within the stage of switching, being projected into a same coordinate where the origin is the center of the two tubes. Further, consider random disturbance with disturbance bounds $\|w\|_\infty = 2, 5$ and 10, respectively, the corresponding cases are also shown in Fig. 5. Clearly, for three cases of $\|w\|_\infty = 1$, $\|w\|_\infty = 2$, and $\|w\|_\infty = 5$, the system states at switching instants can be kept within $\Theta(\eta)$ and the states during the stage of switching are inside E_1 or E_2 as well, showing that the computed $\mathcal{RAS}$-MDT is admissible and the design of switched tube-based MPC is valid. Nevertheless, it can be seen from both Cases II and III that the states are also inside the tube even for a larger bound on the disturbance than the prescribed bound, which indicates the conservatism of the adopted tube due to switching dynamics.

Finally, the $\mathcal{RF}$-FR can be further obtained by determining the $\mathcal{F}$-FR for the nominal system for given MDT τ_m^{giv}. One additional issue is that $\mathcal{X}_{N_m}^m$ varies with different E_m which is linked to $\Theta(\boldsymbol{\eta})$ when $\boldsymbol{\eta}$ varies. Therefore, to obtain a complete $\mathcal{RF}$-FR, it is necessary to check if a $\mathcal{F}$-FR exists at each step of decreasing τ_m^{giv} with $(\boldsymbol{\tau}^{giv})^{[-1]}(\boldsymbol{\eta})$ till the minimum MDT such that $\Theta(\boldsymbol{\eta})$ exists for system (31). Here, only the case of $\tau_m^{giv} \geq \eta_m$ is concerned. Then, the $\mathcal{RF}$-FR will be the union of all the $\mathcal{F}$-FR, in which each $\mathcal{F}$-FR is associated with its own MPC law (i.e., u_k is different since E_m in (32) used in the MPC optimization is different). In the case that more than one $\mathcal{F}$-FR with different E_m cover a same state x_k, then $u^{MPC}(\Omega_m, x_k)$ can be computed based on any one of them.

The above considerations, together with Algorithms 1 and 2 in Section 2.4, yield an algorithm to determine $\mathcal{RF}$-FR for system (29) as below.

Algorithm 4. Determination of $\mathcal{RF}$-FR (Input: τ_m^{giv}, η_m, $\bar{A}_m$, $\forall m \in \mathcal{I}$, $\mathbb{W}$)

(i) Define $\delta_{(k)} := \{(\tau^{giv})^{[-(k-1)]}(\eta)\}$ with $\delta_{(k),r} := \{\delta_{1,(k),r}, \delta_{2,(k),r}, \ldots, \delta_{M,(k),r}\}$ denoting the r^{th} element of $\delta_{(k)}$. Set $k = 1, r = 1$ and $\delta_{(k),r} = \tau^{giv}$. Compute $\Theta(\delta_{(k),r})$ by Algorithm A2 and $E_{m,r}^{(k)}$ by (33), $\forall m \in \mathcal{I}$; set $\mathcal{F}_{MDT}\{\tau^{giv}\} = \varnothing$.
(ii) Update the new constraints $\mathbb{X}_m = \tilde{\mathbb{X}}_m \ominus E_{m,r}^{(k)}$ and the feasible region $\mathcal{X}_{N_m}^m$ by solving (3).
(iii) Compute τ_m based on (11) and (12). Check if $\tau_m \geq \delta_{m,(k),r}$, compute $\mathcal{F}_{MDT}^m\{\delta_{m,(k),r}\}$, $\forall m \in \mathcal{I}$, by Algorithm 2 in which Ψ_m is the feasible region outputted from Algorithm 1; else $\mathcal{F}_{MDT}^m\{\delta_{m,(k),r}\} = \mathcal{X}_{N_m}^m$, $\forall m \in \mathcal{I}$.
(iv) Set $\mathcal{F}_{MDT}\{\tau^{giv}\} = (\bigcup_{m\in\mathcal{I}} \mathcal{F}_{MDT}^m\{\delta_{m,(k),r}\}) \cup \mathcal{F}_{MDT}\{\tau^{giv}\}$. If $\delta_{m,(k),r} = \eta_m$, $\forall m \in \mathcal{I}$, then **exit** and **output** $\mathcal{F}_{MDT}\{\tau^{giv}\}$; else check if $r = 1$, set $k = k+1$, $r = \mathrm{card}\{\delta_{(k)}\}$; else set $r = r - 1$. Update $\Theta(\delta_{(k),r})$ and $E_{m,r}^{(k)}$, $\forall m \in \mathcal{I}$ goto step (ii).

Further, if $T_m(x) = V_\infty^{UC}(\Omega_m, x)$ holds, the $\mathcal{RAS}$-FR can be obtained based on Proposition 2, i.e., modifying Step (iii) in Algorithm 4 to be "Compute $\mathcal{F}_{MDT}^m\{\delta_{m,(k),r}\}$, $\forall m \in \mathcal{I}$, by Algorithm 2 in which $\Psi_m = \mathcal{O}_\infty^\lambda$". The verifications on Algorithm 4 to obtain $\mathcal{RF}$-FR and $\mathcal{RAS}$-FR are similar to Example 3 and omitted here due to space limit.

4. Conclusions and future work

The switched MPC of a class of discrete-time switched linear systems was investigated in this paper. The concept of stage MDT of variable lengths is proposed. By computing the steps over which all the reachable sets of a starting region are contained into a targeting region, the minimum admissible MDT ensuring the persistent feasibility of MPC design was offline determined. Stronger conditions were also developed to ensure asymptotic stability (quadratic stage cost and specific terminal cost would be needed). In addition, based on the proposed concept of ECS and by determining the ECS of a targeting region that can cover the states at the switching instants, the state-dependent MDT was further obtained to reduce the conservatism despite the positions of the states at the switching instants. For given constant MDT, the complete feasible region was also determined such that the switched MPC law can be persistently solved and the resulting closed-loop system is asymptotically stable via ECS approach. Based on the findings for nominal systems, the switched tube-based MPC methodology was further established for the systems with bounded additive disturbance.

The logic operations in the proposed algorithms (including Algorithms A1 and A2) are not heavy, hence the computational complexity of the algorithms behind are essentially dependent on the computation of reachable sets or controllable sets, for which certain sets addition, multiplication, intersection and union are involved (cf. discussions on the question (ii) in Section 2.3). In the linear MPC context, the computation contained in the proposed algorithms are relatively tractable, however the complexity will be dramatically increased in nonlinear setting. Thus one future research direction will be to extend the obtained results to nonlinear systems with unknown switching instants. In addition, the studies on switched MPC for the disturbed systems are based on the tube with constant cross section (rigid tube); recent advances in reducing the conservatism that exists for the rigid tube, such as developed in Raković et al. (2011) can be utilized for further studies on switched tube-based MPC. Finally, the work is based on the assumption that the switching signals are instantly known (detectable within one sampling period). If there is a detection delay of length greater than one sampling period for the switching, the computation of the fundamental reachable/controllable sets in (6), (20) will involve mode-unmatched MPC actions and the related criteria/algorithms should be reestablished. The corresponding asynchronous switched MPC theories will be another future study.

Appendix A. Aided Algorithms

Consider system (15) with MDT set $\mathbf{\Delta} \triangleq \{\Delta_1, \Delta_2, \ldots, \Delta_M\}$, where $\hat{A}_m = A_m + B_m K_m$. For $H \in \mathbb{Z}_+$, let the H-step controllable set, $\mathcal{LP}^m_H(\mathcal{X})$ be defined as $\mathcal{LP}^m_{y+1}(\mathcal{X}) \triangleq \mathcal{LP}^m_1\left(\mathcal{LP}^m_y(\mathcal{X})\right)$, $y \in \mathbb{Z}_{[0,H-1]}$, where $\mathcal{LP}^m_1(\mathcal{X}) \triangleq \{x \in \mathbb{X} : \hat{A}_m x \subseteq \mathcal{X}\}$ and $\mathcal{LP}^m_0(\mathcal{X}) = \mathcal{X}$. Let $\Upsilon^m_{\Delta_m}(\mathcal{X}) := \bigcap_{t\in[\Delta_m, 2\Delta_m-1]} \mathcal{LP}^m_t(\mathcal{X})$, Algorithm A1 to determine a $[\lambda, \mathbf{\Delta}]$-contractive set $\mathcal{O}^\lambda_\infty$ can be obtained as below by combining Algorithm 1 in Dehghan and Ong (2012a) and Algorithm 1 in Dehghan and Ong (2013). Also, consider error system (31) with $w_k \in \mathbb{W}$, MDT set $\boldsymbol{\eta} \triangleq \{\eta_1, \eta_2, \ldots, \eta_M\}$. Based on the definition of H-step reachable set of system (31), let $\Psi^m_{\eta_m}(\mathcal{X}) \triangleq \bigcup_{t\in[\eta_m, 2\eta_m-1]} \mathcal{LR}^m_t(\mathcal{X})$, Algorithm A2, also shown below, to determine a MDT RPI set $\Theta(\boldsymbol{\eta})$ for system (31) with MDT set $\boldsymbol{\eta}$ can be obtained by combining Algorithm 1 in Dehghan and Ong (2013) and Algorithm 1 in Dehghan and Ong (2012b).

Algorithm A1. Computation of $[\lambda, \Delta]$-contractive set $\mathcal{O}^\lambda_\infty$ (Input: $\mathbb{X} = \bigcap_{\forall m\in\mathcal{I}} \mathcal{T}_m, \lambda, \hat{A}_m, \Delta_m, \forall m \in \mathcal{I}$)

(i) Compute $\mathcal{S}_m := \mathbb{X} \cap \left(\bigcap_{t=1,2,\ldots,\Delta_m-1} \mathcal{LP}^m_t(\mathbb{X})\right)$, $\forall m \in \mathcal{I}$, and compute $\Upsilon^m_{\Delta_m}(\lambda\mathcal{S}_m), \forall m \in \mathcal{I}$.
(ii) Set $k = 1$ and let $\mathcal{O}^\lambda_k := \bigcap_{m\in\mathcal{I}} \Upsilon^m_{\Delta_m}(\lambda\mathcal{S}_m)$.
(iii) Compute $\Upsilon^m_{\Delta_m}(\lambda\mathcal{O}^\lambda_k), \forall m \in \mathcal{I}$ and let $\mathcal{O}^\lambda_{k+1} := \mathcal{O}^\lambda_k \cap \left(\bigcap_{m\in\mathcal{I}} \Upsilon^m_{\Delta_m}(\lambda\mathcal{O}^\lambda_k)\right)$.
(iv) If $\mathcal{O}^\lambda_{k+1} \equiv \mathcal{O}^\lambda_k$, then **exit** and **output** $\mathcal{O}^\lambda_\infty = \mathcal{O}^\lambda_k$; else set $k = k+1$, and goto Step (iii).

Algorithm A2. Computation of MDT RPI set $\Theta(\eta)$ (Input: $\bar{A}_m, \eta_m, \forall m \in \mathcal{I}$, $\mathbb{W}$)

(i) Set $k = 0$, $\Theta_{(k)}(\eta) = \varnothing$.
(ii) Compute $\Theta_{(k+1)}(\eta) \triangleq co(\bigcup_{m\in\mathcal{I}} \Psi^m_{\eta_m}(\Theta_{(k)}(\eta)))$.
(iii) If $\Theta_{(k)}(\eta) \equiv \Theta_{(k+1)}(\eta)$, then exit and output $\Theta(\eta) = \Theta_{(k)}(\eta)$; else set $k = k+1$, and goto Step (ii).

Appendix B. Relation among criteria

Section 2.2 ($\mathcal{AS}$-MDT)
Thm. 1 (persistent feasibility) $\Rightarrow$ Cor. 1 $\Rightarrow$ Cor. 2 $\downarrow$ Thm. 2 (attractivity) $\Rightarrow$ Cor. 3 $\downarrow$ Thm. 3 (asymptotic stability) $\Rightarrow$ Cor. 4
Section 2.4 ($\mathcal{AS}$-FR)
Proposition 1 $\longrightarrow$ Algorithm 1 ($\mathcal{F}$-FR) $\downarrow$ Proposition 2 $\longrightarrow$ Algorithm 2 ($\mathcal{AS}$-FR) $\longleftarrow$ Lem. 1
Section 3 ($\mathcal{RAS}$-MDT, $\mathcal{RAS}$-FR)
Proposition 3 ($\mathcal{RF}$-MDT) $\rightarrow$ Proposition 4 ($\mathcal{RFA}$-MDT) $\rightarrow$ Thm. 4 ($\mathcal{RAS}$-MDT)   Proposition 2 $\longrightarrow$ Algorithm 4 ($\mathcal{RAS}$-FR)

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