Techniques for Reallocating Airport Resources during Adverse Weather

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***Abstract*— The decreased airport arrival capacity during adverse weather conditions makes the pre-planned airline flight schedules unachievable, and necessitates the reallocation of arrival resources at the affected airport. This paper considers two different approaches based on market design to resolve the issue of airport landing resource reallocation. Several properties of these techniques are evaluated, including the nature of incentives for airlines to participate, report their true preferences, and the desire to minimize manipulation of the system. The first part of the paper analyzes the problem of slot trading without monetary payments, and presents sufficient conditions for the existence of stable allocations. The second part of the paper proposes the combination of optimization-based slot trading with a payment-based exchange scheme that ensures that the exchange (the FAA) does not operate at a deficit, while minimizing the extent by which airlines could manipulate the system by misrepresenting their preferences.**

I. INTRODUCTION

The growing numbers of aircraft in the skies have resulted in a congested airspace, and put a strain on the resources for takeoffs and landings at airports. Several airports have been designated as *high-density airports*, and are subject to limits on the number of instrument flight takeoffs and landings of aircraft that can be scheduled over a given period of time [1]. Air carrier schedules are based on reservations for instrument flight takeoff or landing, which are also known as *slots*. The limits on the operational capacity of an airport caused by constraints on runway operations, gate availability, and air traffic control, make landing slots scarce resources.

While slots at high-density airports have traditionally been grandfathered, with the success of several combinatorial auctions in recent years, there has been a renewed interest in auction mechanisms for slot allocation [2], including a case study based at the Hartsfield Atlanta airport [3]. The FAA is considering market-based mechanisms for demand, capacity and congestion management in many airports, including New York's LaGuardia [4] and Chicago's O'Hare [5] airports.

An important issue that needs to be addressed is that of slot reallocation when an airline is unable to utilize a slot that it owns, due to circumstances beyond its control. This could be because of over-scheduling of the airspace (for example, departures in O'Hare airport are sometimes slowed because of enroute traffic) or because of adverse weather phenomena. Bad weather can decrease the arrival capacity of the airport, and sometimes even close an airport down temporarily, increasing the rate of arrivals (demand) into the airport at a later time. The FAA is required to respond when the demand over a 15-minute period exceeds the capacity. Small surges in demand are resolved using airborne control procedures, such as hold patterns, re-routes, and variations in speed (within 10%). Due to the high fuel costs of airborne delay, longer surges in demand are resolved by delaying flights at the departure airport, known as a Ground Delay Program or GDP [6]. The disruption of flights in a GDP implies that airlines can no longer fly their original schedules, and requires a reallocation of airport landing slots. Since the implementation of the ground delay program affects the effective allocation of slots, it is necessary to analyze the rules and incentives of GDPs before the airlines can be expected to spend large sums of money in slot auctions.

In developing slot reallocation algorithms for GDPs, we can treat the problem as that of designing an exchange mechanism for slots, where airlines are self-interested agents who wish to maximize their utility, represented by the utility that they derive from the slots that they are allocated. There are several issues that must be considered while designing such market-based mechanisms for slot reallocation. We seek a solution that is *pareto efficient*, that is, there is no other solution that is preferable to all airlines. In other words, we seek a solution such that no airline can receive a better allocation of slots without some other airline being made worse off. While the objective of reallocation is to maximize the overall value of the trades (*allocative efficiency*), we also require that the airlines are not worse off by participating in the slot exchange (*individual rationality* or *voluntary participation*). A stable allocation is also desirable; we would like to find an allocation such that no coalition of airlines would do better by not participating in the exchange, and instead trading amongst themselves. The set of such stable allocations is also called the *core*. To minimize user gaming, we would also like to design a mechanism that encourages airlines to report their true preferences. This could be either expressed by *incentive-compatibility*, which implies that truthful reporting forms a Bayes-Nash equilibrium (that is, if every other airline reports its preferences truthfully, an airline maximizes its utility by reporting truthfully) or by *strategy-proofness*, which implies that truthful reporting is optimal irrespective of the reports of other airlines. We note that the issue of truthful reporting of preferences arises when the airlines are required to state the preferences of flight-slot allocations, or even bid for slots based on their private valuations.

In this paper, we consider two possible approaches to the slot trading problem. The first is a mechanism in which airlines declare the relative priorities of their flights, and also the ranked order of slot preferences for each flight. There are no monetary transfers allowed between airlines.

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As the mechanism designers, we would like to obtain the pareto-efficient reallocation of slots to flights. However, we demonstrate that there may be no stable allocation, that is, the core of the slot trading economy may be empty. This would imply that the airlines would always have incentive to deviate or form coalitions, and the system will not converge to a stable allocation of slots. As a preliminary step to better understand the behavior of this mechanism, we derive sufficient conditions on the preference relations of airlines that guarantee a non-empty core; we also propose an algorithm to determine a core allocation when it exists. The second approach is a mechanism that allows payments between airlines for the slots traded. Airlines report the set of acceptable trades and the associated utility of the trade; we employ the techniques proposed by Parkes et al. [7] to develop a payment scheme that is individual-rational and budget-balanced, that attempts to minimize the ability of airlines to manipulate the payments by deceit, and that determines the pareto-efficient allocation corresponding to the declared utilities of the airlines.

II. Ground Delay Programs

There is currently a move toward Collaborative Decision Making (CDM) for slot allocation during Ground Delay Programs [8]. The chief premise of the CDM program is that an increased data exchange between the FAA and the airlines will result in improved decision making. Before the implementation of CDM-based proposals, GDPs were implemented by assigning flights to slots using a first-come, first-served algorithm known as *Grover Jack*. The two new procedures introduced by the CDM program are known as *Ration-By-Schedule* (RBS) and *Compression*.

The first stage of a GDP is the RBS algorithm, which rations the arrival slots among the airlines on a first-come, first-served basis, according to their *original* scheduled time of arrival (STA). The rationale behind this is that airlines will not forfeit a slot by reporting a delay or cancellation.

Delays, and cancellation of flights by airlines, create gaps in the current schedule. The Compression stage moves flights up to fill in these slots such that an airline that vacates a slot receives the slot belonging to the earliest flight that can utilize the vacant slot. In this manner, there is incentive for airlines to report cancellations.

Compression is essentially an exchange mechanism in which the objective is to minimize delays, and to maximize slot allocation. It assumes that airlines would like to schedule any flight as early as possible, with priorities based on the earliest time of arrival. In reality, we would like to allow airlines the freedom to assign more general priorities and slot preferences (for example, based on their banking strategies). CDM primarily concentrates on incentives for airlines to report delays and cancellations. In this research, we consider the nature of incentives for airlines to participate in a GDP, and when they do, be truthful in their reports to the FAA.

III. Slot reallocation without payments

In this section, we consider a solution based on the Top Trading Cycles mechanism for house allocation [9]. These mechanisms have been shown to, under certain conditions, yield a unique outcome [10] that is pareto-efficient, and in addition, are individually rational and strategy-proof [11].

We fix a priority ordering of flights in the GDP. If an airline cancelled a flight and created a vacant slot, it is allowed to choose one of its flights as the "highest priority flight". If a flight is delayed for reasons not related to the GDP (a mechanical failure, for example), the airline can choose to vacate its slot and use the priority it obtains to choose a later time slot. An ordering is randomly selected for the remaining flights from an exogenous distribution of orderings. Airlines submit a set of preferences for each of their flights. In the absence of any other constraints, the preference set for flight f would be the set of slots ranging from the earliest possible arrival, $e(f)$ (the most preferred slot) to the current slot position, $I(f)$. Given the preference profiles of flights over slots, we find the matching of flights to slots using the Top Trading Cycles algorithm [9].

*Algorithm 1 (**Top Trading Cycles Algorithm**):* We begin with the set of all flights and all slots, and sequentially match slots to flights as follows:

- Each flight f points to its most preferred slot among the remaining slots under its announced preferences, each occupied slot points to its occupant, and each vacant slot points to the flight with the highest priority among those still remaining. Since the numbers of flights and slots are finite, there is at least one cycle. Every flight in the cycle is assigned the slot that it points to, and removed along with its assignment. If there is at least one remaining flight and one remaining slot, we repeat the process.

The algorithm terminates in at most $\min\{|\mathcal{F}|, |\mathcal{S}|\}$ iterations, where $\mathcal{F}$ is the set of flights and $\mathcal{S}$ is the set of slots.

The following theorem describes the properties of the induced Top Trading Cycles (TTC) mechanism.

Theorem 1 (In [11]): For any ordering of flights, the induced TTC mechanism is pareto-efficient, individually rational, and strategy proof.

*Example 1 (**Top Trading Cycles**):* We consider an example scenario of reallocating slots to flights using the Top Trading Cycles algorithm. The initial slot assignments after the Ration-by-Schedule stage is shown in Fig. 1 (a). The solid lines represent initial assignments of slots to flights, while the dotted lines represent the earliest arrival times (STA).

The Top Trading Cycles mechanism allows the airlines to report the order of preference of slots for each flight. Unless the airline states otherwise, we assume that the only objective is to decrease delay, and priorities are determined by the amount of delay incurred. However, we could also consider other scenarios, such as, if UAL2 most prefers the 1602 slot (Fig. 1 (b)), UAL2 most prefers the 1604 slot (Fig. 1 (c)), or if flight AAL2 most prefers the 1606 slot (Fig. 1 (d)).

A. When airlines possess multiple flights in a GDP

It is clear that in the instances in which the flights correspond to agents, the top trading cycles mechanism will find the (single) core allocation. This extends to situations

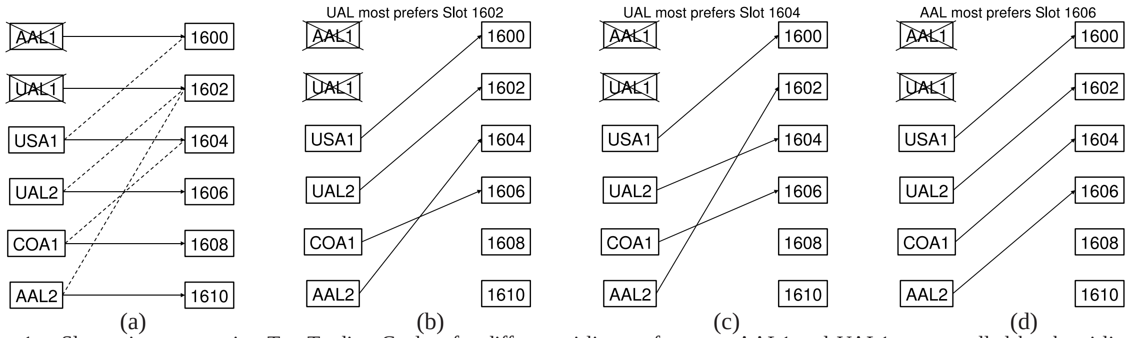

Fig. 1. Slot assignments using Top Trading Cycles, for different airline preferences. AAL1 and UAL1 are cancelled by the airlines.

in which the airlines are the agents, and each airline owns only one flight. Vossen [12] suggests that the case in which airlines may have more than one flight corresponds to a Shapley-Scarf economy with a single type of indivisible good, in which agents may consume multiple units, and by modifying an example illustrated for such an economy by Konishi et al. [13], they construct an example of a GDP with an empty core. We note that slot exchange in a GDP where each airline owns several flights is not quite equivalent to a Shapley-Scarf economy with a single type of indivisible good, in which agents may consume multiple units. The reason for this is that an airline cannot claim equal cost structure over the slots for all its flights, since that would imply that it would ideally like all its flights to be scheduled in the same slot, which is not possible. In reality, since flights are scheduled over different slots in the original schedule, the costs of the slots in the GDP differ from flight to flight.

We show in the following example, modified from [14], that even when airlines only reveal a list of preferences for each flight, if an airline is allowed to own more than one flight, the core may be empty.

*Example 2 (**GDPs and empty cores**):* Let us consider a GDP with 9 slots, 9 flights, and 5 airlines.

$$\mathcal{F} = \{f_1, \cdots, f_9\};\ \mathcal{S} = \{s_1, \cdots, f_9\};\ \mathcal{A} = \{a, b, c, d, e\}$$
$$\mathcal{F}_a = \{1,2\},\ \mathcal{F}_b = \{3,4\},\ \mathcal{F}_c = \{5,6\},\ \mathcal{F}_d = \{7,8\},\ \mathcal{F}_e = \{9\};$$
$$\mathcal{S}_a = \{1,2\},\ \mathcal{S}_b = \{3,4\},\ \mathcal{S}_c = \{5,6\},\ \mathcal{S}_d = \{7,8\},\ \mathcal{S}_e = \{9\};$$

The preference profiles for each of the flights, as given by the airlines are

$$Q(f_1) = (s_7, s_1); Q(f_2) = (s_2, s_3); Q(f_3) = (s_2, s_9, s_5, s_3);$$
$$Q(f_4) = (s_4, s_1); Q(f_5) = (s_6, s_8); Q(f_6) = (s_3, s_8, s_5);$$
$$Q(f_7) = (s_4, s_5, s_8); Q(f_8) = (s_7, s_8); Q(f_9) = (s_9);$$

Clearly, for any core-stable allocation, $\sigma(f_9) = 9$. Each airline, declares its priority order of flights as

$$\mathcal{P}(a) = (f_1, f_2); \mathcal{P}(b) = (f_4, f_3); \mathcal{P}(c) = (f_5, f_6);$$
$$\mathcal{P}(d) = (f_8, f_7); \mathcal{P}(e) = (f_9).$$

Suppose the flights extend their preferences over the rest of their airline using the rule δ such that $(1,2) \succ_\delta (2,1) \succ_\delta (1,3)$. For example, for airline a, with the priorities defined as above, from the perspective of flight f_1, this would translate to $\{s_7, s_3\} \succ \{s_1, s_2\} \succ \{\}$ (there is no preference 3 for f_2). If we were to order the ranking of matchings for each airline ($a = \{f_1, f_2\}$, for example), we would get, for sets of matching defined as $\Sigma^a_{ij} = \{\sigma(f_1) = s_i,\ \sigma(f_2) = s_j\}$, the following ranking of sets:

a	b	c	d
Σ^a_{72}	Σ^b_{24}	Σ^c_{63}	Σ^d_{47}
Σ^a_{73}	Σ^b_{21}	Σ^c_{68}	Σ^d_{57}
Σ^a_{12}	Σ^b_{54}	Σ^c_{83}	Σ^d_{48}
$\cdots$	Σ^b_{34} or Σ^b_{51}	Σ^c_{65}	Σ^d_{78} or Σ^d_{58}

($\downarrow$: decreasing preferences).

We note that for a given airline, both flights have the same ranking of matches. Any core-stable matching σ must be such that

$$\sigma \in A^a = \Sigma^a_{72} \cup \Sigma^a_{73} \cup \Sigma^a_{12}$$
$$\sigma \in A^b = \Sigma^b_{24} \cup \Sigma^b_{21} \cup \Sigma^b_{54} \cup \Sigma^b_{34} \cup \Sigma^b_{51}$$
$$\sigma \in A^c = \Sigma^c_{63} \cup \Sigma^c_{68} \cup \Sigma^c_{83} \cup \Sigma^c_{65}$$
$$\sigma \in A^d = \Sigma^d_{47} \cup \Sigma^d_{57} \cup \Sigma^d_{48} \cup \Sigma^d_{87} \cup \Sigma^d_{58}$$

The intersection of the above sets results in four possible matchings, X_1, X_2, X_3 and X_4, given by

$$X^a_1 = \{s_7, s_3\},\ X^b_1 = \{s_2, s_1\},\ X^c_1 = \{s_6, s_4\},\ X^d_1 = \{s_5, s_8\}$$
$$X^a_2 = \{s_2, s_1\},\ X^b_2 = \{s_5, s_4\},\ X^c_2 = \{s_6, s_3\},\ X^d_2 = \{s_8, s_7\}$$
$$X^a_3 = \{s_2, s_1\},\ X^b_3 = \{s_3, s_4\},\ X^c_3 = \{s_6, s_8\},\ X^d_3 = \{s_5, s_7\}$$
$$X^a_4 = \{s_2, s_1\},\ X^b_4 = \{s_3, s_4\},\ X^c_4 = \{s_6, s_5\},\ X^d_4 = \{s_8, s_7\}$$

We notice that both X_1 and X_4 are blocked by X_3 through $\{c, d\}$, X_2 is blocked by X_1 through $\{a, b, d\}$, and X_3 is blocked by X_2 through $\{b, c\}$. Therefore, the core of this game, where airlines can operate multiple flights, is empty.

IV. Slot Exchanges and the Empty Core Problem

The fact that the core of the slot allocation economy might be empty is an important one, since it implies that there are situations in which there is no stable allocation of slots. We note that this alone does not merit the discarding of game-theoretic approaches in favor of other (for example, optimization-based) techniques, since the problem of coalitions being formed between airlines will exist, even if some other reallocation mechanism (with no payments between airlines) is chosen. Instead, to gain a better understanding of the problem, we try to determine conditions on the airlines' preference profiles that guarantee the existence of a non-empty core, and also propose an algorithm to determine a stable allocation within this core.

In this section, we describe a generalized Shapley-Scarf economy that models GDPs in which each airline (single agent) could own more than one flight. In such a scenario, while side payments between airlines are still not allowed, it is possible for flights *within* the same airline to pool their resources. The initial endowment to an each flight is the slot initially assigned to it. When transfers are allowed within an airline, the budget-feasible outcomes are those allocations in which the total price of the allocated goods to an airline does not exceed the total initial endowment to that airline.

A. Model of a Shapley-Scarf economy for slot reallocation

We broadly follow the notation of Laffond and Laine [14]. We denote the set of positive integers by $\mathbb{N}$. Let $\mathcal{F} = \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$ denote a finite set of flights. Let $S \in \{1, \cdots, n\}$ be the set of (indivisible) slots. A matching σ is a bijection from $\mathcal{F}$ to $\mathcal{S}$, where $\sigma(f) = s$ means that slot s is allocated to flight f. We denote by Σ the set of all possible matchings. Let $\mathcal{A} = \{A_1, \cdots, A_H\}$ be a partition of $\mathcal{F}$ into different airlines. We denote by $A(f)$ the unique airline which operates the flight f.

Individual flight preferences are given by two linear orders: P_f^F is a linear order on $A(f)$, which describes how the flight positions its importance relative to the other flights owned by the same airline; P_f^S is a linear order on $\mathcal{S}$, which describes how f values slots. Therefore, flight f's preferences are denoted by $P_f = (P_f^F, P_f^S)$.

A profile $\pi \in \Pi$ is a vector $(P_f)_{f \in \mathcal{F}}$ of flight preferences. We denote by $r_f^\pi(s)$ the rank given by P_f^S to slot s. $x(f)$ is the flight in $A(f)$ having rank $x \in \{1, \cdots, |A(f)|\}$, according to P_f^F. $\mathbf{1}(A_h)$ denotes the set of flights in A_h that are ranked first (highest priority) by at least one member of A_h. $\mathbf{1}_Z(f)$ is the slot in $Z \subseteq \mathcal{S}$ flight f most prefers.

The set of all preferences on Σ is denoted by Ψ. A preference extension rule is a mapping δ from Π to Ψ. A preference extension rule describes how an airline derives, from the set of all original orderings of its flights and slots, a complete ranking of matchings for a particular flight. We assume that for all flights in an airline, the same extension rule is applied ($\forall f \in a = A(f), \delta_f = \delta_a$).

There are several possible restrictions on the set of preference extension rules. We consider $\pi \in \Pi$, $\sigma \in \Sigma^n$, and let $f \in \mathcal{F}$. The π-ordered rank vector of σ for f is given by $S_f^\pi(\sigma) = (r_{1(f)}^\pi(\sigma), r_{2(f)}^\pi(\sigma), \cdots, r_{|A(f)|(f)}^\pi(\sigma))$.

Definition 1: Let δ be a preference extension rule. Then δ is said to be **neutral** if $\forall \sigma, \sigma', \gamma, \gamma' \in \Sigma$, $\forall \pi, \pi' \in \Pi$, $[S_f^\pi(\sigma) = S_f^{\pi'}(\sigma')$ and $S_f^\pi(\gamma) = S_f^{\pi'}(\gamma')] \Rightarrow \{[\sigma \delta_f(\pi) \gamma] \Leftrightarrow [\sigma' \delta_f(\pi) \gamma']\}$.

A neutral rule describes a specific non-selfish way to evaluate matchings. It implies that the "name" of the slot does not matter when ordering matchings. Each flight (individual) f considers first the well-being of its most preferred flight $1(f)$, then the preferences of $2(f)$, and so on. Therefore the comparison of two matchings by flight f requires the comparison of two vectors in $\Omega_f = \{1, \cdots, n\}^{|A(f)|}$. As a result of neutrality, for $\sigma, \sigma' \in \Sigma^n$, $f \in \mathcal{F}$, and $\pi \in \Pi$, $[\sigma \delta_f(\pi) \sigma']$ will be equivalent to $[S_f^\pi(\sigma) \succeq S_f^\pi(\sigma')]$.

Definition 2: A neutral preference extension rule δ is **monotone** if $\forall \pi \in \Pi$, $\sigma, \sigma' \in \Sigma^n$, $\forall f \in \mathcal{F}$, $\delta(\pi)$ is such that $[S_f^\pi(\sigma) \leq S_f^\pi(\sigma')] \Rightarrow [\sigma \delta(\pi) \sigma']$.

Monotonicity means that an individual (flight) is (strictly) better off when moving from one matching to another Pareto-improves the welfare of the airline to which it belongs.

Definition 3: An **economy** $\mathcal{E}$ is a 5-tuple $(\mathcal{F}, \mathcal{A}, \pi, \delta, \sigma^0)$, where $\mathcal{F}$ is the set of flights, $\mathcal{A}$ is the partition of flights among the airlines, where $(\pi, \sigma^0) \in \Pi \times \Sigma$, where δ is a neutral and monotone preference extension rule, and where σ^0 is called the initial matching of $\mathcal{E}$.

Definition 4: Let $\mathcal{E} = (\mathcal{F}, \mathcal{A}, \pi, \delta, \sigma^0)$ be an economy. A matching σ is said to be unblocked if there exists no $J \subseteq \mathcal{F}$ and no $\sigma' \in \Sigma$ such that:

1) $\forall A \in \mathcal{A}$, $[A \cap J \neq \emptyset] \Rightarrow [A \subseteq J]$
2) $\forall j \in J$, $\sigma' \delta_j(\pi) \sigma$
3) $\forall f \in \mathcal{F} - J$, $\sigma'(f) = \sigma^0(f)$

The **core** of $\mathcal{E}$ is the set $\mathcal{C}(\mathcal{E})$ of unblocked matchings.

Definition 5: A neutral preference extension rule δ is **transfer-consistent** if $\forall \pi \in \Pi$, $\forall \sigma, \sigma'$, $\forall f \in \mathcal{F}$, $[r_{k(f)}^\pi(\sigma) = r_{k(f)}^\pi(\sigma'),\ k \neq k', k'',\ k' < k''$ and $r_{k'(f)}^\pi(\sigma) = r_{k''(f)}^\pi(\sigma') < r_{k''(f)}^\pi(\sigma) = r_{k'(f)}^\pi(\sigma')] \Rightarrow [\sigma \delta_f(\pi) \sigma']$.

Definition 6: A neutral preference extension rule, δ is **strongly separable** if $\forall (w_1, w_2, \cdots, w_K), (z_1, z_2, \cdots, z_K) \in \Omega$ such that all w_is not equal, and all z_is not equal,

$$[(w_1, \cdots, w_K) \succ_\delta (z_1, \cdots, z_K)] \Rightarrow$$
$$[\exists i \text{ such that } \forall x_j \in \{1, \cdots, n\},\ j \neq i,\ (x_1, \cdots, w_i, \cdots, x_K) \succ_\delta (z_1, \cdots, z_K)].$$

Strong separability implies that when comparing two disjoint ordered rank vectors, only one coordinate matters.

B. Guaranteeing a non-empty core

We can guarantee a non-empty core for a slot reallocation economy by placing restrictions on the preference extension rules.

Theorem 2 (Non-empty core): Let $\mathcal{E} = (\mathcal{F}, \mathcal{A}, \pi, \delta, \sigma^0)$ be an economy in which the preference extension rules δ are neutral, monotone, transfer-consistent, and strongly separable. Then, the economy $\mathcal{E}$ has a non-empty core.

Proof: This theorem is proved in the appendix. It is a proof by construction, and describes an algorithm (the Generalized Top Trading Cycles sequence) that determines an allocation that is in the core of the economy. ■

The class of preference extension rules for which we can guarantee a non-empty core is quite restrictive, but include the set of lexicographic rules. Therefore, if airlines order their flights according to priority, and rank allocations lexicographically across the vector, the core is non-empty. Similarly, lexicographic ordering subject to the constraint that no individual flight receives a worse allocation that its current slot is also an acceptable preference relation to determine an allocation that its within the core. If the core is empty, the FAA would have to implement rules to ensure that the market converges; these rules would have to enforce stability by preventing the formation of coalitions by airlines.

The investigation of the incentive-compatibility or strategy-proofness of the Generalized Top Trading Cycles algorithm, under the restricted preference extension rules that guarantee a nonempty core, is a direction for future research. It can be shown that in the most general scenario in which airlines have preferences over the set of all possible matchings, the problem is quite similar to models of voting schemes [15]. In this case, the only matching rules that are strategy-proof and Pareto-efficient are dictatorial rules, in which one airline gets to make the decision.

V. Slot exchanges with payments

We now consider a different approach, namely one in which monetary transfers are allowed between airlines. This is a marked deviation from the current system, and the acceptability of such a scheme to the various stakeholders needs to be studied, but is beyond the scope of this work.

We consider the problem of a general slot exchange mechanism, in which airlines bid for slot trades, and payments are determined when the market is cleared, that is the allocation of slots is determined. Vossen and Ball [6] model the slot trading mechanism as an optimization framework in which the FAA acts as the mediator (in other words, the exchange). They consider the scenario in which airlines make offers in the form (s, T_s), where s is a slot and T_s is the set of all slots that the airline is willing to receive in exchange for relinquishing s. The airline also submits ("bids") the value of each trade, v_{st}, $t \in T_s$, which is the value it would derive by trading slot s for slot t. In addition, the airline is also allowed to define a slot $\rho(s)$ from among the slots it owns, which is the slot that it will retain if all the trades in T_s are denied. We require that the mapping $\rho(s)$ is a one-to-one mapping. The allocation of slots (winner determination) is then determined by value maximization, using the following formulation:

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{s \in \mathcal{S}} \sum_{t \in T_s} v_{st} x_{st} \\ \text{subject to} \quad & \textstyle\sum_{t \in T_s} x_{st} + y_s = 1 \quad \forall\, s \in \mathcal{S} \\ & \textstyle\sum_{t : s \in T_t} x_{ts} + y_{\rho^-(s)} = 1 \quad \forall\, s \in \mathcal{S} \\ & x_{st},\, y_s \in \{0, 1\} \quad \forall\, s \in \mathcal{S},\, t \in T_s \end{aligned} \tag{1}$$

where $\rho^-(s) = \{t : \rho(t) = s\}$. The variable $x_{st} = 1$ for some $t \in T_s$ when the airline receives slot t in exchange for slot s. The variable $y_s = 1$ if the offer (s, T_s) is rejected, and 0 if it is accepted. The above formulation would determine the efficient allocation if the airlines bid truthfully.

The next task is to determine the payments by (or to) the airlines, depending on their declared value for the trade. As before, we would like the payment rules to satisfy individual rationality, that is, no airline should be worse off by participating in the exchange. Since the FAA will be operating this mechanism, we require that the exchange does not run at a loss, that is, we require budget balance. In addition, since the payments are being computed based on the values (bids) declared by the airlines, we would like to minimize the extent to which the airlines can manipulate the allocation and payments by misrepresenting their values (i.e., we would like strategy-proofness).

It is well-known that there is no exchange mechanism that can be efficient, budget-balanced, and individual rational [16]. As an alternative, we follow the approach proposed by Parkes et al. [7], wherein we enforce budget-balance and individual rationality, and try to achieve a fairly efficient and fairly incentive-compatible scheme.

A. Vickrey payments

Vickrey-Clarke-Groves (VCG) pricing mechanisms support efficient, individual rational and strategy-proof exchanges that are, however, frequently not budget-balanced.

We consider the formulation shown in (1). The optimal trade (for the reported values) is given by x^*_{st}, and the optimal value is $V^* = \sum_{s \in \mathcal{S}} \sum_{t \in T_s} v_{st} x^*_{st}$. We denote by $(V_{-a})^*$ the optimal value without any participation from airline a. $V^*_{-a} = \sum_{s \in \mathcal{S} \setminus \mathcal{S}_a} \sum_{t \in T_s} v_{st} x^*_{st}$ is the value of trade x^*_{st} to all airlines except a. Then, the Vickrey payment [17] to airline a is computed as

$$p_{\text{vick,a}} = (V_{-a})^* - V^*_{-a}. \tag{2}$$

Negative payments imply that the airline receives money from the exchange (FAA). The Vickrey discount is defined as the difference between the bid and the payment, that is, $\Delta_{\text{vick},a} = \sum_{s \in \mathcal{S}_a} \sum_{t \in T_s} v_{st} x^*_{st} - p_{\text{vick,a}}$. The Vickrey payment is individual rational, because the discount is always non-negative. Similarly, the allocation is also individual rational because the airlines only bid on preferable trades, and are allowed to reserve a slot $\rho(s)$ for a flight in the event that none of the preferred trades is accepted. Vickrey payments can also be shown to be strategy-proof [17]. We now demonstrate using a slot trading example that the VCG mechanism can result in a failure of budget-balance.

Example 3 ***(Slot trading with Vickrey payments)****:* Consider 3 airlines $\mathcal{A} = \{A, B, C\}$, which operate 6 flights, such that $\mathcal{F}_A = \{f_1, f_6\}$, $\mathcal{F}_B = \{f_2, f_5\}$, and $\mathcal{F}_C = \{f_3, f_4\}$, and the initial slot assignments are $\mathcal{S}_A = \{s_1, s_6\}, \mathcal{S}_B = \{s_2, s_5\}$, and $\mathcal{S}_C = \{s_3, s_4\}$. Suppose f_1 is cancelled (airline A will trade slot s_1 for any other slot). $T_{s_1} = \{s_2, s_3, s_4, s_5, s_6\}$, $T_{s_2} = \{s_1\}$, $T_{s_3} = \{s_1, s_2\}$, $T_{s_4} = \{s_2, s_3\}$, $T_{s_5} = \{s_1, s_2, s_3\}$, $T_{s_6} = \{s_2\}$. In the following discussion, we use the symbol '\$' to denote the unit of currency used. The airlines report the value of trades as $v_{s_1 t} = 0$ for all t, $v_{s_2 s_1} = \$10$, $v_{s_3 s_1} = \$20$, $v_{s_3 s_2} = \$10$, $v_{s_4 s_2} = \$20$, $v_{s_4 s_3} = \$10$, $v_{s_5 s_1} = \$40$, $v_{s_5 s_2} = \$30$, $v_{s_5 s_3} = \$20$, and $v_{s_6 s_2} = \$40$. Then, the optimal flight-slot assignments and Vickrey payments are given by

Airline	**Flight**	**Initial**	**Final**	$\mathbf{p}_{\text{vick}}$
A	F1 F6	S1 S6	– S2	-\$10
B	F2 F5	S2 S5	S1 S5	-\$10
C	F3 F4	S3 S4	S3 S4	0

which implies that the FAA (exchange) would run at a loss of \$20.

B. Approximate Vickrey-based payments

We employ the approach proposed in [7], and determine a payment scheme that minimizes the distance from the Vickrey payments, subject to the budget balance constraints. In other words, we would like to minimize the ($\mathcal{L}_2$ or $\mathcal{L}_\infty$) distance from the Vickrey payments, but constrain the payment scheme to be budget-balanced (that is, the sum of payments from all the airlines must be nonnegative). Therefore, in conjunction with Problem (1), we solve the following problem, written in terms of the discounts, Δ:

$$\begin{aligned} \text{minimize} \quad & ||\Delta_{\text{vick}} - \Delta||_2 \\ \text{such that} \quad & \textstyle\sum_{a \in \mathcal{A}^*} \Delta_a \leq V^* && \text{[Budget-balance]} \\ & \Delta_a \leq \Delta_{\text{vick},a},\ \forall a \in \mathcal{A}^* && [\text{Discount} \leq \Delta_{\text{vick}}] \\ & \Delta_a \geq 0,\ \forall a \in \mathcal{A}^* && \text{[Indiv. rationality]} \end{aligned}$$

where $\mathcal{A}^*$ is the set of airlines that participate in the optimal trade. It has been shown in [7] that the solution to the above problem can be written analytically as $\Delta_a = \max(0, \Delta_{\text{vick},a} - C)$, where $C = \frac{\sum_{a \in \mathcal{A}^*} \Delta_{\text{vick},a} - V^*}{|\mathcal{A}^*|}$ (Threshold rule).

It has been shown empirically that the Threshold rule applied to exchanges results in a mechanism that has a relatively high level of truth-revelation by the agents, and therefore a high level of efficiency [7].

*Example 4 (**Slot trading with the Threshold rule**):* We return to the scenario in Example 3, and apply payments according to the Threshold rule. $C = (50+20-50)/2 = 10$. This implies that $\Delta_A = \$40$, $\Delta_B = \$10$, $\Delta_C = 0$, and $p_A = p_B = p_C = 0$, which is budget-balanced.

It must be borne in mind that while the Threshold rule maintains a level of truth-revelation by the airlines, especially when they do not have knowledge of their competitors true valuations, the payment scheme is not strategy-proof. For example, let us consider the case where airline A falsely reports the value of trading s_6 for s_2 as $v(s_6 s_2) = \$30$, when its true valuation is \$40 (as in Example 3). Then, the optimal allocation will remain unchanged, while the optimal value will be calculated as $V^* = \$40$. The Vickrey payments would be $p_{\text{vick},A} = -\$10$ (unchanged, since Vickrey payments are strategy proof) and $p_{\text{vick},B} = 0$. But the exchange still runs at a deficit, and the budget-balanced payments can be shown to be $p_A = -\$5$, $p_B = \$5$ and $p_C = 0$. This means that the total payoff to airline A (its true utility minus the payment) increases from \$40 when it reported truthfully, to \$45 when it reported falsely; in other words, the payment scheme using the Threshold rule is not strategy-proof.

1) Fiat currency vs. commodity currency: When considering sales, trades and compensations, it is necessary to decide on a unit of exchange, or currency. One possibility is the use of "slot currency", issued in fixed quantities to each airline [18]. These would play the role of vouchers that could be freely bought or sold among airlines, but could only be redeemable in airport slots, perhaps during some future GDP. Another alternative is the use of a "fiat currency" instead of commodity currency (such as slots) for the trading.

VI. Conclusion

The focus of this paper has been the design of slot reallocation mechanisms for the Ground Delay Programs that are adopted at airports during adverse weather situations. In contrast to the techniques that are currently in use, we would like to accommodate a wide range of airline strategies in the prioritization of flights. The possibility that there may be no stable allocation poses a challenge in this market, and we have derived conditions on the manner in which airlines extend their individual flight's slot preferences over the entire fleet that guarantee the existence of a core allocation. In particular, if airlines choose priority orders of their flights and slots to their flights, a lexicographic ordering of allocations based on these priorities will ensure a nonempty core. We believe that this analysis is an important first step to understanding the behavior of the slot trading market.

Monetary transfers between airlines during slot reallocation is a significant shift from the current paradigm, and requires a detailed analysis from a policy and stakeholder perspective to determine acceptability. However, it is clear that any such system would have to ensure that the FAA does not operate at a loss, and that manipulation of the allocation by the airlines be limited. For this reason, we have proposed the combination of a payment mechanism that uses approximate Vickrey payments in combination with an optimization-based approach to slot exchanges. Future research will involve an extensive empirical study of the incentive-compatible properties of this mechanism when applied to typical airline slot valuations.

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Appendix

Proof of Theorem 2

Proof: We first formalize the notion of a weak budget set.

Definition 7: Let $p = (p_s)_{s \in \mathcal{S}} \in \mathbb{N}^n$ be a price vector. Let $f \in \mathcal{F}$. The **weak budget set** of f facing p is the subset of matchings $\Delta(p, f) = \{\sigma \in \Sigma : \sum_{i \in A(f)} p_{\sigma(i)} \leq \sum_{i \in A(f)} p_{\sigma^0(i)}\}$.

We now define weak equilibrium concepts for the economy $\mathcal{E} = (\mathcal{F}, \mathcal{A}, \pi, \delta, \sigma^0)$.

Definition 8: Let $p = (p_s)_{s \in \mathcal{S}} \in \mathbb{N}^n$ be a price vector. A **weak** $p-$**optimum** for $A(f)$ is a matching $\sigma \in \Delta(p, f)$ such that there is no $\sigma' \in \Delta(p, f) - \{\sigma\}$ which satisfies $\sigma' \delta_i(\pi) \sigma \ \forall i \in A(f)$. We denote by $\mathcal{W}(p, A(f))$ the set of all weak $p-$optima for $A(f)$.

Definition 9: A **cooperative weak price equilibrium** for $\mathcal{E}$ is a pair $(\sigma, p) \in \Sigma \times \mathbb{N}^n$ such that $\forall A \in \mathcal{A}$, $\sigma \in \mathcal{W}(p, A)$. We denote by $\mathcal{W}_{coop}(\mathcal{E})$ the set of all matchings that can be implemented as a weak price equilibrium.

Allowing for transfers between the flights owned by the same airline, we now consider the benefit to the airline, instead of the individual flights. This results in economies with no equilibrium, as well as economies in which the Top Trading Cycles mechanism results in non-equilibrium matchings [14].

Lemma 1: Let $\mathcal{E}$ be an economy. Then $\mathcal{W}_{coop} \subseteq \mathcal{C}(\mathcal{E})$.

Proof: Let $\sigma \in \mathcal{W}_{coop}(\mathcal{E})$ and let $p = (p_s)_{s \in \mathcal{S}}$ be the associated equilibrium price vector. Suppose $\sigma \notin \mathcal{C}(\mathcal{E})$. Let J and σ' be defined as in Definition 4. Let $J = \cup_{h=1,\cdots,H'} A_h$, where $A_h \in \mathcal{A}$. Condition (2) implies that $\forall h, \forall i \in A_h, \ \sigma' \delta_i(\pi) \sigma$. Therefore, from the definition of a weak $p-$optimum (Definition 8), $[\forall h, \sigma \in \mathcal{W}(p, A_h)] \Rightarrow [\forall h, \forall f \in A_h, \sigma' \notin \Delta(p, f)]$. From the definition of weak budget sets (Definition 7), $\sum_{i \in A_h} p_{\sigma'(i)} > \sum_{i \in A_h} p_{\sigma^0(i)}$. Summing over h, one gets $\sum_h \sum_{i \in A_h} p_{\sigma'(i)} > \sum_h \sum_{i \in A_h} p_{\sigma^0(i)}$, which is impossible because Condition (3) of Definition 4 implies that $\cup_{f \in A_h} \sigma'(f) = \cup_{f \in A_h} \sigma^0(f)$. Therefore, $\sigma \in \mathcal{C}(\mathcal{E})$, proving that $\mathcal{W}_{coop} \subseteq \mathcal{C}(\mathcal{E})$. ∎

We begin by considering the following lemma. Let us suppose an airline a is faced with several possible choices of allocations (all feasible) to its flights, and must evaluate them. Let the airline operate K flights, and let the ordered set of ranks accessible to flight f from among the allocation choices be denoted $M^{(f)}$.

Lemma 2: Let Θ be the set of all economies where the extension rule δ is neutral, monotone and transfer-consistent. Suppose $\delta_a \in \Theta$ is strongly separable. Let $M^{(i)} = \{r_1^{(i)}, r_2^{(i)}, \cdots, r_X^{(i)}\}$, $i \in \{1, \cdots, K\}$, be K subsets of $\{1, \cdots, n\}$ such that $|M^{(1)}| = \cdots = |M^{(K)}| = X$, and $r_1^{(i)} < \cdots < r_X^{(i)}, \forall i$. Let

$$R^{(i)} = \Big\{ r^{(i)} \in M^{(i)} : (x^{(1)}, \cdots r^{(i)} \cdots, x^{(K)}) \succ_{\delta_a} (y^{(1)} \cdots y^{(K)}), \\ \forall x^{(j)}, y^{(j)} (j \neq i) \in M^{(j)}, \ y^{(i)} \in M^{(i)} - \{r^{(i)}\} \Big\}.$$

Then, $\cup_i R^{(i)} = \{r_1^{(j)}\}$, for some value of $j \in \{1, \cdots, K\}$. We denote this unique integer in the set $\cup_i M^{(i)}$ by $R(M^{(1)}, M^{(2)}, \cdots, M^{(K)})$.

Proof: Suppose $\exists r \in R^{(i)} - \{r_1^{(i)}\}$. Then, by definition, $(r_X^{(1)}, r_X^{(2)}, \cdots, r, \cdots, r_X^{(K)}) \succ_{\delta_a} (r_X^{(1)}, r_X^{(2)}, \cdots, r_1^{(i)}, \cdots, r_X^{(K)})$, which, however, violates the monotonicity of δ_a. Since this is true for all $i \in K$, $\nexists r \in R^{(i)} - \{r_1^{(i)}\}$, for any i. This implies that $\cup_i R^{(i)} \subseteq \{r_1^{(1)}, r_1^{(2)}, \cdots, r_1^{(K)}\}$. In fact, for any $i, j \in \{1, \cdots, K\}$, $R^{(i)} \cup R^{(j)} \subseteq \{r_1^{(i)}, r_1^{(j)}\}$. We can assume, w.l.o.g, that $i < j$.

Suppose $R^{(i)} \cup R^{(j)} = \{r_1^{(i)}, r_1^{(j)}\}$. This implies that $(r_1^{(1)} \cdots r_1^{(i)} \cdots r_j^{(j)} \cdots r_K^{(K)}) \succ_{\delta_a} (r_1^{(1)} \cdots r_i^{(i)} \cdots r_1^{j}, \cdots r_K^{(K)})$, and $(r_1^{(1)} \cdots r_i^{(i)} \cdots r_1^{(j)} \cdots r_K^{(K)}) \succ_{\delta_a} (r_1^{(1)} \cdots r_1^{(i)} \cdots r_j^{j} \cdots r_K^{(K)})$.

This is not possible, since the two conditions contradict each other. Therefore, $R^{(i)} \cup R^{(j)} = \{r_i\}$, or $\{r_j\}$, or $\emptyset$. Using induction, it is clear that $\bigcup_{i=1:K} R^{(i)} = \{r_1^{(j)}\}$, for some j, or $\emptyset$. Suppose $\bigcup_i R^{(i)} = \emptyset$. This implies $R^{(i)} = \emptyset, \forall i$. This implies that there does not exist an $r^{(i)}$ such that

$$(x^{(1)}, \cdots, r^{(i)}, \cdots, x^{(K)}) \succ_{\delta_a} (y^{(1)}, \cdots, y^{(K)}), \\ \forall x^{(j)}, y^{(j)} \in M^{(j)}, y^{(i)} = M^{(i)} - \{r^{(i)}\}.$$

$$\Rightarrow (r_X^{(1)}, \cdots, r_1^{(i)}, \cdots, r_X^{(K)}) \prec_{\delta_a} (r_1^{(1)}, \cdots, r_2^{(i)}, \cdots, r_1^{(K)})$$
$$\Rightarrow (n, \cdots, r_1^{(i)}, \cdots, n) \prec_{\delta_a} (r_1^{(1)}, \cdots, r_2^{(i)}, \cdots, r_1^{(K)}).$$

This, in turn, implies that either $(n, n, \cdots, r_2^{(i)}, \cdots, n) \succ_{\delta_a} (n, n, \cdots, r_1^{(i)}, \cdots, n)$, which violates monotonicity, or it implies that $\exists j \neq i$ such that

$$(n, n, \cdots r_1^{(j)}, \cdots n, \cdots, n) \succ_{\delta_a} (n, n, \cdots n, \cdots r_1^{(i)}, \cdots, n). \quad (3)$$

Let $\mathcal{J}(i) =$
$\{j : (n, n, \cdots r_1^{(j)} \cdots, n \cdots, n) \succ_{\delta_a} (n, n, \cdots n \cdots, r_1^{(i)} \cdots, n)\}$.
If $\mathcal{J}(i) = \emptyset$ for some i, it implies that $\bigcup_i R^{(i)} \neq \emptyset$.
Let us assume that $\mathcal{J}(i) \neq \emptyset, \ \forall i$. Suppose $j \in \mathcal{J}(i)$ and $j' \in \mathcal{J}(j)$. Eqn. (3) implies that

$$(n, n, \cdots, r_1^{(j')}, \cdots, n, \cdots, n) \succ_{\delta_a} (n, n, \cdots, n, \cdots, r_1^{(j)}, \cdots, n) \\ \succ_{\delta_a} (n, n, \cdots, n, \cdots, r_1^{(i)}, \cdots, n).$$

Consider $\mathcal{J}^{(k)}(i) = \mathcal{J}(\mathcal{J}(\cdots \mathcal{J}(i) \cdots))$.
Then, $j \in \mathcal{J}^{(k)}(i) \Rightarrow j \notin \{\bigcup_{l=0}^{k-1} \mathcal{J}^{(l)}(i)\}$, where $\mathcal{J}^{(0)}(i) = i$.
Suppose $|\bigcup_{l=0}^{(k-1)} \mathcal{J}^{(l)}(i)| = K - 1$. Then,
$\exists$ (only one) $s \notin \{\bigcup_{l=0}^{k-1} \mathcal{J}^{(l)}(i)\}$ such that $s \in \mathcal{J}^{(k)}(i)$.
$\Rightarrow (n, n, \cdots r_1^{(s)} \cdots, n, \cdots, n) \succ_{\delta_a} (n, n, \cdots, n \cdots r_1^{(t)}, \cdots, n)$.

However, (3) implies that $\exists t' \neq s$, such that $(n, n, \cdots, r_1^{(t')}, \cdots, n, \cdots, n) \succ_{\delta_a} (n, n, \cdots, n, \cdots, r_1^{(s)}, \cdots, n)$, $t' \notin \{\bigcup_{l=0}^{k-1} \mathcal{J}^{(l)}(i)\}$. This is a contradiction, implying that $\bigcup_i R^{(i)} \neq \emptyset$. ∎

We consider $t \in \mathbb{N}$, and let $\mathcal{S}_t \subseteq \mathcal{S}$ be a subset of slots and $\mathcal{F}_t \subseteq \mathcal{F}$ be a subset of flights such that $|\mathcal{S}_t| = |\mathcal{F}_t|$. Let $\mathcal{E}$ be an economy such that $\delta \in \Theta$, and $i \in \mathcal{F}_t$. We define

1) $Q_t^i = \{r \in \{1, \cdots, n\} : \exists s \in \mathcal{S}_t$ such that $r = r_i^{\pi}(s)\}$.
2) $b_t^i = \min\{r \in Q_t^i\} = r_i^{\pi}(\mathbf{1}_{\mathcal{S}_t}(i))$.
3) Suppose $b_t^{k(i)} = R(Q_t^{\mathbf{1}(i)}, \cdots, Q_t^{\mathbf{K}(i)})$. Then,
$$\theta_t^i = \begin{cases} k(i), & k(i) \in \mathcal{F}_t \\ i, & k(i) \notin \mathcal{F}_t \end{cases}$$
4) For $A = A(i) = (f_1, \cdots, i, \cdots, f_K)$, $\theta_t(A) = \{\theta_t(f_1), \cdots, \theta_t(i), \cdots, \theta_t(f_K)\}$ with the convention that if $f_j \notin \mathcal{F}_t$, $\theta_t(f_j) = \theta_t(k)$, $k = m(j)$ such that $m = \min\{m : \mathbf{m}(j) \in \mathcal{F}_t\}$, according to P_j^F.
5) $\theta_t(\mathcal{F}_t) = \cup_{i \in \mathcal{F}_t} \theta_t(i)$.

$\theta_t(i)$ is well-defined for any $i \in \mathcal{F}_t$, from Lemma 2, if δ is strongly separable.

Definition 10: Let $\mathcal{E}$ be an economy where $\delta \in \Theta$. Let $\mathcal{S}_t \subseteq \mathcal{S}$, $\mathcal{F}_t \subseteq \mathcal{F}$, such that $|\mathcal{S}_t| = |\mathcal{F}_t|$. Let σ and γ be two one-to-one mappings from $\mathcal{F}$ to $\mathcal{S}$. Then, γ is said to be $\sigma-$**feasible** if for any $f \in \mathcal{F}_t$ such that $\gamma(f) \neq \sigma(f)$, and $A = A(f)$:
$f \in \theta_t(A) \Longrightarrow [\gamma(f) = \mathbf{1}_{\mathcal{S}_t}(f)]$.
$f \notin \theta_t(A) \Longrightarrow [j \in \theta_t(A(f))$, for some $j \in A(f), \ f \in \mathcal{F}_t, \ j \neq f$ and $\gamma(f) = \sigma(j)]$.
The set of $\sigma-$feasible matchings from $\mathcal{F}_t$ to $\mathcal{S}_t$ is denoted by $\mathbb{F}(\sigma, \mathcal{F}_t, \mathcal{S}_t)$.
The above definition of $\sigma-$feasible matchings specifies the procedure of moving from one matching (σ) to another (γ). In particular, if a flight f is called in a claim (some flight in the same airline calls for f to get its most preferred slot), then f is assigned its best slot among those available. If flight f is assigned a new slot without being called in a claim, then some other flight owned by its airline (denoted flight j) is given its top slot, and f is assigned the slot $\sigma(j)$ originally assigned to flight j.

Lemma 3: Let $\mathcal{E}$ be an economy in which $\delta \in \Theta$. Let $\mathcal{S}_t \subseteq \mathcal{S}$, $\mathcal{F}_t \subseteq \mathcal{F}$, such that $|\mathcal{S}_t| = |\mathcal{F}_t|$. Let σ^t be any one-to-one mapping from $\mathcal{F}$ to $\mathcal{S}$. Suppose that $\forall f \in \mathcal{F}_t \cap \theta_t(A(f))$, $\sigma^t(f) \neq \mathbf{1}_{\mathcal{S}_t}(f)$. Then, there exists a matching $\sigma^{t+1} \in \mathbb{F}(\sigma, \mathcal{F}_t, \mathcal{S}_t) - \{\sigma\}$.

Proof: We define, for flight j and airline $a = A(j)$, $\mathcal{C}_j^{(a)} = \{f_i \in a : \theta_t(f_i) = j, \ f_i \notin \theta_t(A(j))\} \cup \{j\}$, $\forall j$ such that $|\theta_t(A(j))| < |A(j)|$, $|\mathcal{C}_j^{(a)}| > 1$. We note that $\sum_j |\mathcal{C}_j^{(a)}| = |A(j)|$.

Let $\mathcal{C}^1 = \cup_a \cup_{j \in a} \mathcal{C}_j^{(a)}$. Let $j \in \mathcal{C}_j^{(A(j))} = \theta_t(\mathcal{C}_j^{(A(j))})$. We note that each element in the set $\mathcal{F}_t - \mathcal{C}^1$ is a singleton set. Let $\mathcal{G}$ be a

directed graph defined on $\{\mathcal{C}^1 \cup [\mathcal{F}_t - \mathcal{C}^1]\}^2$, with arcs described by $\forall C \in \mathcal{C}^1$, $\forall f \in \mathcal{F}_t - \mathcal{C}^1$, $[(C,f) \in \mathcal{G} \Leftrightarrow \sigma^t(f) = \mathbf{1}_{\mathcal{S}_t}(\theta_t(C))]$, and $[(f,C) \in \mathcal{G} \Leftrightarrow \mathbf{1}_{\mathcal{S}_t}(f) \in \sigma^t(C)]$.
$\forall C, C' \in \mathcal{C}^1$, $(C, C') \in \mathcal{G} \Leftrightarrow \mathbf{1}_{\mathcal{S}_t}(\theta_t(C)) \in \sigma^t(C')$
$\forall f, f' \in \mathcal{F}_t - \mathcal{C}^1$, $(f, f') \in \mathcal{G} \Leftrightarrow \sigma^t(f') = \mathbf{1}_{\mathcal{S}_t}(f)$.

Since $\forall f \in \mathcal{F}_t \cap \theta_t(A(f))$, $\sigma^t(f) \neq \mathbf{1}_{\mathcal{S}_t}(f)$, each vertex in $\mathcal{G}$ has a single outgoing arc (*i.e.*, the out-score is one). Therefore, the directed graph $\mathcal{G}$ contains a cycle. Let us denote this cycle by $T = \{c_j\}_{j=1,\cdots,J}$ where $(c_j, c_{j+1}) \in \mathcal{G}$ for all j, and $c_{J+1} = c_1$. The new mapping from $\mathcal{F}$ to $\mathcal{S}$, denoted σ^{t+1}, is given by
$\forall j$, $[c_j \in \mathcal{C}^1] \Rightarrow [\sigma^{t+1}(\theta_t(c_j)) = \mathbf{1}_{\mathcal{S}_t}(\theta_t(c_j)) \in \sigma^t(c_{j+1})]$, and
$[f \in c_j - \{\theta_t(c_j)\}$, and $\sigma^t(f) \in \{\mathbf{1}(\theta_t(c_{j-1})), \mathbf{1}(c_{j-1})\}] \Rightarrow [\sigma^{t+1}(f) = \sigma^t(\theta_t(c_j))]$.
$\forall j$, $[c_j \notin \mathcal{C}^1] \Rightarrow [\sigma^{t+1}(c_{j+1}) = \mathbf{1}_{\mathcal{S}_t}(a_j)]$.
The mapping σ^{t+1} satisfies the conditions of Defn. 10, and therefore $\sigma^{t+1} \in \mathbb{F}(\sigma, \mathcal{F}_t, \mathcal{S}_t) - \{\sigma^t\}$ (*i.e.*, σ^{t+1} is σ^t-feasible). ■

The Generalized Top Trading Cycles (GTTC) algorithm [14] consists of constructing successive σ^t-feasible matchings, starting from an initial matching σ^0.

Algorithm 2: **(Generalized Top Trading Cycles Algorithm):** Let $\mathcal{E} = (\mathcal{F}, \mathcal{A}, \pi, \delta, \sigma^0)$ be an economy where $\delta \in \Theta$. The generalized top trading cycles sequence is the sequence $\{\mathcal{F}_t, \mathcal{S}_t, \sigma^t\}_{t=0,\cdots,T}$ defined by

1) $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{S}_0 = \mathcal{S}$
2) $\forall t \in \{0, \cdots, T\}$,
   if $\mathbb{A}_t = \{f \in \theta(\mathcal{F}_t) : \sigma^t(f) = \mathbf{1}_{\mathcal{S}_t}(f)\} \neq \emptyset$, then
   a) $\mathcal{F}_{t+1} = \mathcal{F}_t - \mathbb{A}_t$
   b) $\sigma^{t+1} = \sigma^t$
   c) $\mathcal{S}_{t+1} = \mathcal{S}_t - \sigma^t(\mathbb{A}_t)$

   if $\mathbb{A}_t = \emptyset$, then
   a) $\mathbb{B}_t = \{f \in \theta(\mathcal{F}_t) : \psi_{\sigma^t}(f) \neq \sigma^t(f)\}$
   b) $\mathcal{F}_{t+1} = \mathcal{F}_t - \mathbb{B}_t$
   c) $\sigma^{t+1}(f) = \begin{cases} \sigma^t(f), \ f \notin \mathcal{F}_t \\ \psi_{\sigma^t}(f), \ f \in \mathcal{F}_t \end{cases}$
   d) $\mathcal{S}_{t+1} = \mathcal{S}_t - \psi_{\sigma^t}(\mathbb{B}_t) = \sigma^{t+1}(\mathcal{F}_{t+1})$
   where ψ_{σ^t} is the maximal element in $\mathbb{F}(\sigma^t, \mathcal{F}_t, \mathcal{S}_t)$
3) $\mathcal{F}_{T+1} = \mathcal{S}_{T+1} = \emptyset$

Proposition 1: Let $\mathcal{E} = (\mathcal{F}, \mathcal{A}, \pi, \delta, \sigma^0)$ be an economy in which $\delta \in \Theta$, δ strongly separable. Let σ^T be the matching obtained through the GTTC sequence given by $\mathbb{T} = \{\mathcal{F}_t, \mathcal{S}_t, \sigma^t\}_{t=0,\cdots,T}$. Then, $\sigma^T \in \mathcal{W}_{coop}(\mathcal{E})$.

Proof: Let K be the number of flights belonging to the largest airline in the GDP, *i.e.*, $K = \max_i |A(i)|$. Let $t(f)$ be the stage at which $\sigma^{t+1}(f) = \sigma^T(f)$. Let $p = (p_s)_{s \in \mathcal{S}}$ be the price vector defined by $p_s = \frac{1}{K^t}$, $\forall s \in \psi_{\sigma^t}(\mathbb{B}_t) \cup \mathbb{A}_t$. Consider any $A = (f_1, \cdots, f_{K'}) \in \mathcal{A}$. Then,
$\sum_{i=1}^{K'} p_{\sigma^T(f_i)} = \sum_{i=1}^{K'} \frac{1}{K^{t(f_i)}} = \sum_{i=1}^{K'} p_{\sigma^0(f_i)}$.
Therefore, by Definition 7, $\forall i$, $\sigma^T \in \Delta(p, i)$. Let us suppose, without loss of generality, that $t(f_1) \leq t(f_2) \leq \cdots \leq t(f_{K'})$.

We need to prove that $\sigma^T \in \mathcal{W}_{coop}(\mathcal{E})$. Let us assume to the contrary, that it, that $\sigma^T \notin \mathcal{W}(p, A)$. Then, there exists $\sigma' \in \Delta(p, f_1)$ such that $\sigma' \delta_j(\pi) \sigma^T$, $\forall j \in A(f_1)$.
Claim: $\sigma'(A) \subseteq \mathcal{S}_t(f_1)$
Suppose not. Then, there exists $m' \in \sigma'(A) \cap \mathcal{S}_t$, $t < t(f_1)$. Therefore,
$p_{m'} \geq \frac{1}{K^t} \geq \frac{1}{K^{t(f_1)-1}} = \frac{K}{K^{t(f_1)}} \geq \frac{K'}{K^{t(f_1)}} \geq \sum_{i=1}^{K'} \frac{1}{K^{t(f_i)}} = w(A)$, say. Now, if $p_{m'} = w(A)$, the budget constraint (Definition 7) implies that $\sum_{j \in \sigma'(A) - \{m'\}} p_j = 0$, which in turn means that $\forall j \in \sigma'(A) - \{m'\}$, $p_j = 0$, which is not possible. Similarly, $p_{m'} > w(A)$ implies that $\sum_{j \in A} p_{\sigma'(j)} \geq \sum_{j \in A} p_{\sigma^0(j)}$, which in turn contradicts $\sigma' \in \Delta(p, f_1)$. This proves the claim that $\sigma'(A) \subseteq \mathcal{S}_t(f_1)$. Let $i^* \in A$ such that $f_1 = \theta_{t(f_1)}(i^*)$. From the definition of the GTTC sequence $\mathbb{T}$, $S_{i^*}^{\pi}(\sigma^T) = (b_{t(f_1)}^{f_1}, \cdots, b_{t(f_{K'})}^{f_{K'}})$, or some permutation of it, where $b_{t(f_i)}^{f_i} = R(Q_{t(1)}^{\mathbf{1}(i^*)}, \cdots, Q_{t(K')}^{\mathbf{K'}(i^*)})$, $t^{(1)} \neq \cdots \neq t^{(K')} \in \{t(f_1), \cdots, t(f_{K'})\}$. Suppose (without loss of generality) that $S_{i^*}^{\pi}(\sigma^T) = (b_{t(f_2)}^{f_2}, \cdots, b_{t(f_1)}^{f_1}, \cdots, b_{t(f_{K'})}^{f_{K'}})$, *i.e.*, $t^{(i)} = t(f_1)$. Then, $\sigma' \delta_{i^*}(\pi) \sigma^T \Rightarrow S_{i^*}^{\pi}(\sigma') \succeq_\delta S_{i^*}^{\pi}(\sigma^T)$

$$\begin{aligned} \Rightarrow S_{i^*}^{\pi}(\sigma') &= (r_{\mathbf{1}(i^*)}^{\pi}(\sigma'), \cdots, r_{\mathbf{i}(i^*)}^{\pi}(\sigma'), \cdots, r_{\mathbf{K'}(i^*)}^{\pi}(\sigma')) \\ &\succeq (b_{t(f_2)}^{f_2}, \cdots, b_{t(f_1)}^{f_1}, \cdots, b_{t(f_{K'})}^{f_{K'}}). \end{aligned} \quad (4)$$

But $b_{t(f_1)}^{f_1} = R(Q_{t(f_2)}^{\mathbf{1}(i^*)}, \cdots, Q_{t(f_1)}^{\mathbf{i}(i^*)}, \cdots, Q_{t(f_{K'})}^{\mathbf{K'}(i^*)})$. Therefore, by definition, $\forall x^{(j)}, y^{(j)} \in Q_{t(f_j)}^{\mathbf{j}(i^*)}$, $\forall y^{(i)} \in Q_{t(f_1)}^{\mathbf{i}(i^*)}$,

$$(x^{(1)}, \cdots, b_{t(f_1)}^{f_1}, \cdots, x^{(K')}) \succ_\delta (y^{(1)}, \cdots, y^{(i)}, \cdots, y^{(K')}). \quad (5)$$

Therefore, from Eqns. (4-5), it follows that $\sigma'(a_1) = \sigma^T(f_1)$.
Let $A' = A - \{f_1\}$. Then, $t(f_2) = \min_{j \in A'} t_j$.
Claim: $\sigma'(A') \subseteq \mathcal{S}_t(f_2)$.
Suppose not. Then, there exists $m' \in \sigma'(A') \cap \mathcal{S}_t$, $t < t(f_2)$. Therefore,
$p_{m'} \geq \frac{1}{K^t} \geq \frac{1}{K^{t(f_2)-1}} = \frac{K}{K^{t(f_2)}} \geq \sum_{i=2}^{K'} \frac{1}{K^{t(f_i)}} = w(A')$, say. Now, if $p_{m'} = w(A')$, the budget constraint (Definition 7) implies that $\sum_{j \in \sigma'(A') - \{m'\}} p_j = 0$, which in turn means that $\forall j \in \sigma'(A') - \{m'\}$, $p_j = 0$, which is not possible. Similarly, $p_{m'} > w(A')$ implies that $\sum_{j \in A'} p_{\sigma'(j)} \geq \sum_{j \in A'} p_{\sigma^0(j)}$, which in turn contradicts $\sigma' \in \Delta(p, f_1)$.

As before, by the definition of $\mathbb{T}$, $\sigma'(f_2) = \sigma^T(f_2)$. Similarly, it is possible to show, for $i = 1, \cdots, K'-1$, that $\sigma'(f_i) = \sigma^T(f_i)$. Since $\sigma' \in \Delta(p, f_1)$, then, $p_{\sigma'(f_{K'})} \leq p_{\sigma^T(f_{K'})} = \frac{1}{K^{t(f_{K'})}}$. The definition of $\mathbb{T}$ implies that $\sigma^T(f_{K'}) = \mathbf{1}_{\mathcal{S}_{t(f_{K'})}}(f_{K'})$. Since $\forall i = 1, \cdots, K'-1$, $\sigma'(f_i) = \sigma^T(f_i)$, and $\sigma^T(f_{K'}) = \mathbf{1}_{\mathcal{S}_{t(f_{K'})}}(f_{K'})$, $\sigma' \delta_{i^*} \sigma^T$ contradicts the monotonicity of δ (Definition 2).

Therefore, for any $A \in \mathcal{A}$, $\sigma^T \in \mathcal{W}(p, A)$. ■

We have shown that in the case of economies in which the preference extension rule δ is neutral, monotone, transfer-consistent and strongly separable, the GTTC sequence terminates in an allocation that is a cooperative weak price equilibrium, and is therefore in the core of the economy from Lemma 1. ■

*Example 5 (**A GDP instance using the GTTC algorithm**):* Consider a GDP with 6 flights, 6 slots, and 4 airlines.
Let $\mathcal{F}_0 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$, $\mathcal{S}_0 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $\mathcal{A} = \{a, b, c, d\}$.
$\mathcal{F}_a = \{f_1, f_2\}$, $\mathcal{F}_b = \{f_3\}$, $\mathcal{F}_c = \{f_4\}$, and $\mathcal{F}_d = \{f_5, f_6\}$.
$\mathcal{S}_a = \{s_1, s_2\}$, $\mathcal{S}_b = \{s_3\}$, $\mathcal{S}_c = \{s_4\}$, and $\mathcal{S}_d = \{s_5, s_6\}$.

The preference profiles for each of the flights, as given by the airlines are
$Q(f_1) = (s_2, s_3, s_6, s_4, s_5, s_1)$; $Q(f_2) = (s_2, s_3, s_6, s_4, s_5, s_1)$;
$Q(f_3) = (s_4, s_5, s_3, s_2, s_1, s_6)$; $Q(f_4) = (s_6, s_3, s_4, s_2, s_1, s_5)$;
$Q(f_5) = (s_2, s_5, s_4, s_6, s_3, s_1)$; $Q(f_6) = (s_2, s_5, s_4, s_6, s_3, s_1)$.
Suppose flights extend their preferences over assignments to the other flights in their airline in a lexicographic manner. We now apply the Generalized Top Trading Cycles Algorithm (Algorithm 2) to the exchange of slots in this GDP.
Iteration **t=0:** $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{S}_0 = \mathcal{S}$, $\sigma^0(f_i) = s_i$.
$\theta_0(a) = \{f_1, f_2\}$, $\theta_0(b) = \{f_3\}$, $\theta_0(c) = \{f_4\}$, $\theta_0(d) = \{f_5, f_6\}$.
$\mathbb{A}_0 = \{f_2\} \neq \emptyset$.
t=1: $\mathcal{F}_1 = \{f_1, f_3, f_4, f_5, f_6\}, \mathcal{S}_1 = \{s_1, s_3, s_4, s_5, s_6\}, \sigma^1(f_i) = s_i$.
$\theta_1(a) = \{f_1\}, \theta_1(b) = \{f_3\}, \theta_1(c) = \{f_4\}, \theta_1(d) = \{f_5, f_6\}$.
$\mathbb{A}_1 = \{f_5\} \neq \emptyset$.
t=2: $\mathcal{F}_2 = \{f_1, f_3, f_4, f_6\}$, $\mathcal{S}_2 = \{s_1, s_3, s_4, s_6\}$, $\sigma^2(f_i) = s_i$.
$\theta_2(a) = \{f_1\}$, $\theta_2(b) = \{f_3\}$, $\theta_2(c) = \{f_4\}$, $\theta_2(d) = \{f_6\}$.
$\mathbb{A}_2 = \emptyset$, $\mathbb{B}_2 = \{f_4, f_6\}$. $\gamma(f_1) = s_3$, $\gamma(f_3) = s_4$

t=3: $\mathcal{F}_3 = \{f_1, f_3\}, \mathcal{S}_3 = \{s_1, s_3\}, \sigma^3(f_i) = \begin{cases} s_i, i \in \{1,2,3,5\} \\ s_6, i = 4 \\ s_4, i = 6 \end{cases}$

$\theta_3(a) = \{f_1\}$, $\theta_3(b) = \{f_3\}$; $\mathbb{A}_3 = \{f_3\} \neq \emptyset$.
t=4: $\mathcal{F}_4 = \{f_1\}, \mathcal{S}_4 = \{s_1\}, \sigma^4(f_i) = \sigma^3(f_i)$, $\forall i$.
$\theta_4(a) = \{f_1\}$; $\mathbb{A}_4 = \{f_1\} \neq \emptyset$.
The final assignment of slots to flights (airlines) is given by
$\mathcal{S}_a = \{s_1, s_2\}$, $\mathcal{S}_b = \{s_3\}$, $\mathcal{S}_c = \{s_6\}$, and $\mathcal{S}_d = \{s_5, s_4\}$, and is in the core of the economy.