

# Near-Optimal Mechanisms for Resource Allocation Without Monetary Transfers

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## Abstract

We study the problem in which a central planner sequentially allocates a single resource to multiple strategic agents using their utility reports at each round, but without using any monetary transfers. We consider general agent utility distributions and two standard settings: a finite horizon  $T$  and an infinite horizon with  $\gamma$  discounts. We provide general tools to characterize the convergence rate between the optimal mechanism for the central planner and the first-best allocation if true agent utilities were available. This heavily depends on the utility distributions, yielding rates anywhere between  $1/\sqrt{T}$  and  $1/T$  for the finite-horizon setting, and rates faster than  $\sqrt{1-\gamma}$ , including exponential rates for the infinite-horizon setting as agents are more patient  $\gamma \rightarrow 1$ . On the algorithmic side, we design mechanisms based on the promised-utility framework to achieve these rates and leverage structure on the utility distributions. Intuitively, the more flexibility the central planner has to reward or penalize any agent while incurring little social welfare cost, the faster the convergence rate. In particular, discrete utility distributions typically yield the slower rates  $1/\sqrt{T}$  and  $\sqrt{1-\gamma}$ , while smooth distributions with density typically yield faster rates  $1/T$  (up to logarithmic factors) and  $1-\gamma$ .

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## 1 Introduction

Resource allocation or auction design is one of the foundational problems in mechanism design and has been thoroughly studied across economics, operations research, and computer science. In this problem, a planner aims to allocate resources to  $n$  strategic agents while maximizing some notion of global welfare, using reports/bids from the agents on their utility for the resources. Importantly, agent reports are strategic hence the planner only has limited or partial information about the agents’ preferences. To achieve positive results, fundamental tools in mechanism design are money transfers between the agents and the planner. The large majority of approaches heavily rely on the use of such money payments to align incentives between the strategic users and the central planner. For instance, the celebrated Vickrey-Clarke-Groves (VCG) mechanism [Clarke, 1971, Groves, 1973, Vickrey, 1961] ensures incentive-compatibility while implementing optimal allocations by carefully selecting payment mechanisms. In many applications, however, money transfers may be impractical. Enforcing payments from beneficiaries to the central planner is typically undesirable when allocating public services: from designing antipoverty programs [Bardhan and Mookherjee, 2005], choosing school assignments [Ashlagi and Shi, 2016, Russell and Ebbert, 2011], allocating scientific equipment [Linden, 2022], to distributing food to food banks. Cloud computing is another application in which memory or computing resources may be scarce, hence enforcing a scheduling policy is necessary, but monetary transfers may be unpopular or inadequate, especially when allocating internal resources [Ng, 2011].

**Resource allocation without monetary transfers.** Following these motivations, we consider the setting in which monetary transfers are not allowed and focus on single resource allocation for simplicity. The reports of the agents are strategic and hence form a perfect Bayesian equilibrium. In a full-information setting where the planner has direct access to all agent utilities, the optimal policy is to allocate the resource to an agent with maximum utility, also known as the *first-best* allocation. Unfortunately, prohibiting the use of money severely limits approximating the first-best allocation, in light of negative results arising from Arrow’s impossibility theorem [Arrow, 2012]. For a single allocation, whenever there are at least 3 or more alternative allocations, it is impossible to design an incentive-compatible mechanism that is non-dictatorial [Gibbard, 1973, Satterthwaite, 1975]. Indeed, without penalization from monetary transfers, agents can report as large utilities as possible for the resource: utility reports from the agents

carry virtually no information.<sup>1</sup> We note that mechanism design without monetary transfers can be successful in specific settings, e.g. if agent preferences are single-peaked, when matching users pairwise in a bipartite graph (see Schummer and Vohra [2007] for an overview of classical results), or in interdomain routing [Feigenbaum et al., 2007, Levin et al., 2008]; these exploit specific properties of the problem that are difficult to generalize.

In a repeated allocation setting, however, the central planner can use future allocations to align incentives by favoring or disfavoring agents. We consider two classical repeated settings in the literature: an infinite-horizon setting with a fixed utility time discount factor  $\gamma \in [0, 1)$  and a finite-horizon  $T$  setting without discounts. To illustrate the benefit of repeated allocations, the central planner can for instance fix budgets for each agent that serve as virtual currency. This is routinely used in practice [Walsh, 2014] including for allocating computing resources in distributed systems [Ng, 2011] or food to food banks [Prendergast, 2022].

**Characterizing the convergence to first-best allocation.** From the folk theorem [Friedman, 1971, Fudenberg et al., 2009], it is known that one can approximate the optimal first-best allocation arbitrarily well as agents are more patient  $\gamma \rightarrow 1$ , or when the horizon grows  $T \rightarrow \infty$ . However, this result is not quantitative in nature and does not provide direct insights into which allocation mechanisms could be optimal. The goal of this paper is to provide answers to the following questions: (1) What is the welfare gap for the central planner between the full-information case (first-best allocation) and the strategic resource allocation setting for general utility distributions? In particular, how does it depend on the utility distributions of the agents, and the parameters  $\gamma$  and  $T$ ? (2) On the algorithmic side, what are (near-)optimal allocation mechanisms without monetary transfers?

## 1.1 Related works

**Single-time allocation without money.** We first discuss the literature on one-shot resource allocation without money. When there are two resources and agents share the same utility distribution, Miralles [2012] characterizes optimal mechanisms and shows that these can be constructed as combinations of lotteries and an auction-like strategies. For several resources and additive utilities for the agents, Cole et al. [2013], Guo and Conitzer [2010], Han et al. [2011] give characterizations on the competitive ratio of strategy-proof mechanisms compared to the first-best allocation, without making priors on the agent utility distributions. Slightly different from our setting without money, the line of work including Chakravarty and Kaplan [2013], Condorelli [2012], Hartline and Roughgarden [2008], Hoppe et al. [2009] studies the setting in which the central planner can impose payments from the agents but cannot redistribute these between agents. This setting is often referred to *money burning*: any utility signal in the form of payment to the central planner is lost. As discussed above, impossibility results due to Arrow’s theorem generally prohibit the central planner from converging to the first-best allocation for a single allocation, while our focus is on designing mechanisms that converge to this optimal allocation as fast as possible.

**Repeated allocation without money.** Closest to our work, we now discuss the literature on the repeated allocation setting without transfers. While the folk theorem Fudenberg et al.

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<sup>1</sup>In this case, agent reports can at most signal whether agents have negative, zero, or positive utility for the resource.

[2009] guarantees the existence of a mechanism that asymptotically converges to the first-best allocation, Guo et al. [2009] explicitly provides a mechanism that achieves at least 75% of the optimal welfare in the infinite-horizon setting provided agents are sufficiently patient. On the other hand, their mechanism may not converge to the optimal first-best allocation as  $\gamma \rightarrow 1$ . Guo and Hörner [2015] characterize the optimal allocation in the stylized setting in which the central planner has a fixed cost and decides at each iteration whether to allocate the resource to a single agent whose utilities evolve according to a two-state Markov chain.

We previously introduced budget-based mechanisms, also called *artificial currency* or *script* strategies Friedman et al. [2006], Johnson et al. [2014], Kash et al. [2007, 2015], in which each agent is given an initial budget that can be used as currency to bid on resources at each round. These have the advantage of being very simple and interpretable and their implementation in several real-world applications have motivated the study of their theoretical performance including [Budish, 2011, Gorokh et al., 2021, Jackson and Sonnenschein, 2007]. For instance, Gorokh et al. [2021] shows that in the finite-horizon  $T$  setting and for discrete utility distributions their convergence to the first-best allocation decreases as  $1/T$ . However, their results hold under a weaker notion of approximate Nash equilibrium in which incentive-compatibility constraints are only satisfied approximately up to  $\sqrt{\log T/T}$  terms. In a similar spirit to artificial currencies, Elokda et al. [2022] recently proposed the use of karma mechanisms and discussed its applications in ride-hailing platforms, in which agents can receive an additional budget reward for accepting to give the resource to other agents.

The work of Balseiro et al. [2019] is closest to the present paper. They consider the discounted infinite-horizon setting where agent utility distributions admit densities bounded above and below by some fixed constant. With these distributional assumptions and for two agents, they provide allocation mechanisms that converge to the first-best allocation as  $\Theta(1 - \gamma)$  for  $\gamma \rightarrow 1$  and show that this is the best possible among all mechanisms up to constants. In comparison, they show that budget-based mechanisms have a convergence rate of at most  $\Omega(\sqrt{1 - \gamma})$ , hence budget-based strategies are suboptimal under these utility distribution assumptions. For  $n \geq 3$  agents, they prove a  $\mathcal{O}((1 - \gamma)^{1/(n+4)})$  convergence rate for their mechanism, leaving a significant gap compared to their  $\Omega(1 - \gamma)$  lower bound.

Compared to these previous works, we give tools to prove convergence rates for *general* utility distributions. As an example, within the specific assumptions from Balseiro et al. [2019], we provide an allocation mechanism that achieves the optimal convergence rate  $\Theta(1 - \gamma)$  for any number of agents  $n$ . More precisely, the upper (resp. lower) bound holds whenever the utility distributions have densities lower (resp. upper) bounded by a constant. Our work also aims to reconcile the analysis for the finite-horizon and infinite-horizon settings by using the same techniques for both. Further details on our results are given in the following section.

## 1.2 Overview of the contributions

Instead of focusing on the gap between the first-best allocation objective and the optimal allocation mechanism to maximize social welfare, we aim to characterize the complete region of achievable utilities in the strategic setting. In particular, this would also give the gap between the full-information and strategic settings for any choice of non-negative utility weights for the agents (the standard social welfare objective corresponds to the equalitarian choice of giving all agents the same weight). This allows for formulating the problem in a dynamic programming form, referred to as the *promised utility* framework [Abreu et al., 1990, Balseiro et al., 2019, Spear and Srivastava, 1987, Thomas and Worrall, 1990]. In the finite-horizon setting,

a Bellman-like operator links the achievable region at times  $T \geq 1$  and  $T + 1$ , and in the infinite-horizon setting the achievable region satisfies a fixed-point equation for this Bellman operator. Our techniques build upon those of the promised utility literature and in particular of [Balseiro et al. \[2019\]](#), which focuses on the infinite-horizon discounted setting. With this reformulation, the question becomes how to characterize for any choice of non-negative weights  $\alpha$  the gap  $\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}} \alpha^\top \mathbf{x}$  where  $\mathcal{U}$  and  $\mathcal{U}^*$  denote the achievable region in the strategic setting and in the full-information setting respectively.

**Two geometrical tools.** We first develop two main geometrical tools for the  $\gamma$ -discounted infinite-horizon setting, Lemmas 3 and 6, to respectively prove lower bounds and upper bounds on the achievable region  $\mathcal{U}$ , and hence respectively prove upper and lower bounds on the gap between optimal mechanisms without money and the first-best allocation. In the finite-horizon  $T$  setting, we show that the same tools can be used up to logarithmic factors by setting  $1 - \gamma \approx 1/T$  (see Lemmas 12 and 14), which unifies the analysis for both settings.

Importantly, all tools to prove upper bounds on this gap are constructive hence correspond to an implementable mechanism described within the paper.

**General bounds for the optimal convergence rate to first-best allocation.** As a consequence of these tools, we show (Theorem 7) a universal  $\mathcal{O}(\sqrt{1 - \gamma})$  (resp.  $\mathcal{O}(1/\sqrt{T})$ ) upper bound on the convergence rate to the first-best allocation for the  $\gamma$ -discounted infinite-horizon (resp. finite-horizon  $T$ ) setting. In particular, this holds for any number of agents and any utility distributions (under the only assumption that these have support bounded in  $[0, \bar{v}]$  for some fixed utility value  $\bar{v}$  known a priori).

However, this uniform upper bound on the convergence rate may not be optimal in general. Depending on the profile of agent utility distributions, arbitrary rates faster than  $\sqrt{1 - \gamma}$  are possible for the infinite-horizon setting. In fact, while the distributional assumptions from [Balseiro et al. \[2019\]](#) prohibited faster rates than  $1 - \gamma$ , we give non-trivial examples of exponential convergence rates as fast as  $(1 - \gamma)e^{-c/\sqrt{1 - \gamma}}$  (see Theorem 30 which essentially corresponds to the case when any agent  $i \in [n]$  can have ties for the first-best allocation with non-zero probability). For the finite-horizon case, faster rates than  $1/\sqrt{T}$  are possible but they cannot converge faster than  $1/T$  (Lemma 13) as a result of the boundary at time  $T$ . In particular, the exponential rates from the infinite-horizon setting do not carry to the finite-horizon case.

We then provide general upper and lower bounds for arbitrary distributions (Theorem 11) that can be used to prove convergence rates between  $\sqrt{1 - \gamma}$  and  $1 - \gamma$  (resp.  $1/\sqrt{T}$  and  $1/T$  up to logarithmic factors). While the upper and lower bounds do not match in general, both share the same form and give insights into (1) the dependency of the convergence rate in terms of the utility distributions and (2) corresponding optimal allocation mechanisms. Intuitively, the main driver is whether the central planner has enough flexibility to reward or penalize any agent in future rounds, without incurring large social welfare costs. Accordingly, our mechanisms use this flexibility to align incentives: the larger the flexibility, the closer one can approach the optimal first-best allocation.

**Special cases and examples.** In particular, we can give necessary (Theorem 10) and sufficient (Theorem 8) conditions to reach the faster rate  $1 - \gamma$  from [Balseiro et al. \[2019\]](#) in the infinite-horizon setting, which significantly generalize their results (the necessary and sufficient conditions are both satisfied under their distributional assumptions). In the finite-horizon set-

ting, this corresponds to the fastest rate  $1/T$  up to logarithmic factors. At a high level, to achieve these faster rates it suffices that any agent  $i \in [n]$  shares some common utility support with some other agent  $j \neq i$ : the mechanism can easily then implement future penalties on agent  $i$  by allocating to agent  $j$  when their utilities are sufficiently close (and optimal within all agents) in following rounds, and similarly implement future rewards by allocating to agent  $i$  when the utility of agent  $j$  is sufficiently close (and optimal). The sufficient conditions can therefore be written in terms of a graph between agents: an edge  $i \rightarrow j$  is formed whenever the mechanism can use agent  $j$  to easily reward agent  $i$ . The condition becomes whether this graph can be partitioned into non-trivial strongly connected components.

The previous example covers the case of smooth distributions (Corollaries 15 and 16). On the other extreme, we also take the example of discrete utility distributions (Theorem 17 and Corollary 18). In this case, the optimal convergence rate is usually  $\sqrt{1-\gamma}$  (resp.  $1/\sqrt{T}$ ) for the infinite-horizon (resp. finite-horizon) setting. In specific cases for which every agent  $i \in [n]$  shares in its utility support a non-trivial value with some other agent  $j \neq i$ , the rate of convergence is exponential  $\mathcal{O}((1-\gamma)e^{-c/\sqrt{T-\gamma}})$  for the infinite-horizon setting, and reaches the optimal  $1/T$  rate up to logarithmic factors for the finite-horizon  $T$  case.

In comparison, Gorokh et al. [2021] showed that for discrete distributions, one can reach a convergence rate  $1/T$  to the first-best allocation with artificial currencies but in a weaker form of Bayesian equilibrium in which incentive-compatibility constraints are only satisfied up to  $\sqrt{\log T/T}$  terms. Our previous results show that the optimal rate at the perfect Bayesian equilibrium is exactly  $\Theta(1/\sqrt{T})$  in general, when incentive-compatibility constraints are perfectly enforced (this can be potentially much faster if the distributions have convenient geometry). Intuitively, this implies that the incentive-compatibility gap from Gorokh et al. [2021] can be traded for the social welfare gap from the central planner. Unlike Gorokh et al. [2021], our upper bound holds for arbitrary distributions.

### 1.3 Organization of the paper

After defining the model and the framework used for our constructions in Section 2, we give a detailed summary of our main results in Section 3, explaining how the geometric tools can be used to derive results in both the discounted infinite-horizon and the finite-horizon settings, and giving examples for simple choices of utility distributions. The two geometric tools are proved in Section 4, which also presents their strongest form. We then derive our characterizations for the discounted infinite-horizon setting in Section 5 and the finite-horizon setting in Section 6.

### 1.4 Notations

For  $n \geq 1$ , we use the notation  $[n] := \{1, \dots, n\}$ . We also write  $\mathbb{R}_+ := [0, \infty)$  and  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ . By default, we use  $\|\cdot\|$  for the Euclidean norm  $\|\cdot\|_2$ , otherwise, we will always specify the norm within the paper. For any  $\mathbf{x} \in \mathbb{R}^n$  and  $r > 0$ , we denote  $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| \leq r\}$  and  $B^\circ(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| < r\}$ , the closed and open balls centered at  $\mathbf{x}$  respectively. We let  $S_{n-1} = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y}\| = 1\}$  be the unit  $n$ -dimensional sphere. For a compact  $C \subset \mathbb{R}^n$  and any  $\mathbf{x} \in \mathbb{R}^n$ , we define  $d(\mathbf{x}, C) = \max_{\mathbf{y} \in C} \|\mathbf{y} - \mathbf{x}\|$ , where the maximum is attained since  $C$  is compact. Next, for any subset  $S \subset [n]$  and  $\mathbf{x} \in \mathbb{R}^n$ , we denote  $\mathbf{x}_S := (x_i)_{i \in S}$ . For  $i \in [n]$  we also denote  $\mathbf{x}_{-i} = (x_j)_{j \neq i}$ . We denote  $\Delta_n := \{\mathbf{x} \in \mathbb{R}_+^n, \sum_{i \in [n]} x_i \leq 1\}$  the probability simplex (augmented with a do-nothing option). For a distribution  $\mathcal{D}$  on  $\mathbb{R}$ , we denote by  $\text{Supp}(\mathcal{D})$  its support, that is, the complement of the union of all open sets  $O$  such that  $\mathbb{P}_{X \sim \mathcal{D}}(X \in O) = 0$ .

We use standard Landau notations. For two functions  $f$ , and  $g$ , we write  $f(x) = \mathcal{O}(g(x))$  if there exists a constant  $C$  such that  $f(x) \leq Cg(x)$  for all  $x$  in the support of  $f$ . We may specify  $f(x) \underset{x \rightarrow a}{=} \mathcal{O}(g(x))$  when the inequality  $f(x) \leq Cg(x)$  only holds in a neighborhood of  $a$  within the support of  $f$ . We write  $f(x) = \Omega(g(x))$  when  $g(x) = \mathcal{O}(f(x))$ , and  $f(x) = \Theta(g(x))$  if both  $f(x) = \mathcal{O}(g(x))$  and  $f(x) = \Omega(g(x))$  hold. Last, we write  $f(x) \underset{x \rightarrow a}{=} \omega(g(x))$  if  $0 \leq h(x)g(x) \leq f(x)$  where  $h(x) \rightarrow \infty$  as  $x \rightarrow a$ .

## 2 Problem formulation and preliminaries

In this section, we detail the considered model and give useful preliminaries about the promised utility framework. Most of the statements in this section can be considered as relatively standard in the literature. We include their proofs in Appendix A for completeness.

### 2.1 Model definitions

We consider the sequential resource allocation problem in which a central planner repeatedly allocates a single resource between  $n$  agents. At every round  $t \geq 1$ , each strategic agent  $i \in [n]$  observes their private realized utility  $u_i(t)$ , then reports a value  $v_i(t)$  to the central planner only, and last, having observed the reported values  $\mathbf{v}(t) = (v_i(t))_{i \in [n]}$  the central planner then allocates the single resource to one or none of the agents, which we represent by the index  $\hat{i}(t) \in \{0, \dots, n\}$  where  $\hat{i}(t) = 0$  signifies that the resource was not allocated at time  $t$ . Alternatively, the central planner selects an probabilistic allocation  $\mathbf{p}(t) = (p_i(t))_{i \in [n]} \in \Delta_n = \{\mathbf{y} \geq \mathbf{0} : \sum_{i \in [n]} y_i \leq 1\}$ . Last, at the end of the round, the reported values  $\mathbf{v}(t)$  and allocation  $\mathbf{p}(t)$  are made public to all agents.

Crucially, we suppose that the true utilities are independent between agents and independent and identically distributed (i.i.d.) for each agent. That is, for any  $i \in [n]$ , we have  $(u_i(t))_{t \geq 1} \overset{iid}{\sim} \mathcal{D}_i$ . We assume that the fixed distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  have bounded support in  $[0, \bar{v}]$  for some fixed value  $\bar{v} > 0$ , and are publicly known to the central planner and to all agents  $i \in [n]$ . Last, we consider the case of myopic agents, that is, at time  $t$ , agent  $i$  does not have access to their realized utilities at future times  $u_i(t')$  for  $t' > t$ . Formally, write the history available at time  $t$  as  $\mathbf{h}(t) = (v_j(t'), p_j(t'), j \in [n], t' < t)$ . At time  $t$ , the central planner strategy is given by a function  $S_t : [0, \bar{v}]^{n(t-1)} \times \Delta_n^{t-1} \times [0, \bar{v}]^n \rightarrow \Delta_n$  and the allocation is given by  $\mathbf{p}(t) = S_t(\mathbf{h}(t), \mathbf{v}(t))$ , while the strategy of agent  $i \in [n]$  is given by a function  $\sigma_{i,t} : [0, \bar{v}]^{n(t-1)} \times \Delta_n^{t-1} \times [0, \bar{v}] \rightarrow [0, \bar{v}]$  and their report is given by  $v_i(t) = \sigma_{i,t}(\mathbf{h}(t), u_i(t))$ . For convenience, throughout the paper,  $\mathbf{u}$  represents a random utility vector with independent coordinates with  $u_i \sim \mathcal{D}_i$ .

The goal of the central planner is to maximize the social welfare or equivalently, to minimize the regret compared to the optimal allocation in hindsight  $i^*(t) \in \arg \max_{i \in [n]} u_i$ . We consider two different settings.

1. In the  $\gamma$ -discounted infinite-horizon setting, the central planner and all agents share the same discount factor  $\gamma \in [0, 1)$ . That is, the central planner aims to minimize the expected regret

$$(1 - \gamma) \mathbb{E} \left[ \sum_{t \geq 1} \gamma^{t-1} \left( \max_{i \in [n]} u_i(t) - u_{\hat{i}(t)}(t) \right) \right].$$

while agent  $i \in [n]$  aims to maximize their total discounted utility  $(1 - \gamma) \mathbb{E} \left[ \sum_{t \geq 1} \gamma^{t-1} u_i(t) p_i(t) \right]$ .

2. In the *finite horizon*  $T$  setting, there is no discount factor: the central planner aims to minimize the average regret

$$\frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^T \max_{i \in [n]} u_i(t) - u_{\hat{i}(t)}(t) \right],$$

while agent  $i \in [n]$  aims to maximize their total utility  $\frac{1}{T} \mathbb{E} \left[ \sum_{t \in [T]} u_i(t) p_i(t) \right]$ .

We now specify the equations corresponding to the strategic agents' reports forming a perfect Bayesian equilibrium. Using the same notations as in [Balseiro et al. \[2019\]](#), we denote by  $V_{i,t}$  the expected remaining utility of agent  $i$  starting from time  $t$ , given its own utility  $u_i(t)$  and the history  $\mathbf{h}(t)$ , with the allocation strategy  $\mathbf{S} = (S_{t'})_{t' \geq 1}$ , all other agents employing their strategy  $\boldsymbol{\sigma}_{-i} = (\sigma_{j,t'})_{j \neq i, t' \geq 1}$  and agent  $i$  using strategy  $\tilde{\sigma}_i = (\tilde{\sigma}_{i,t'})_{t' \geq 1}$ :

$$V_{i,t}(\mathbf{S}, \boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i \mid u_i(t), \mathbf{h}(t)) := (1 - \gamma) \mathbb{E} \left[ \sum_{t' \geq t} \gamma^{t'-t} u_i(t') p_i(t') \mid u_i(t), \mathbf{h}(t) \right]$$

for the  $\gamma$ -discounted setting, and

$$V_{i,t}(\mathbf{S}, \boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i \mid u_i(t), \mathbf{h}(t)) := \frac{1}{T - t + 1} \mathbb{E} \left[ \sum_{t'=t}^T u_i(t') p_i(t') \mid u_i(t), \mathbf{h}(t) \right]$$

for the finite horizon  $T$  setting. In both definitions,  $\mathbf{p}(t')$  is induced by the central planner strategy  $S_{t'}$  for  $t' \geq t$  starting from the history  $\mathbf{h}(t)$ , using the reports of agents  $j \neq i$  according to  $\sigma_j$  and the reports of agent  $i$  according to  $\tilde{\sigma}_i$ .

The perfect Bayesian equilibrium constraints write

$$\forall i, t, u_i(t), \mathbf{h}(t), \tilde{\sigma}_i, \quad V_{i,t}(\mathbf{S}, \boldsymbol{\sigma}_{-i}, \sigma_i \mid u_i(t), \mathbf{h}(t)) \geq V_{i,t}(\mathbf{S}, \boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i \mid u_i(t), \mathbf{h}(t)). \quad (\text{PBE})$$

## 2.2 Achievable utility sets

As an alternative reformulation, letting  $\mathbf{U} = (U_i)_{i \in [n]}$  be the vector of realized utilities where  $U_i$  is the total expected utility gained by agent  $i$ , the original problem is to maximize the total utility  $\mathbf{1}^\top \mathbf{U}$  among realizable utility vectors  $\mathbf{U}$ . For our purposes we will consider the more general problem in which agents can have weights  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  and the goal of the central planner becomes maximizing the total utility  $\boldsymbol{\alpha}^\top \mathbf{U}$ . Equivalently, letting  $V_i(\mathbf{S}, \boldsymbol{\sigma}) := \mathbb{E}_{u_i(1)}[V_{i,1}(\mathbf{S}, \boldsymbol{\sigma} \mid u_i(1))]$ , we aim to characterize the region of achievable utilities:

$$\{\mathbf{U} = (V_i(\mathbf{S}, \boldsymbol{\sigma}))_{i \in [n]} \in \mathbb{R}_+^n : \mathbf{S}, \boldsymbol{\sigma} \text{ satisfy } (\text{PBE})\},$$

which we denote  $\mathcal{U}_\gamma$  for the  $\gamma$ -discounted setting, and  $\mathcal{V}_T$  for the finite horizon  $T$  setting.

**Full-information set.** We first characterize the optimal allocation baseline, which corresponds to the *full-information* setting in which the central planner has direct access to all agent utilities  $(u_i(t))_{i \in [n]}$  before deciding its allocation. This also corresponds the setting in which agents are non-strategic. The set of achievable utilities in this full-information setting is

$$\mathcal{U}^* = \{(\mathbb{E}_{u_j \sim \mathcal{D}_j}[u_j p_i(\mathbf{u})])_{i \in [n]}, \mathbf{p} : [0, \bar{v}]^n \rightarrow \Delta_n\}.$$



We can also easily characterize the Pareto-optimal points in this set of achievable utilities. This boundary is parametrized by weights  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  with the corresponding maximizing utility vectors  $\arg \max_{\mathbf{U} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{U}$ . Such  $\boldsymbol{\alpha}$ -optimal utility vectors exactly correspond to first-best allocation functions that always allocate the resource to some agent  $i \in \arg \max_{j \in [n]} \alpha_j u_j$ :

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} = \mathbb{E} \left[ \max_{i \in [n]} \alpha_i u_i \right]. \quad (1)$$

**No-information set.** We next define the set of utilities that can be achieved *without reports*, that is without information on the utility realizations. This corresponds to a setting in which the central planner selects a fixed allocation  $\mathbf{p} \in \Delta_n$ , which gives the following frontier

$$\mathcal{U}^{NI} = \{ (p_i \mathbb{E}_{u_i \sim \mathcal{D}_i} [u_i])_{i \in [n]}, \mathbf{p} \in \Delta_n \}.$$

For intuition, if we have  $\mathbb{P}(u_i = 0) = 0$  for all  $i \in [n]$ , we have  $\mathcal{U}^{NI} = \mathcal{U}_0 = \mathcal{V}_1$ . We will see however, that if this does hold we may have  $\mathcal{U}^{NI} \subsetneq \mathcal{U}_0 = \mathcal{V}_1$ .

Using mixed allocation strategies, we can easily check the following.

**Fact 1.** *The sets  $\mathcal{U}^*$ ,  $\mathcal{U}^{NI}$ ,  $\mathcal{U}_\gamma$ , and  $\mathcal{V}_T$  are convex sets of  $\mathbb{R}_+^n$  for all  $\gamma \in [0, 1)$  and  $T \geq 1$ .*

### 2.3 The promised utility framework

Before giving our results, we present a convenient reformulation of the sequential allocation problem in terms of dynamic programming, referred to as the *promised utility framework* [Abreu et al., 1990, Balseiro et al., 2019, Spear and Srivastava, 1987, Thomas and Worrall, 1990]. We give below an introduction and summary of the framework for the  $\gamma$ -discounted infinite-horizon setting as detailed in Balseiro et al. [2019].

We focus on the  $\gamma$ -discounted infinite-horizon setting for now. In its direct form, maximizing the social welfare requires solving an infinite-horizon perfect Bayes equilibrium. However, the symmetry of the problem in time allows to drastically simplify the formulation. In light of the revelation principle, we can without loss of generality suppose that the mechanism is incentive-compatible hence agents report truthfully. Say we want to design an allocation strategy to ensure that the final utility vector is  $\mathbf{U}$ . The first decision is to decide of the allocation function for the first time step  $t = 1$  which we denote  $\mathbf{p}(\cdot | \mathbf{U}) : [0, \bar{v}]^n \rightarrow \Delta_n$ . It takes as input the agent reports  $\mathbf{v}(t = 1) = \mathbf{u}(1)$  and outputs the allocation distribution  $\mathbf{p}(1)$ . We then denote by  $\mathbf{W}(\cdot | \mathbf{U}) : [0, \bar{v}]^n \rightarrow \mathbb{R}^n$  the remaining utility to be gathered by the agents at future times  $t \geq 2$ , that is,

$$W_i(\mathbf{v} | \mathbf{U}) = (1 - \gamma) \mathbb{E} \left[ \sum_{t \geq 2} \gamma^{t-2} u_i \mathbb{1}[\hat{i}_t = i] \mid \mathbf{u}(1) = \mathbf{v} \right].$$

Note that the history also includes the allocation  $\mathbf{p}(1)$  but this is completely subsumed by the knowledge of  $\mathbf{u}(1) = \mathbf{v}$  via  $\mathbf{p}(1) = \mathbf{p}(\mathbf{v} | \mathbf{U})$ . The problem at time  $t = 2$  is now perfectly equivalent to the original problem except that now the central planner aims to ensure that the utility vector to realize is  $\mathbf{W}(\mathbf{u}(1) | \mathbf{U})$ .

This motivates focusing only on the allocation function and the promised utility function: to achieve a region  $\mathcal{R} \subset \mathcal{U}^*$ , it suffices to specify two functions  $\mathbf{p}(\cdot | \mathbf{U}) : [0, \bar{v}]^n \rightarrow \Delta_n$  and  $\mathbf{W}(\cdot | \mathbf{U}) : [0, \bar{v}]^n \rightarrow \Delta_n$  for all  $\mathbf{U} \in \mathcal{R}$ , where the later is viewed as a utility promise to the

agents. These should satisfy the following constraints. The *promise keeping* constraint checks that the target utility  $\mathbf{U}$  is met:

$$\forall \mathbf{U} \in \mathcal{R}, \forall i \in [n], \quad U_i = \mathbb{E}[(1 - \gamma)u_i p_i(\mathbf{u} | \mathbf{U}) + \gamma W_i(\mathbf{u} | \mathbf{U})]. \quad (2)$$

Second, the central planner should ensure that the promised utilities can be fulfilled for any possible reports of the agent. That is,

$$\forall \mathbf{U} \in \mathcal{R}, \forall \mathbf{v} \in [0, \bar{v}]^n, \quad \mathbf{W}(\mathbf{v} | \mathbf{U}) \in \mathcal{R}. \quad (3)$$

The last constraint is to ensure that (PBE) is satisfied, which can be simplified since we supposed that the mechanism is incentive-compatible. Having defined the interim allocation  $P_i(v | \mathbf{U}) := \mathbb{E}_{\mathbf{u}_{-i}}[p_i(v, \mathbf{u}_{-i} | \mathbf{U})]$  and the interim promised utility  $W_i(v | \mathbf{U}) = \mathbb{E}_{\mathbf{u}_{-i}}[W_i(v, \mathbf{u}_{-i} | \mathbf{U})]$  for all  $i \in [n]$ , the perfect Bayesian incentive-compatibility equations write

$$\forall \mathbf{U} \in \mathcal{R}, \forall i \in [n], \forall u \in [0, \bar{v}], \quad u \in \arg \max_{v \in [0, \bar{v}]} (1 - \gamma)u P_i(v | \mathbf{U}) + \gamma W_i(v | \mathbf{U}). \quad (4)$$

**Fact 2.** *A region  $\mathcal{R}' \subset \mathcal{U}^*$  can be achieved in the infinite-horizon  $\gamma$ -discounted setting if and only there exists a superset  $\mathcal{R} \supseteq \mathcal{R}'$ , functions  $\mathbf{p}(\cdot | \mathbf{U})$  and  $\mathbf{W}(\cdot | \mathbf{U})$  for all  $\mathbf{U} \in \mathcal{R}$  satisfying Eqs. (2) to (4).*

When  $\gamma > 0$ , we can further characterize the form of the allocation and promised utility functions following an argument from Myerson [1981] standard in the literature. Given the incentive-compatibility constraint Eq. (4), without loss of generality, we can suppose that the functions  $P_i(\cdot | \mathbf{U})$  are non-decreasing. For convenience, we say that a choice of allocation function  $\mathbf{p}(\cdot | \mathbf{U})$  is non-decreasing if the corresponding interim allocation function  $P_i(\cdot | \mathbf{U})$  is non-decreasing for all  $i \in [n]$ . Then, the incentive-compatibility constraint Eq. (4) implies

$$W_i(v_i | \mathbf{U}) - W_i(0 | \mathbf{U}) = \frac{1 - \gamma}{\gamma} \left( \int_0^{v_i} P_i(v | \mathbf{U}) dv - P_i(v_i | \mathbf{U}) v_i \right). \quad (5)$$

Combining this with the constraint from Eq. (2), we obtain for all  $\mathbf{U} \in \mathcal{R}$ ,  $i \in [n]$ , and  $v_i \in [0, \bar{v}]$ ,

$$W_i(v_i | \mathbf{U}) = \frac{1}{\gamma} \left[ U_i + (1 - \gamma) \left( \int_0^{v_i} P_i(v | \mathbf{U}) dv - P_i(v_i | \mathbf{U}) v_i - \int_0^{\bar{v}} F_i(v) P_i(v | \mathbf{U}) dv \right) \right]. \quad (6)$$

This formula can be found in Balseiro et al. [2019] as Eq (10). This equation effectively replaces the promise keeping constraint Eq. (2) and the incentive-compatibility constraint Eq. (4). Further, we can check that  $W_i(v_i | \mathbf{U})$  is non-increasing in  $v_i$  by re-writing

$$\int_0^{v_i} P_i(v | \mathbf{U}) dv - P_i(v_i | \mathbf{U}) v_i = \int_0^{v_i} (P_i(v | \mathbf{U}) - P_i(v_i | \mathbf{U})) dv.$$

The promised utility function only needs to satisfy the following validity equation for the interim promise:

$$\forall \mathbf{U} \in \mathcal{R}, \forall i \in [n], \forall v \in [0, \bar{v}], \quad W_i(v | \mathbf{U}) = \mathbb{E}_{\mathbf{u}_{-i}}[W_i(v, \mathbf{u}_{-i} | \mathbf{U})]. \quad (7)$$

As a summary, we have the following characterization.

**Fact 3.** *Let  $\gamma \in (0, 1)$ . A region  $\mathcal{R}' \subset \mathcal{U}^*$  can be achieved in the infinite-horizon  $\gamma$ -discounted setting if and only if there exists a superset  $\mathcal{R} \supseteq \mathcal{R}'$ , functions  $\mathbf{p}(\cdot | \mathbf{U})$  and  $\mathbf{W}(\cdot | \mathbf{U})$  for all  $\mathbf{U} \in \mathcal{R}$  satisfying Eqs. (3) and (7), where the interim future promise is defined as in Eq. (6).*

## 2.4 A convenient choice of promised utility functions.

Having decided of the allocation function  $\mathbf{p}(\cdot | \mathbf{U})$ , the remaining choice of the mechanism design is in the form of the promised utility function  $\mathbf{W}(\cdot | \mathbf{U})$  so that it satisfies Eqs. (3) and (7). We introduce a specific choice of form for these promised utility functions that will be useful for our mechanism design constructions.

The specific construction is somewhat more general than the present sequential allocation setting. Consider the following problem: given  $n$  independent random variables  $\tilde{Z}_1, \dots, \tilde{Z}_n$  and a direction  $\boldsymbol{\alpha} \in \mathbb{R}^n$ , we aim to construct a coupling  $(Z_1, \dots, Z_n)$  such that (1) for all  $i \in [n]$ ,  $\mathbb{E}[Z_i | \tilde{Z}_i] = \tilde{Z}_i$ , and (2) that is as far along the direction  $\boldsymbol{\alpha}$  as possible in the worst-case sense. That is, we aim to solve the problem

$$\max_{\mathbf{Z}} \min_{\omega \in \Omega} \boldsymbol{\alpha}^\top \mathbf{Z}(\omega), \quad \text{s.t.} \quad \forall i \in [n], \mathbb{E}[Z_i | \tilde{Z}_i] = \tilde{Z}_i. \quad (8)$$

Taking the expectation, any coupling  $\mathbf{Z}$  satisfying the marginal constraints must satisfy

$$\min_{\omega \in \Omega} \boldsymbol{\alpha}^\top \mathbf{Z}(\omega) \leq \mathbb{E}[\boldsymbol{\alpha}^\top \mathbf{Z}] = \boldsymbol{\alpha}^\top \mathbb{E}[\tilde{\mathbf{Z}}]. \quad (9)$$

This value can also be reached for instance using the following coupling from [Balseiro et al. \[2019\]](#). For  $i \in [n]$  such that  $\alpha_i = 0$ , we can let  $Z_i = \tilde{Z}_i$ . Otherwise, we define

$$Z_i = \tilde{Z}_i - \frac{1}{n-1} \sum_{j \neq i} \frac{\alpha_j}{\alpha_i} (\tilde{Z}_j - \mathbb{E}[\tilde{Z}_j]). \quad (10)$$

We can easily check that this coupling reaches the upper-bound from Eq. (9) since the coordinates of  $\tilde{\mathbf{Z}}$  are independent. This coupling forces the vector  $\mathbf{Z}$  to lie in the hyperplane  $\{\mathbf{x} : \boldsymbol{\alpha}^\top (\mathbf{x} - \mathbb{E}[\tilde{\mathbf{Z}}]) = 0\}$ .

[Balseiro et al. \[2019\]](#) used this construction to specify the future promise functions: for any  $\mathbf{U}$ , letting  $Z_i := W_i(v_i | \mathbf{U})$ , they use the promise function  $\mathbf{W}(\mathbf{v} | \mathbf{U}; \boldsymbol{\alpha}(\mathbf{U}))$  defined as the previous coupling for some pre-specified vector  $\boldsymbol{\alpha}(\mathbf{U}) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ . For our purposes, we will use a slightly different coupling. The term  $\alpha_i$  in the denominator leads to instabilities that we can mitigate by using the coupling from Eq. (10) only for the variables  $i$  with largest value of  $|\alpha_i|$ . Let  $i_1$  and  $i_2$  be the indices of the largest and second-largest components of  $\boldsymbol{\alpha}$  respectively. We pose

$$Z_i = \begin{cases} \tilde{Z}_{i_1} - \frac{\alpha_{i_2}}{\alpha_{i_1}} (\tilde{Z}_{i_2} - \mathbb{E}[\tilde{Z}_{i_2}]) - \sum_{i \notin \{i_1, i_2\}} \frac{\alpha_i}{\alpha_{i_1}} (\tilde{Z}_i - \mathbb{E}[\tilde{Z}_i]) & i = i_1, \\ \tilde{Z}_{i_2} - \frac{\alpha_{i_1}}{\alpha_{i_2}} (\tilde{Z}_{i_1} - \mathbb{E}[\tilde{Z}_{i_1}]) & i = i_2, \\ \tilde{Z}_i & i \notin \{i_1, i_2\}. \end{cases} \quad (11)$$

Both couplings defined in Eqs. (10) and (11) are optimal for the previous optimization problem.

**Lemma 1.** *For any  $\boldsymbol{\alpha} \in \mathbb{R}^n$ , the couplings defined in Eqs. (10) and (11) are solutions of the problem in Eq. (8) with value  $\boldsymbol{\alpha}^\top (\mathbb{E}[\tilde{\mathbf{Z}}])_{i \in [n]}$ .*

With this construction at hand, to specify an allocation strategy for the central planner, we may only specify the allocation function  $\mathbf{p}(\cdot | \mathbf{U})$  and the directions  $\boldsymbol{\alpha}(\mathbf{U})$ . By defining  $\mathbf{W}(\cdot | \mathbf{U})$  via Eq. (11), the constraint Eq. (7) is automatically satisfied. The only constraint that remains to be checked to have a valid allocation is Eq. (3), which asks that the future promises stay within the constructed region.

### 3 Summary of results

We start by characterizing separately the cases when the regions  $\mathcal{U}_0 = \mathcal{V}_1$  already contain an  $\alpha$ -optimal utility vector for some direction  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . The following lemma also gives an optimal mechanism in this case.

**Lemma 2.** *First, we have  $\mathcal{U}_0 = \mathcal{V}_1$ . Next, fix a direction  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . The following statements are equivalent.*

1. *There exists an  $\alpha$ -optimal vector in  $\mathcal{U}_0$ , that is,  $\max_{\mathbf{x} \in \mathcal{U}_0} \alpha^\top \mathbf{x} = \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x}$ . In particular, for all  $\gamma \in [0, 1)$ ,  $\mathcal{U}_\gamma$  (resp.  $T \geq 1$ ,  $\mathcal{V}_T$ ) has an  $\alpha$ -optimal vector.*
2. *For all  $i \in [n]$ , either*
  - (a) *there exists  $j \neq i$  such that  $\mathbb{P}(\alpha_j u_j \geq \alpha_i u_i) = 1$ ,*
  - (b) *or there exists an interval  $[m_i, M_i]$  such that  $\mathbb{P}(\alpha_i u_i \in [m_i, M_i] \cup \{0\}) = 1$  but  $\mathbb{P}(\alpha_j u_j \in (m_i, M_i)) = 0$  for all  $j \neq i$ .*
3. *Either one of these scenarios holds*

- (a) *There exists  $i \in [n]$  such that for all  $j \neq i$ ,  $\mathbb{P}(\alpha_i u_i \geq \alpha_j u_j) = 1$ . In which case,  $\mathbb{E}[u_i] \mathbf{e}_i \in \mathcal{U}_0$  is  $\alpha$ -optimal, and is reached by the mechanism that always allocates to agent  $i$ .*
- (b) *There exists a subset of agents  $S = \{i_1, \dots, i_k\} \subset \{i \in [n] : \alpha_i > 0\}$  and a sequence  $m_0 = \infty > m_1 \geq m_2 \geq \dots m_{k-1} > 0$  for which*
  - *If  $l < k$ ,  $\mathcal{D}_{i_l}$  is supported on  $\{0\} \cup [m_l/\alpha_{i_l}, m_{l-1}/\alpha_{i_l}]$  and  $\mathbb{P}_{u \sim \mathcal{D}_{i_l}}(u = 0) > 0$ .*
  - *$\mathcal{D}_{i_k}$  is supported on  $[0, m_{k-1}/\alpha_{i_k}]$ .*
  - *For  $i \notin S$ , we have  $\mathbb{P}(\alpha_i u_i \leq \alpha_{i_k} u_{i_k}) = 1$ .*

*In this case,  $\mathbf{U} = \sum_{l \in [k]} \mathbb{P}(u_{i_l} = 0, l' < l) \mathbb{E}[u_{i_l}] \mathbf{e}_i \in \mathcal{U}_0$  is  $\alpha$ -optimal, and is reached by the following allocation mechanism for reported utilities  $\mathbf{v} \in [0, \bar{v}]^n$ :*

$$\mathbf{p}(\mathbf{v}) = \mathbf{e}_{i_l(\mathbf{v})}, \quad l(\mathbf{v}) = \min\{l : v_{i_l} > 0\} \cup \{k\}.$$

The two scenarios of Lemma 2 are depicted in Fig. 1. The first one corresponds to a trivial allocation problem in which one agent completely dominates all others, while the second one involves some slightly-more complex hierarchy between agents. The proof is given in Section 4. In light of the second characterization of Lemma 2, and even when  $\mathcal{U}_0$  does not contain an  $\alpha$ -optimal vector, we note that indices  $i \in [n]$  satisfying either condition 2(a) or 2(b) are somewhat superfluous (see Lemma 19 for more details). Indeed, if condition 2(a) holds, then it is possible to ignore agent  $i$ , never allocating to them, and still yield an  $\alpha$ -optimal utility vector by allocating to  $j$  instead. On the other hand, if condition 2(b) holds, the reports of agent  $i$  are also not really necessary to reach an  $\alpha$ -optimal vector. Knowing the utilities  $u_j$  for the other agents  $j \neq i$  and whether  $u_i > 0$  is sufficient: if this is the case, we can allocate to agent  $i$  when  $\max_{j \neq i} \alpha_j u_j < M_i$  and to an agent in  $\arg \max_{j \neq i} \alpha_j u_j$  otherwise.

This motivates the definition of the complementary set of agents:

$$\begin{aligned} I &:= \{i \in [n] : i \text{ does not satisfy 2(a) nor 2(b)}\} \\ &= \{i : \forall j \neq i, \mathbb{P}(\alpha_i u_i > \alpha_j u_j) > 0\} \\ &\quad \cap \{i : \nexists m_i \leq M_i : \mathbb{P}(\alpha_i u_i \in [m_i, M_i] \cup \{0\}) = 1 \text{ and } \forall j \neq i, \mathbb{P}(\alpha_j u_j \in (m_i, M_i)) = 0\}, \end{aligned} \tag{12}$$

which will be useful to state our results.

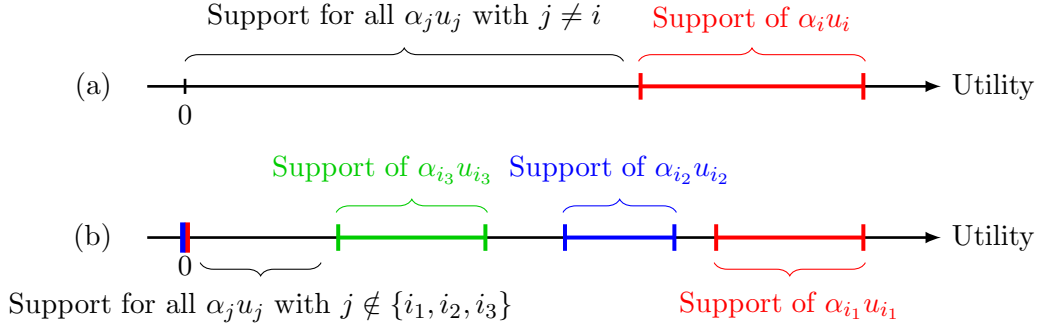


Figure 1: Illustration of the two scenarios (a) and (b) from Lemma 2 which correspond to cases in which there is already an optimal allocation strategy for  $\gamma = 0$  (resp.  $T = 1$ ) in the infinite-horizon (resp. finite-horizon) setting. Figure (a) corresponds to the scenario in which agent  $i$  dominates all other agents. Figure (b) corresponds to the scenario in which agent  $i_1$  and  $i_2$  have non-zero utility mass at 0 and the support of  $\alpha_i u_i$  for  $i \in \{i_1, i_2, i_3\}$  follow a hierarchy.

### 3.1 Two geometric lemmas to construct upper and lower bounds on achievable regions

In this section, we present our major tools to construct lower bounds (resp. upper bounds) on  $\mathcal{U}_\gamma$ , that is, to show that some regions belong to  $\mathcal{U}_\gamma$  (resp. cannot be achieved within  $\mathcal{U}_\gamma$ ).

The following result shows that if a ball  $B(\mathbf{x}, r)$  has a sufficiently large margin within the optimal region  $\mathcal{U}^*$ , it is achievable in the  $\gamma$ -discounted setting.

**Lemma 3.** *Fix  $\gamma \in (0, 1)$ . Let  $\mathbf{x} \in \mathcal{U}^*$  and  $r, \delta > 0$  such that  $B(\mathbf{x}, r + \delta) \subset \mathcal{U}^*$ . Then, if*

$$\frac{r\gamma}{1-\gamma} \geq \delta \geq C \frac{1-\gamma}{\gamma r}, \quad (13)$$

*we have  $B(\mathbf{x}, r) \subset \mathcal{U}_\gamma$ , where  $C = 4n^2 \bar{v}^2 \left(1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{i \in [n]} (\mathbb{E}[u_i] - x_i - r)}\right)^2$ .*

The proof is given in Section 4.1. The left figure of Fig. 2 gives an illustration of this geometrical lemma. As an important remark, the term  $C$  should be viewed as a constant. Indeed, suppose we aim to understand the regret of the central planner in the strategic versus non-strategic settings for a weighted  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  objective. That is, we aim to give an upper bound on

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{x}.$$

We can then use Lemma 3 with small balls locally close to an  $\boldsymbol{\alpha}$ -optimal utility vector  $\mathbf{U} \in \mathcal{U}^*$ . For sufficiently small balls and close to this optimum point, the term  $C$  becomes

$$C \asymp 4n^2 \bar{v}^2 \left(1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{i \in [n]} (\mathbb{E}[u_i] - U_i)}\right)^2,$$

which is independent from  $\gamma$ . In fact, the term  $C$  from Lemma 3 can be greatly simplified if one has additional structure in the problem. To give an example, suppose that we have the following assumption.

**Assumption 4.** *The distributions of the valuations of the agents  $i \in [n]$  have support in  $[\underline{v}, \bar{v}]$ , where  $\bar{v} \geq \underline{v} > 0$ .*

Then, we can simplify the previous bound as follows.

**Lemma 5.** *Under Assumption 4, we can choose  $C = \bar{v}^2(2n + \bar{v}/\underline{v})^2$  in the statement of Lemma 3.*

The sufficient condition from Lemma 3 can be generalized by splitting agents into groups and applying the lemma to each group separately. For simplicity, we defer the complete statement of our lower bound tool to Section 4 (Lemma 21).

We now turn to the second tool which can be used to show that some regions of  $\mathcal{U}^*$  cannot be achieved by  $\mathcal{U}_\gamma$ . The statement is local, that is, we focus on the regions of  $\mathcal{U}^*$  that are close to being  $\alpha$ -optimal for some fixed direction  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . To do so, we need to assume that the scenarios from Lemma 2 for which  $\alpha$ -optimal vectors can already be achieved within  $\mathcal{U}_0$  do not arise (otherwise tight upper bounds are trivial).

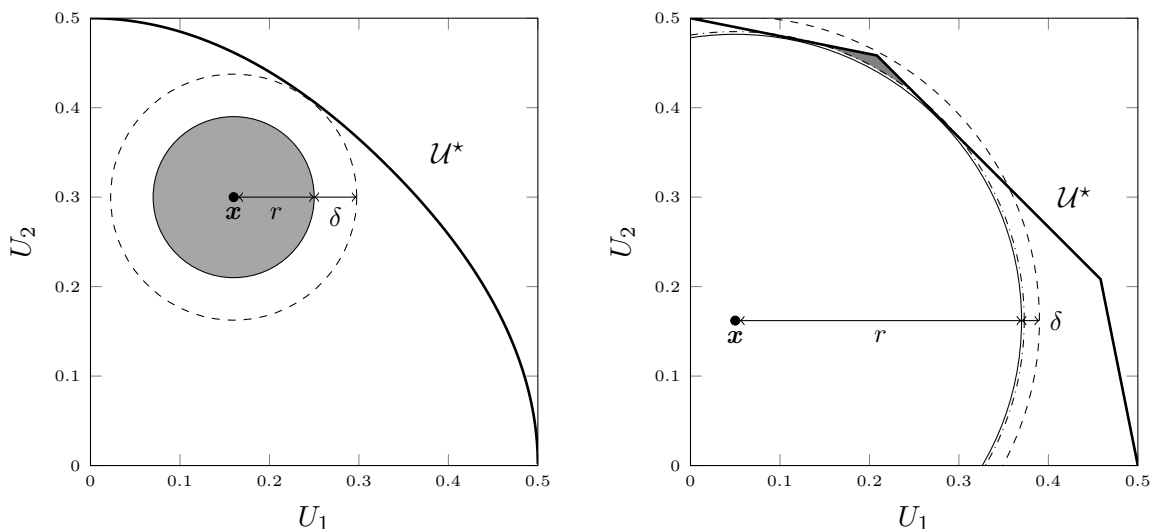


Figure 2: Illustration of Lemma 3 (left) and Lemma 6 (right) with 2 agents. (1) On the left, if the gray ball  $B(\mathbf{x}, r)$  is within the full-information utility set  $\mathcal{U}^*$  with a margin  $\delta$  satisfying Eq (13), then  $B(\mathbf{x}, r)$  is contained in  $\mathcal{U}_\gamma$ . The region  $\mathcal{U}^*$  here corresponds to agents having uniform utility distributions on  $[0, 1]$ . (2) On the right, the ball  $B(\mathbf{x}, r)$  separates the region  $\mathcal{U}^*$  in two connected components, the one  $\mathcal{C}$  closest to the boundary being included within a margin  $\delta$ . If  $\delta$  satisfies the constraints from Lemma 6, then the gray region is outside  $\mathcal{U}_\gamma$  (note that the region is slightly smaller than the connected component  $\mathcal{C}$  (corresponding to the dash-dotted ball). The region  $\mathcal{U}^*$  here corresponds to agents having uniform utility distributions on  $\{1/6, 5/6\}$ .

**Lemma 6.** *Fix a direction  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  for which  $\mathcal{U}_0$  does not already contain an  $\alpha$ -optimal vector (see Lemma 2 for a characterization). There exist two constants  $c_1, c_2 > 0$  such that the following holds. For  $\gamma \in (0, 1)$  and  $r > 0$ , suppose that  $\mathcal{U}^* \setminus B^\circ(\mathbf{x}, r)$  has a connected component  $\mathcal{C} \subset \{\mathbf{y} : \alpha^\top \mathbf{y} \geq \max_{z \in \mathcal{U}^*} \alpha^\top z - c_1\}$ . If*

$$\mathcal{C} \subset B(\mathbf{x}, r + \delta), \quad \delta = \min \left( c_2 \frac{1 - \gamma}{\gamma r}, r \right),$$

then we have

$$\mathcal{U}_\gamma \cap \left( \mathcal{C} \setminus B^\circ \left( \mathbf{x}, r + \frac{1-\gamma}{\gamma} \delta \right) \right) = \emptyset.$$

In particular, for any  $\mathbf{U} \in \mathcal{C}$ , writing  $\mathbf{U} = \mathbf{x} + \|\mathbf{U} - \mathbf{x}\| \mathbf{d}$ ,

$$\max_{\mathbf{y} \in \mathcal{U}_\gamma} \mathbf{d}^\top (\mathbf{y} - \mathbf{x}) < \left( r + \frac{1-\gamma}{\gamma} \delta \right) \|\mathbf{d}\|.$$

The proof of this result is given in Section 4.2 and the right figure of Fig. 2 gives an illustration of this geometrical lemma. Roughly speaking, Lemma 6 is a natural counterpart of Lemma 3: if a ball  $B(\mathbf{x}, r)$  does not have enough margin within  $\mathcal{U}^*$ , then some regions in this outer margin cannot be achieved. The first condition  $\mathcal{C} \subset \{\mathbf{y} : \boldsymbol{\alpha}^\top \mathbf{y} \geq \max_{\mathbf{z} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{z} - c_1\}$  localizes the region of interest. We recall that  $c_1 > 0$  is independent from  $\gamma$ , hence this assumption is very mild (especially since it is possible to reach points that are  $c_1$ -close to optimal for sufficiently small  $\gamma$  as we will see shortly).

As for Lemma 3, the statement can be refined by considering a subset of agents. In particular, it will be useful to consider the setting in which a single agent  $i \in [n]$  competes for the allocation against all other agents  $[n] \setminus \{i\}$ . The precise statement is deferred to Section 4 (Lemma 24) for simplicity.

### 3.2 Main results for the discounted infinite-horizon setting

In this section, we give an overview of some of the results that we can obtain using the two previous tools Lemmas 3 and 6.

**A universal rate  $\mathcal{O}(\sqrt{1-\gamma})$ .** Using Lemma 3, we first show that we can always ensure a  $\mathcal{O}(\sqrt{1-\gamma})$  regret from the central planner in the  $\gamma$ -discounted sequential allocation problem.

**Theorem 7.** For any  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ ,

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{x} = \mathcal{O}(C\sqrt{1-\gamma}).$$

For a more precise statement specifying the constant  $C > 0$ , we refer to Theorem 27, which we prove in Section 5.2. This stronger statement in fact shows that any vector  $\mathbf{U} \in \mathcal{U}^*$  is at distance at most  $\mathcal{O}(\sqrt{1-\gamma})$  from  $\mathcal{U}_\gamma$ . Intuitively, the proof of Theorem 27 is a simple consequence of Lemma 3: in the standard case where  $\mathbf{U} \in \mathcal{U}^*$  satisfies  $0 < U_i < \mathbb{E}[u_i]$  for all  $i \in [n]$ , Fig. 3 gives a visual proof of the result. We first note that  $\prod_{i \in [n]} [0, U_i] \subset \mathcal{U}^*$ , which is enough to construct a ball  $B(\mathbf{x}, r)$  within this region and satisfying the constraints Eq (13) with  $r = \delta = \sqrt{C \frac{1-\gamma}{\gamma}}$ .

The actual convergence rate of the boundary of  $\mathcal{U}_\gamma$  to  $\mathcal{U}^*$  may however be significantly faster than  $\mathcal{O}(\sqrt{1-\gamma})$ , precisely when the boundary of  $\mathcal{U}^*$  is smooth. This is also captured by Lemma 3: the smoother the local boundary of  $\mathcal{U}^*$ , the larger the radius  $r + \delta$  of the ball that can be fit close to that boundary, and the smaller the margin  $\delta \approx (1-\gamma)/r$  that is needed.

As an important remark, the fastest convergence rates from  $\mathcal{U}_\gamma$  to  $\mathcal{U}^*$  that one can prove using the tools from Lemma 3 is  $\mathcal{O}(1-\gamma)$ . Indeed, in order to apply Lemma 3, we need  $B(\mathbf{x}, r) \subset \mathcal{U}^* \subset \prod_{i \in [n]} [0, \mathbb{E}[u_i]]$ . In particular, this requires

$$r \leq \frac{1}{2} \min_{i \in [n]} \mathbb{E}[u_i] \quad \text{and} \quad \delta \geq C \frac{1-\gamma}{\gamma r} \geq \frac{8n^2 \bar{v}^2}{\min_{i \in [n]} \mathbb{E}[u_i]} \frac{1-\gamma}{\gamma} = \Omega(1-\gamma).$$

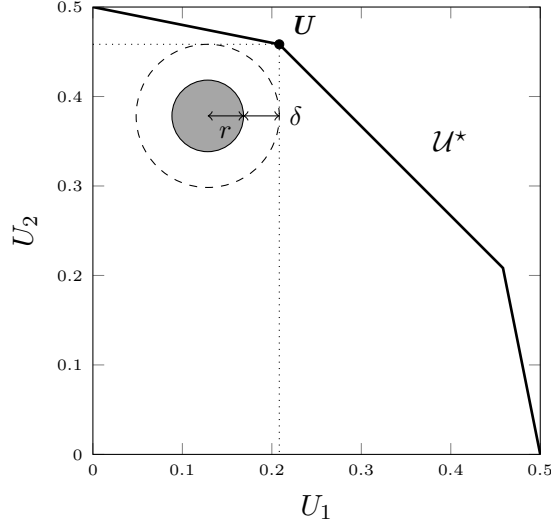


Figure 3: Proof of Theorem 27 in the standard case  $0 < U_i < \mathbb{E}[u_i]$  for all  $i \in [n]$ . For any  $U \in \mathcal{U}^*$ , the dotted region satisfies  $\prod_{i \in [n]} [0, U_i] \subset \mathcal{U}^*$ . We can then fit a ball  $B(\mathbf{x}, r)$  as in the figure with  $r = \delta = \sqrt{C \frac{1-\gamma}{\gamma}}$ . These satisfy Eq (13) for  $1 - \gamma$  sufficiently small. Hence  $B(\mathbf{x}, r) \subset \mathcal{U}_\gamma$  and as a result  $d(U, \mathcal{U}_\gamma) = \mathcal{O}(\sqrt{1 - \gamma})$ .

**Sufficient conditions for faster rates  $\mathcal{O}(1 - \gamma)$ .** We now give conditions in terms of the distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$ —intuitively characterizing the local smoothness of the boundary of  $\mathcal{U}^*$ —under which the faster rates  $\mathcal{O}(1 - \gamma)$  can be achieved.

**Theorem 8.** Fix  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . Define  $I \subset [n]$  as in Eq. (12) and let  $\tilde{I} = I \cup \{i \in [n] : \mathbb{P}(\alpha_i u_i = \max_{j \neq i} \alpha_j u_j) > 0\}$ .

For any subset  $S \subset [n]$ ,  $\mathbf{z} \in \mathbb{R}^S$  and  $r > 0$ , we denote by  $B_S(\mathbf{z}, r)$  the corresponding ball within  $\mathbb{R}^S$ . We also denote by  $P_S : \mathbb{R}^N \rightarrow \mathbb{R}^S$  the projection onto the  $S$  coordinates. Suppose that there exists a partition  $\tilde{I} = I_1 \sqcup I_2 \sqcup \dots \sqcup I_q$  with  $|I_s| \geq 2$  for all  $s \in [q]$ , an  $\alpha$ -optimal vector  $U \in \mathcal{U}^*$  and  $r > 0$ , such that (SC1) taking  $\mathbf{x} := (U_i - r \sum_{s \in [q]} \frac{\alpha_i}{\|\alpha_{I_s}\|} \mathbb{1}_{i \in I_s})_{i \in [n]}$ , we have

$$\mathbb{P} \left( \max_{i \in I_s} \alpha_i u_i = \max_{i \in I_{s'}} \alpha_i u_i = \max_{i \in [n]} \alpha_i u_i > 0 \right) = 0, \quad (14)$$

and for all  $s \in [q]$ ,

$$B_{I_s}(\mathbf{x}_{I_s}, r) \subset P_{I_s}(\mathcal{U}^* \cap \{\mathbf{y} : \forall i \notin I_s, y_i = U_i\}).$$

Then

$$\max_{\mathbf{y} \in \mathcal{U}^*} \alpha^\top \mathbf{y} - \max_{\mathbf{y} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{y} = \mathcal{O}(1 - \gamma).$$

The sufficient condition (SC1) is equivalent to the following. Let  $Z = \max_{i \in [n]} \alpha_i u_i$  and for any  $\emptyset \subsetneq B \subsetneq \tilde{I}$ , we denote  $Z_B = \max_{i \in B} \alpha_i u_i$ . The equivalent condition (SC2) is Eq. (14) holds and

$$\forall s \in [q], \forall \emptyset \subsetneq B \subsetneq I_s, \quad \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\eta)Z_B} \right] \underset{\eta \rightarrow 0}{=} \Omega(\eta).$$



We also include a sufficient condition (SC3) that involves fewer equations (and that implies the previous condition). Consider the directed graph  $\mathcal{G}$  on  $\tilde{I}$  such that for  $i, j \in \tilde{I}$ ,

$$i \rightarrow j \iff f(\eta; i, j) := \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_j u_j = Z} \mathbb{1}_{\alpha_j u_j \in [\alpha_i u_i, (1+\eta)\alpha_i u_i]} \right] \underset{\eta \rightarrow 0}{=} \Omega(\eta).$$

The sufficient condition is that there exists a partition of  $\tilde{I} = I_1 \sqcup I_2 \sqcup \dots \sqcup I_q$  such that  $|I_s| \geq 2$  and  $I_s$  is strongly connected in  $\mathcal{G}$  for all  $s \in [q]$ .

The proof is given in Section 5.3. The conditions from Theorem 8 allow for grouping agents according to any suitable partition, which gives flexibility to the central planner. For intuition, let us focus on the simpler case when these hold for the trivial partition  $\tilde{I} = I_1$ . In this simple case, the geometric condition (SC1) asks that one can fit a ball  $B(\mathbf{x}, r_0) \subset \mathcal{U}^*$  that is tangent to its boundary at an  $\alpha$ -optimal point. The rate  $\mathcal{O}(1 - \gamma)$  then results from applying Lemma 3 with the parameters  $\mathbf{x}$ , and  $r, \delta > 0$  such that  $\delta = C \frac{1-\gamma}{\gamma r}$  (to satisfy Eq. (13)) and  $r + \delta = r_0$ . This can be achieved for  $1 - \gamma$  sufficiently small, which gives a margin  $\delta = \Theta(1 - \gamma)$ .

We next give insights about the conditions (SC2) and (SC3). At the high level, because of the incentive-compatibility constraints Eq. (4), the central planner needs to be able to penalize agent  $i$  on future rounds if they reported a high utility  $v_i(t)$  at time  $t$ , and similarly reward agent  $i$  on future rounds if they reported a low utility  $v_i(t)$ . This can also be observed from the constraint Eq. (7): the interim future promise  $W_i(v_i(t) \mid \mathbf{U})$  is given via Eq. (6), which is decreasing in the report  $v_i(t)$ . To be able to implement these penalties/rewards for agent  $i$  without incurring large social welfare cost, convenient scenarios are when the utilities of several agents including  $i$  are close to being  $\alpha$ -optimal. The central planner then has flexibility on which of these agents to allocate to.

This is precisely the intuition behind the conditions (SC2) and (SC3): (SC2) intuitively asks that for any subgroup of agents  $B$ , with sufficiently large probability their best  $\alpha$ -weighted  $Z_B$  utility is almost as good as that of the complementary subgroup  $Z_{I_s \setminus B}$  so that rewarding this subgroup can be performed without large social welfare cost. (SC3) goes one step further and ensures that for any agent  $i \in I_s$  there is an element  $i \neq j \in I_s$  within the same group  $I_s$  that can be used to reward (resp. penalize) agent  $i$  without incurring large welfare cost:  $i \rightarrow j$  (resp.  $j \rightarrow i$ ). We note that when  $|\tilde{I}| \geq 4$ , the strongly connected condition from (SC3) is stronger than simply assuming that for all  $i \in \tilde{I}$ , there exists  $j, j' \in \tilde{I}$  such that  $i \rightarrow j$  and  $j' \rightarrow i$ , as can be seen in Fig. 4.

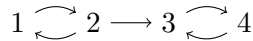


Figure 4: Example of graph for which each node has an incoming and outgoing edge but does not admit a partition into strongly connected components.

Note that (SC2) and (SC3) are very close in nature. (SC3) is slightly more general in the following sense. To ensure the faster rates  $\mathcal{O}(1 - \gamma)$ , it suffices that for any sufficiently small  $1 - \gamma$ , there is an index  $i \neq j^+(\gamma) \in \tilde{I}$  that can be used to reward  $i$  without incurring large social welfare cost, and similarly an index  $j^-(\gamma)$  for penalties. For the proofs, it is not necessary however that these indices be constant as  $\gamma \rightarrow 1^-$ . Precisely, this additional flexibility is encompassed by condition (SC2) (and (SC1) since they are equivalent) but not (SC3).

While the sufficient conditions from Theorem 8 are somewhat complex, we argue that it is satisfied under standard distributional assumptions. For instance, Theorem 8 implies that the

convergence rate  $\mathcal{O}(1 - \gamma)$  for the  $\alpha$  objective can be reached whenever the boundary of  $\mathcal{U}^*$  is twice-differentiable locally at an  $\alpha$ -optimal point  $\mathbf{U}$ : in this case, we can fit a ball  $B(\mathbf{U})$  within  $\mathcal{U}^*$  and tangent to its boundary at  $\mathbf{U}$ , that has radius  $r(\mathbf{U}) = \mathcal{O}(1/\|H(\mathbf{U})\|)$  where  $H(\mathbf{U})$  is the directional Hessian at  $\mathbf{U}$  and  $\|\cdot\|$  is the operator norm. The following result gives additional examples. We include its proof in Section 5.3.

**Corollary 9.** Fix  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  and define  $\tilde{I}$  as in Theorem 8. Let  $m$  denote the infimum of the support of  $\max_{i \in [n]} \alpha_i u_i$ . Suppose that for any  $i \in \tilde{I}$ , there exists  $j \in \tilde{I}$  with  $j \neq i$  such that either

- there exists  $m_i > 0$  such that  $\mathbb{P}(\alpha_i u_i = m_i), \mathbb{P}(\alpha_j u_j = m_i) > 0$ , and  $\mathbb{P}(\alpha_k u_k \leq m_i) > 0$  for all  $k \notin \{i, j\}$ ,
- or there exist  $b_i > a_i > m$  such that both  $\alpha_i u_i$  and  $\alpha_j u_j$  have an absolutely-continuous part that has density bounded away from zero on  $[a_i, b_i]$  by some constant.

Then,

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} = \mathcal{O}(1 - \gamma).$$

As a comparison, the main result from [Balseiro et al., 2019] implies that if the distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  have a density on  $[0, \bar{v}]$  that is bounded above and below by constants, one can achieve a  $\mathcal{O}(1 - \gamma)$  convergence rate. In particular, Corollary 9 implies that this holds even if one only has densities bounded below.

**Necessary conditions for faster rates  $\mathcal{O}(1 - \gamma)$**  We now provide some necessary conditions for having the faster rates  $\mathcal{O}(1 - \gamma)$ . These do not exactly match the sufficient conditions from Theorem 8, however, they can be written in similar terms. Intuitively, the result below implies that if the graph  $\mathcal{G}$  from Theorem 8 has an isolated node, then one cannot reach convergence rates  $\mathcal{O}(1 - \gamma)$ .

**Theorem 10.** Fix  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  for which  $\mathcal{U}_0$  does not already contain an  $\alpha$ -optimal vector (see Lemma 2 for a characterization). Let  $I \subset [n]$  be defined as in Theorem 8 and  $Z_i = \max_{j \neq i} \alpha_j u_j$  for  $i \in [n]$ . If for some  $i \in \tilde{I}$ , we have (NC1)

$$\mathbb{E} \left[ u_i \mathbb{1}_{\frac{\alpha_i u_i}{1+\eta} \leq Z_i \leq (1+\eta)\alpha_i u_i} \right] \stackrel{\eta \rightarrow 0}{=} o(\eta),$$

then

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \stackrel{\gamma \rightarrow 1}{=} \omega(1 - \gamma).$$

In particular, (NC1) is satisfied if (NC2) there exists  $i \in \tilde{I}$  such that for all  $j \neq i$  with  $\alpha_j > 0$ ,

$$\mathbb{E} \left[ u_i \mathbb{1}_{\alpha_j u_j = Z} \mathbb{1}_{\alpha_j u_j \in [\alpha_i u_i, (1+\eta)\alpha_i u_i]} \right] \stackrel{\eta \rightarrow 0}{=} o(\eta) \quad \text{and} \quad \mathbb{E} \left[ u_j \mathbb{1}_{\alpha_i u_i = Z} \mathbb{1}_{\alpha_i u_i \in [\alpha_j u_j, (1+\eta)\alpha_j u_j]} \right] \stackrel{\eta \rightarrow 0}{=} o(\eta).$$

The proof of this result is given in Section 5.4. Similarly to the interpretation of (SC2) and (SC3), the previous result shows that having enough flexibility to penalize/reward each agent (as quantified in (NC1) and (NC2)) is necessary to achieve the faster rates  $\mathcal{O}(1 - \gamma)$ .

**General results for proving convergence rates.** We are now ready to give our most general characterizations. While Theorem 8 and Theorem 10 focused on characterizing conditions for reaching the faster rate  $\mathcal{O}(1-\gamma)$ , the methodology can be used to reach any rate between  $\sqrt{1-\gamma}$  and  $1-\gamma$  more generally as shown in the result below.

**Theorem 11.** Fix  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  for which  $\mathcal{U}_0$  does not already contain an  $\alpha$ -optimal vector (see Lemma 2 for a characterization). Define  $\tilde{I}$  as in Theorem 8. For any  $i \in [n]$  we define  $Z_i = \max_{j \neq i} \alpha_j u_j$ , and for any  $\emptyset \subsetneq B \subsetneq \tilde{I}$  we denote  $Z_B = \max_{i \in B} \alpha_i u_i$ . We also let  $Z = \max_{i \in [n]} \alpha_i u_i$ .

There exists a constant  $C > 0$  such that following holds. Fix any partition  $\tilde{I} = I_1 \sqcup I_2 \sqcup \dots \sqcup I_q$  with  $|I_s| \geq 2$  for all  $s \in [q]$  such that Eq. (14) holds. We define the function

$$f(\eta) := \min_{s \in [q]} \min_{\emptyset \subsetneq B \subsetneq I_s} \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\eta)Z_B} \right], \quad \eta \geq 0. \quad (15)$$

Then, for any  $\gamma \in [1/2, 1)$ ,

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \leq C \sqrt{1-\gamma} \inf \{1\} \cup \left\{ \eta \in (0, 1] : \forall \eta' \in [\eta, 1], f(\eta') \geq C_\eta \sqrt{1-\gamma} \frac{\eta'}{\eta} \right\} + C(1-\gamma). \quad (16)$$

On the other hand, there exists constants  $c_\eta, \eta_0, c > 0$  such that the following holds. For any  $i \in \tilde{I}$ , we define the function

$$g_i(\eta) := \mathbb{E} \left[ u_i \mathbb{1}_{\frac{\alpha_i u_i}{1+\eta} \leq Z_i \leq (1+\eta)\alpha_i u_i} \right].$$

Then, for any  $\gamma \in [9/10, 1)$ ,

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \geq c \sqrt{1-\gamma} \sup \{0\} \cup \{ \eta \in [0, \eta_0] : g_i(\eta) \leq c_\eta \sqrt{1-\gamma} \}. \quad (17)$$

The proof is given in Section 5.4. We make a few remarks about the bounds in Theorem 11, starting with Eq. (16). Denote by  $\eta(\gamma)$  the infimum term of this bound. First, as expected from our previous discussions, it can only prove convergence rates up to  $\mathcal{O}(1-\gamma)$  because of the second term  $C(1-\gamma)$ . As a preview, faster rates are certainly possible in the  $\gamma$ -discounted setting but they will not carry to the finite-horizon  $T$  setting.

Next, note that Eq. (16) recovers the universal convergence rate  $\mathcal{O}(\sqrt{1-\gamma})$  since  $\eta(\gamma) \leq 1$ , and also recovers conditions from Theorem 8 under which rates  $\mathcal{O}(1-\gamma)$  can be achieved. Indeed, (SC2) corresponds to the condition  $f(\eta) \stackrel{\eta \rightarrow 0}{=} \Omega(\eta)$ , which precisely implies  $\eta(\gamma) = \mathcal{O}(\sqrt{1-\gamma})$ .

The term involving  $\eta(\gamma)$  is relevant when  $\eta(\gamma) \in [\sqrt{1-\gamma}, 1]$  and can lead to any convergence rate between  $\mathcal{O}(1-\gamma)$  and  $\mathcal{O}(\sqrt{1-\gamma})$ : the larger the function  $f(\eta)$ —hence the more flexibility the planner has to implement rewards/penalties—the faster the resulting convergence.

The bound Eq. (16) involving  $f(\eta)$  has also a natural analog of (SC3) involving a graph on  $\tilde{I}$ , which we did not include in the statement of Theorem 8 for simplicity. Precisely, for any  $\eta \geq 0$ , we consider the weighted directed graph  $\mathcal{G}_s(\eta)$  on  $I_s$  such that each edge  $i \rightarrow j$  is associated with the weight  $f(\eta; i, j)$  as defined in Theorem 8. Then, instead of the function  $f(\cdot)$  we could also use the function

$$\tilde{f}(\eta) := \min_{s \in [q]} \min_{i \neq j \in I_s} (\max \text{ flow from } i \text{ to } j \text{ on } \mathcal{G}_s(\eta)), \quad \eta \geq 0, \quad (18)$$

since we can check that  $c_1 f(\eta) \leq \tilde{f}(\eta) \leq c_2 f(\eta)$  for all  $\eta \geq 0$  for some constants  $c_1, c_2 > 0$  (see Lemma 28 for a proof). The function  $\tilde{f}$  can have computational advantages over  $f$  since it only requires solving at most  $n^2$  max flow problems.

On the other hand, Eq. (17) gives lower bounds on the convergence rate, that can be anywhere from  $\mathcal{O}(\sqrt{1-\gamma})$  to arbitrarily fast rates (potentially  $\mathcal{U}_\gamma$  may even contain an  $\alpha$ -optimal utility vector for sufficiently small  $1-\gamma$ ). The function  $g_i(\cdot)$  for  $i \in \tilde{I}$  is the same as that appearing in the statement of Theorem 10. It quantifies the flexibility that the central planner has to reward/penalize agent  $i$  in future rounds. Note that while Eqs. (16) and (17) do not match in general, they share a similar form and can be tight in some cases as discussed in Section 3.4.

The difference between the two bounds are two-fold. First, the lower bounds Eq. (17) are derived by considering the setting in which agent  $i$  is competing for the resource against all other agents. While this allows for simpler formulations, it can only encompass interactions between  $\alpha_i u_i$  and the best utility within  $I \setminus \{i\}$ , that is,  $Z_i$ . This contrasts with the function  $f(\cdot)$  which involves interactions between groups  $B$  and  $I_s \setminus B$ . Interestingly, note that when there are only  $n = 2$  agents, this first difference vanishes. Second, the function  $g_i$  dominates both the terms  $f(\eta; i, j)$  for  $j \neq i$  and the terms  $f(\eta; j, i)$  for  $j \neq i$ . In other terms, Eq. (17) requires that the central planner does not have enough flexibility to *both* reward and penalize agent  $i$ . Ideally, one would aim to prove that if the central planner does not have enough flexibility to *either* reward or penalize agent  $i$ , then one should obtain lower bounds on the convergence rate. However, this is not true in general: one cannot replace the function  $g_i(\cdot)$  by

$$g'_i(\eta) = \min \left( \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_i u_i \leq Z_i \leq (1+\eta)\alpha_i u_i} \right], \mathbb{E} \left[ u_i \mathbb{1}_{Z_i \leq \alpha_i u_i \leq (1+\eta)Z_i} \right] \right)$$

within Eq. (17) in general. There are simple examples for which either one of the terms in  $g'_i$  is identically zero for sufficiently small  $\eta \geq 0$  (but not both, otherwise Eq. (17) applies) but the faster rates  $\mathcal{O}(1-\gamma)$  are still achievable.

### 3.3 Links between the discounted infinite-horizon and the finite-horizon settings

Fortunately, up to minor changes, all the tools discussed earlier to give upper bounds and lower bounds on the achievable region  $\mathcal{U}_\gamma$  in the discounted setting also carry to the finite-horizon  $T$  setting. To begin the analysis, we first note that the utility regions  $\mathcal{V}_T$  are linked by a recursive equation that can be rewritten within the promised utility framework. Fix  $T \geq 1$  and let  $\gamma(T) = 1 - 1/T$ . For  $T = 1$ , the achievable region is exactly  $\mathcal{V}_1 = \mathcal{U}_0$ . For  $T \geq 2$ , the promised utility framework can be applied almost by definition using the parameter choice  $\gamma = \gamma(T)$  as detailed below. Its proof is deferred to Appendix A.

**Fact 4.** Fix  $\mathcal{R} \subset \mathcal{U}^*$  and  $T \geq 2$ . Then  $\mathcal{R} \subset \mathcal{V}_T$  if and only if there exists functions  $\mathbf{p}(\cdot | \mathbf{U})$  and  $\mathbf{W}(\cdot | \mathbf{U})$  for all  $\mathbf{U} \in \mathcal{R}$  satisfying Eqs. (2) and (4) for  $\gamma = \gamma(T)$ , and they satisfy

$$\forall \mathbf{U} \in \mathcal{R}, \forall \mathbf{v} \in [0, \bar{v}]^n, \quad \mathbf{W}(\mathbf{v} | \mathbf{U}) \in \mathcal{V}_{T-1}, \quad (19)$$

instead of the original constraint Eq. (3).

Equivalently,  $\mathcal{R} \subset \mathcal{V}_T$  if and only if the promised utility function satisfies Eq. (7) where the interim promised utility was defined via Eq. (6) with  $\gamma = \gamma(T)$ , and satisfies Eq. (19).

Using this reformulation we can now compare the regions  $\mathcal{V}_T$  for  $T \geq 1$  to the region  $\mathcal{U}_{\gamma(T)}$ .

**Lemma 12.** *For any  $T \geq 1$ , we have  $\mathcal{V}_T \subset \mathcal{U}_{\gamma(T)}$ .*

This is proved in Section 6. As a direct consequence of Lemma 12, all techniques developed earlier to give upper bounds on  $\mathcal{U}_{\gamma(T)}$  also apply to  $\mathcal{V}_T$ . In particular, Lemmas 6 and 24, Theorem 10, and the second claim Eq. (17) from Theorem 11 all hold by changing  $\mathcal{U}_\gamma$  to  $\mathcal{V}_T$  and  $\gamma$  to  $\gamma(T) = 1 - 1/T$ .

In addition, we can easily prove that in the finite-horizon  $T$  setting, one cannot reach faster rates than  $\mathcal{O}(1 - \gamma(T)) = \mathcal{O}(1/T)$ . Intuitively, this is because on the last allocation iteration, the reports of the agents carry no information, hence the central planner can only reach a utility vector within  $\mathcal{U}_0$  for this iteration, which incurs a non-zero (constant) regret. We include the proof in Section 6.

**Lemma 13.** *Let  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  such that  $\mathcal{U}_0$  does not already contain an  $\alpha$ -optimal vector (see Lemma 2 for a characterization). Then, there exists  $C > 0$  such that for all  $T \geq 1$ ,*

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{V}_T} \alpha^\top \mathbf{x} \geq \frac{C}{T}.$$

We now turn to the lower bounds on  $\mathcal{U}_\gamma$  and show that the main lemmas to construct lower bounds still hold in this setting up to logarithmic factors. The proof is again constructive and proceeds by building a path from  $\mathcal{V}_1 = \mathcal{U}_0$  to the desired region (see Fig. 6 for an illustration).

**Lemma 14.** *[Informal version of Lemma 31] Let  $T \geq 2$ . Up to changing the constants, Lemma 3 holds by replacing  $\mathcal{U}_\gamma$  with  $\mathcal{V}_T$  and  $\gamma$  with  $1 - \ln T/T$ , and adding the constraint  $r_0 \geq r \geq \delta$  for some constant  $r_0$  that only depends on  $\alpha$  and the utility distributions.*

A similar result can be obtained for the generalization Lemma 21 (see Lemma 32). The precise statements and corresponding proofs are given in Section 6. Contrary to Lemma 12, Lemma 14 requires a sub-optimal choice  $\gamma = 1 - \ln T/T$ ; ideally, this should hold for  $\gamma = \gamma(T)$ . The more complete statements show that this term in fact vanishes in some cases (see Corollary 18 for an example). However, this may not vanish, typically when utility distributions are somewhat smooth. While we believe that the form of this additional term in Lemma 31 is essentially necessary, we leave this question open. The core of the problem is captured in the following simple scenario.

**Open Question 1.** *Consider the  $n = 2$  agent finite-horizon  $T$  resource allocation setting where the utility distributions are uniform  $\mathcal{U}([0, 1])$  for both agents. Characterize the asymptotic behavior (in  $T$ ) of*

$$\max_{\mathbf{x} \in \mathcal{U}^*} \mathbf{1}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{V}_T} \mathbf{1}^\top \mathbf{x}.$$

Our results show that the convergence rate is between  $1/T$  and  $\ln T/T$ . We conjecture that the correct rate is  $\ln T/T$ , which would imply that there is indeed a separation between the infinite-horizon and finite-horizon settings.

Practically speaking, Theorems 8 and 27, Corollary 9, and the first claim Eq. (16) from Theorem 11 all hold by changing  $\mathcal{U}_\gamma$  to  $\mathcal{V}_T$  and  $\gamma$  to  $1 - \ln T/T$ . As a remark, in all proofs which used Lemmas 3 and 21, the constraint  $r \geq \delta$  was already enforced (this is the natural choice in practice), and the constraint  $r_0 \geq r$  is without loss of generality by considering smaller radius balls.

### 3.4 Examples and special cases

In this section, we give examples of how the methods described above can be used for some utility distributions of interest.

We start with the case of smooth utility distributions. [Balseiro et al. \[2019\]](#) shows that if the utility distributions have density upper and lower bounded by a constant on the domain  $[0, \bar{v}]$  then the convergence rate is  $\Theta(1 - \gamma)$ . Using the previous results, we can give further details about this convergence: in [Corollary 9](#) we showed that having densities bounded below is sufficient to have a convergence rate  $\mathcal{O}(1 - \gamma)$ , and [Theorem 11](#) implies that having densities bounded above is sufficient to prove that the rate of convergence is  $\Omega(1 - \gamma)$ .

**Corollary 15.** *Suppose that the utility distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  admit a density upper bounded by some constant on  $[0, \bar{v}]$ . Then, for any  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ ,*

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \underset{\gamma \rightarrow 1}{=} \Omega(1 - \gamma).$$

*If they admit a density lower bounded by some constant on  $[0, \bar{v}]$ , then, for any  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ ,*

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \underset{\gamma \rightarrow 1}{=} \mathcal{O}(1 - \gamma).$$

**Proof** We aim to use [Theorem 11](#). From the distributional assumptions, for any  $i \in [n]$ ,  $Z_i$  admits a density upper bounded by some constant fixed over  $[0, \bar{v}]$ . This implies that  $g_i(\eta) \underset{\eta \rightarrow 0}{=} \mathcal{O}(\eta)$ . Hence  $\sup\{\eta \in [0, \eta_0] : g_i(\eta) \leq c_\eta \sqrt{1 - \gamma}\} = \Omega(\sqrt{1 - \gamma})$  as  $\gamma \rightarrow 1$  and [Theorem 11](#) provides the desired first claim. The second claim is taken directly from [Corollary 9](#). ■

For the finite-horizon setting, we have the following, which we prove in [Section 6](#).

**Corollary 16.** *Suppose that the utility distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  admit a density upper bounded by some constant on  $[0, \bar{v}]$ . Then, for any  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ , if  $\mathcal{U}_0$  does not already include an  $\alpha$ -optimal vector (see [Lemma 2](#) for a characterization), there exist constants  $c, C > 0$  such that*

$$\frac{c}{T} \leq \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{V}_T} \alpha^\top \mathbf{x} \leq \frac{C \ln(T + 1)}{T}, \quad T \geq 1.$$

We next turn to other extreme case of discrete utility distributions. In this case, [Theorem 11](#) implies that most of the time the rate of convergence is  $\Omega(\sqrt{1 - \gamma})$ , which can be reached by [Theorem 7](#). In some cases, however, the convergence can be much faster as detailed below.

**Theorem 17.** *Suppose that all utility distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  are discrete. Let  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  such that  $\mathcal{U}_0$  does not already contain an  $\alpha$ -optimal vector (see [Lemma 2](#) for a characterization). Let  $\tilde{I} \subset [n]$  and  $Z$  be defined as in [Theorem 8](#).*

*If there exists  $i \in \tilde{I}$  such that for all  $j \neq i$ ,  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0) = 0$ , then*

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \underset{\gamma \rightarrow 1}{=} \Theta(\sqrt{1 - \gamma}).$$

*Otherwise, there exists a constant  $c(\alpha) > 0$  such that*

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \underset{\gamma \rightarrow 1}{=} \mathcal{O}\left((1 - \gamma)e^{-c(\alpha)/\sqrt{1 - \gamma}}\right).$$

The proof of this result is given in Section 5.5. The second case of Theorem 17 corresponds to the case when with non-zero probability there are (significant) ties between different agents. In this case, the frontier of  $\mathcal{U}^*$  is flat along direction  $\alpha$ , that is the set of  $\alpha$ -optimal utility vectors forms a  $(|\tilde{I}| - 1)$ -dimensional convex set of  $\mathbb{R}_+^n$ . In practice, the set of directions  $\alpha$  for which this happens and  $I \neq \emptyset$  has zero Lebesgue measure. A more general statement of when these rates can be achieved beyond discrete distributions can be found in Theorem 30.

To prove these faster rates, we need an additional tool compared to Lemma 3. Indeed, applying directly this result (or rather the generalization from Lemma 21) would only prove a convergence rate  $\mathcal{O}(1 - \gamma)$ : as discussed earlier, this is the fastest rate that can be proved with these tools in their direct form. In practice, however, these can be combined with *gluing* arguments to yield faster rates when the frontier of  $\mathcal{U}^*$  is particularly flat.

More precisely, Lemma 3 gives results when a ball  $B(\mathbf{x}, r + \delta)$  can be fit within  $\mathcal{U}^*$ . To decrease  $\delta$  while satisfying Eq. (13), one needs to use balls with larger radius  $r$ , which is for instance limited by the diameter of  $\mathcal{U}^*$ . However, as long as we ensure that some boundary of a spherical cap is also within the acceptable region  $\mathcal{U}_\gamma$ , one can potentially increase this radius very significantly, as illustrated in Fig. 5. We refer to this as a *gluing* strategy: first use the standard strategy from Lemma 3 or Lemma 21 to show that some regions are within  $\mathcal{U}_\gamma$  then successively build from there using balls of larger radius such that the boundary of their local spherical cap is within the previous region.

The second case of Theorem 17 is an example for which the gluing method yields faster rates. In this simple case when the optimal boundary of  $\mathcal{U}^*$  is exactly flat— $(n - 1)$ -dimensional, the achievable frontier constructed via the gluing strategy essentially satisfies the differential equation  $\delta'' = \Theta(\delta/(1 - \gamma))$  where  $\delta$  denotes the distance of the frontier to the optimal frontier of  $\mathcal{U}^*$  (this corresponds to the differential equation  $\delta \approx (1 - \gamma)/r$  where  $r = 1/\delta''$  is the radius of the ball kissing the constructed curve at the considered point, which exactly satisfies Eq. (13)). In turn, this shows that we can achieve utility vectors within  $\mathcal{U}^*$  that are at distance  $\approx e^{-c/\sqrt{1-\gamma}}$  for some constant  $c$  describing the size of the flat optimal boundary of  $\mathcal{U}^*$ .

By Lemma 13, such exponential rates cannot generalize to the finite-horizon setting. In this case, we obtain the following convergence rates. The proof is given in Section 6.

**Corollary 18.** *Suppose that all utility distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  are discrete. Let  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  such that  $\mathcal{U}_0$  does not already contain an  $\alpha$ -optimal vector (see Lemma 2 for a characterization). Let  $\tilde{I} \subset [n]$  and  $Z$  be defined as in Theorem 8. There exist constants  $c, C > 0$  such that the following holds.*

*If there exists  $i \in \tilde{I}$  such that for all  $j \neq i$ ,  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0) = 0$ , then*

$$\frac{c}{\sqrt{T}} \leq \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{V}_T} \alpha^\top \mathbf{x} \leq \frac{C}{\sqrt{T}}, \quad T \geq 1.$$

*Otherwise,*

$$\frac{c}{T} \leq \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{V}_T} \alpha^\top \mathbf{x} \leq \frac{C \ln(T + 1)}{T}, \quad T \geq 1.$$

## 4 Main tools to analyze the achievable utility regions

We start by proving Lemma 2 which gives a characterization of cases when  $\mathcal{U}_0 = \mathcal{V}_1$  already contains an  $\alpha$ -optimal utility vector.

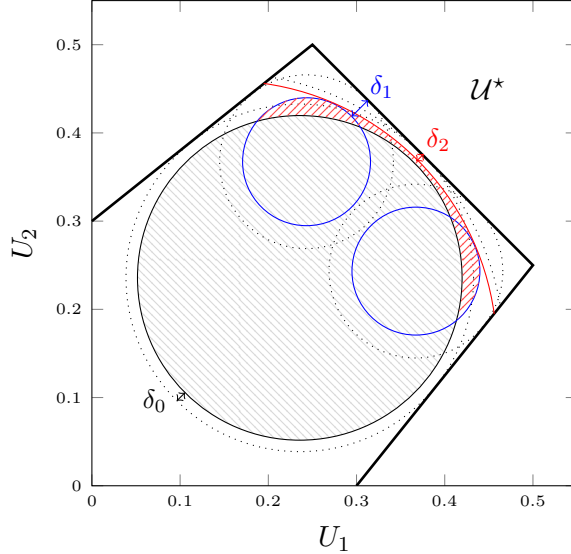


Figure 5: Illustration of how the *gluing* strategy achieves faster convergence rates when the frontier of the full-information region  $\mathcal{U}^*$  is close to flat. The figure depicts a region  $\mathcal{U}^*$  with an exaggerated piece-wise affine boundary for illustration purposes. The dashed gray region corresponds to the region that can be shown to be achievable in  $\mathcal{U}_\beta$  using Lemma 3 directly, which is limited by the maximum-radius ball tangent to the region of optimal utility vectors that can be fit within  $\mathcal{U}^*$  (represented with a dotted circle). The necessary margin  $\delta_0$  is inversely proportional to the radius of the corresponding black ball as per Eq. (13). The gluing strategy effectively increases this radius as shown in the red arc, resulting in a smaller margin  $\delta_2$ . To avoid the region from stepping outside of  $\mathcal{U}^*$ , we can glue the corresponding region with the blue balls which can be proved to be within  $\mathcal{U}_\beta$  directly using Lemma 3. This results in the extra red-dashed region which can then be proved to be within  $\mathcal{U}_\beta$ . Note that since the radius of blue balls is necessarily smaller than that of the black ball, we have  $\delta_1 \geq \delta_0$ . The main benefit of the gluing strategy is that at the second layer of the construction, the effective radius of the boundary of the new red-dashed region has been artificially increased. The gluing strategy can be used with an arbitrary number of layers to reach exponential convergence rates to the boundary of  $\mathcal{U}^*$ .

**Proof of Lemma 2** First note that the perfect Bayesian equations Eq. (PBE) characterizing  $\mathcal{V}_1$  for  $t = T = 1$  are identical to those for  $\mathcal{U}_0$  for  $t = 1$ , which is the only time step that matters for the total utility  $\mathbf{V}(\mathcal{S}, \boldsymbol{\sigma})$  since  $\gamma = 0$ . Hence we directly have  $\mathcal{U}_0 = \mathcal{V}_1$ .

We now turn to the main claim of the result. Throughout the proof, we fix a direction  $\boldsymbol{\alpha}$  as in the statement.

**Proving 1.  $\Rightarrow$  2.** We can introduce the preorder  $\succeq$  such that  $i \succeq j$  if and only if  $\mathbb{P}(\alpha_i u_i \geq \alpha_j u_j) = 1$  (it is reflexive and transitive, but not necessarily antisymmetric). Let  $T = \{i \in [n], \nexists j \neq i, j \succeq i\}$ , which gives

$$\forall i \in T, \forall j \neq i, \quad \mathbb{P}(\alpha_i u_i > \alpha_j u_j) > 0. \quad (20)$$



This implies in particular that  $\alpha_i > 0$  for all  $i \in T$ , and that  $\mathbb{P}(u_i = 0) < 1$ . For every  $i \in T$ , recall the notation  $\text{Supp}(\mathcal{D}_i)$  for the support of  $\mathcal{D}_i$ . We define

$$m_i := \alpha_i \inf(\text{Supp}(\mathcal{D}_i) \setminus \{0\}) \quad \text{and} \quad M_i := \alpha_i \sup(\text{Supp}(\mathcal{D}_i) \setminus \{0\}). \quad (21)$$

Note that these are well-defined since  $\text{Supp}(\mathcal{D}_i) \setminus \{0\} \neq \emptyset$ , otherwise  $\mathbb{P}(u_i = 0) = 1$ . In particular,  $M_i > 0$ .

By assumption, there exists an  $\alpha$ -optimal vector  $\mathbf{U}_0 \in \mathcal{U}_0$ . Hence, by the revelation principle, there is an allocation function  $\mathbf{p}(\cdot)$  that reaches  $\mathbf{U}_0$  and is incentive-compatible as per Eq. (4). Now fix  $i \in T$  and denote  $Z_i = \max_{j \neq i} \alpha_j u_j$ . Because it is  $\alpha$ -optimal, with probability one on  $\mathbf{u}$ , we have  $p_i(\mathbf{u}) > 0$  only if  $i \in \arg \max_{j \in [n]} \alpha_j u_j$ . We also recall that  $P_i(\cdot)$  is non-decreasing by hypothesis. As a result, for all  $u \in \text{Supp}(\mathcal{D}_i)$  we have

$$\mathbb{P}(Z_i < \alpha_i u) \leq P_i(u) \leq \mathbb{P}(Z_i \leq \alpha_i u). \quad (22)$$

In particular, this holds for  $u \in \{m_i/\alpha_i, M_i/\alpha_i\}$ . In our setting, because  $\gamma = 0$ , the incentive-compatibility constraint writes

$$\forall u \in (0, \bar{v}], \quad u \in \arg \max_{v \in [0, \bar{v}]} u P_i(v),$$

where  $P_i(v_i) = \mathbb{E}[p_i(v_i, \mathbf{u}_{-i})]$ . This implies that  $P_i$  is constant on  $(0, \bar{v}]$  for all  $i \in [n]$ . Together with Eq. (22), this shows that

$$\mathbb{P}(Z_i < M_i) \leq P_i(M_i/\alpha_i) = \lim_{x \rightarrow m_i^+} P_i(x/\alpha_i) \leq \lim_{x \rightarrow m_i^+} P_i(Z_i \leq x) = \mathbb{P}(Z_i \leq m_i). \quad (23)$$

Hence, either  $m_i = M_i$ , or we obtained  $\mathbb{P}(Z_i \leq m_i) = \mathbb{P}(Z_i < M_i)$ . In both cases, we obtained  $\mathbb{P}(Z_i \in (m_i, M_i)) = 0$ , while  $\text{Supp}(\mathcal{D}_i) \subset [m_i, M_i] \cup \{0\}$  hence  $\mathbb{P}(\alpha_i u_i \in [m_i, M_i] \cup \{0\}) = 1$ .

Now note that for any  $j \neq i$ ,

$$\begin{aligned} 0 = \mathbb{P}(Z_i \in (m_i, M_i)) &\geq \mathbb{P}(\alpha_j u_j \in (m_i, M_i) \text{ and } \forall k \notin \{i, j\}, \alpha_k u_k < M_i) \\ &= \mathbb{P}(\alpha_j u_j \in (m_i, M_i)) \prod_{k \notin \{i, j\}} \mathbb{P}(\alpha_k u_k < M_i). \end{aligned}$$

However, Eq. (20) implies that for all  $k \neq i$ , we have  $\mathbb{P}(\alpha_k u_k < M_i) > 0$ . Together with the previous equation this gives  $\mathbb{P}(\alpha_j u_j \in (m_i, M_i)) = 0$ , which holds for all  $j \neq i$ .

**Proving 2.  $\Rightarrow$  3.** We suppose that the second statement of Lemma 2 holds. Further, suppose that scenario 3(a) does not hold, that is,  $([n], \succeq)$  does not have a maximum element. We now construct the set  $M$  as the set of maximal elements, that is

$$M = \{i \in [n] : j \succeq i \Rightarrow i \succeq j\}.$$

On  $M$ , the preorder  $\succeq$  is symmetric, hence is an equivalence relation and induces equivalence classes. We define  $S$  as a set containing exactly one representative for each equivalence class of  $\succeq$  on  $M$ . Note that for any  $i \neq j \in S$  we have neither  $i \succeq j$  nor  $j \succeq i$ . By assumption, we also have  $|S| \geq 2$ . For any  $i \in M$ , we have  $\alpha_i > 0$  and  $\mathbb{P}(u_i = 0) < 1$ , otherwise for all  $j \in M$ ,  $j \succeq i$  and there would be only one equivalence class.

Let  $T$  be the union of equivalence classes that are singletons. One can note that  $T$  coincides with the definition from the proof that  $1. \Rightarrow 2.$ , hence elements of  $T$  do not satisfy assumption 2(a) as seen in Eq. (20). Hence they must satisfy assumption 2(b): defining  $m_i, M_i$  as in Eq. (21),

$$\mathbb{P}(\alpha_i u_i \in [m_i, M_i] \cup \{0\}) = 1 \quad \text{and} \quad \mathbb{P}(\alpha_j u_j \in (m_i, M_i)) = 0, \quad j \neq i. \quad (24)$$

For  $i \in M \setminus T$ , there exists  $j \in M$  with  $j \neq i$  in its equivalence class. Then,  $i \succeq j$  and  $j \succeq i$ , which implies that the distributions  $\mathcal{D}_i$  and  $\mathcal{D}_j$  are identical and are equal to a Dirac. We can define  $m_i = M_i > 0$  as the value such that  $\mathbb{P}(\alpha_i u_i = m_i) = 1$ . Note that with these values, Eqs. (21) and (24) hold for all  $i \in M$ . Note that  $M \setminus T$  only contains one equivalence class, otherwise we would have  $i, j \in M \setminus T$  with say  $m_i < m_j$  which implies that  $i$  is not maximal. In summary,  $|S \setminus T| \leq 1$ .

We next prove that for all  $i, j \in S$ ,  $M_j \notin (m_i, M_i)$ . Otherwise, by definition of  $M_j$  in Eq. (21) for all  $\epsilon > 0$  we have  $\mathbb{P}(\alpha_j u_j \in (m_i, M_j)) > 0$  contradicting Eq. (24). Similarly, we have  $m_j \notin (m_i, M_i)$ . This implies that all intervals  $(m_i, M_i)$  for  $i \in S$  are disjoint. We order them by increasing order  $S = \{i_1, \dots, i_k\}$  where  $M_{i_1} \geq m_{i_1} \geq M_{i_2} \geq m_{i_2} \geq \dots \geq M_{i_k} \geq m_{i_k}$ .

For any  $i \in T$  and  $j \in S \setminus T$ , we must have  $m_j \leq m_i$ , otherwise  $m_j \geq M_i$  which implies  $j \succeq i$  since  $\mathbb{P}(\alpha_j u_j = m_j) = 1$ . This shows that the element  $j \in S \setminus T$  if it exists, appears last  $j = i_k$ . For any  $l \in [k-1]$ , we have  $i_l \in T$  so  $\text{Supp}(\mathcal{D}_{i_l}) \subset [m_{i_l}, M_{i_l}] \cup \{0\}$ . Then, we must have  $\mathbb{P}(u_{i_l} = 0) > 0$ , otherwise we would have  $i_l \succeq i_{l+1}$ . We also note that  $m_{i_{k-1}} \geq M_{i_k} > 0$ .

By construction of  $S$ , for any  $i \notin S$  there must exist  $j \in S$  such that  $j \succeq i$ . If  $j = i_l$  where  $l \in [k-1]$  we must have  $\mathbb{P}(\alpha_i u_i = 0) = 1$  since we just proved that  $\mathbb{P}(\alpha_{i_l} u_{i_l} = 0) > 0$ . Hence, in all cases  $i_k \succeq i$ .

Up to renaming  $m_{i_l}$  by  $m_l$ , this ends the proof of the characterization 3(b) of  $\mathcal{D}_1, \dots, \mathcal{D}_n$ . It is straightforward to check that the proposed allocation function  $\mathbf{p}(\cdot)$  is incentive-compatible and reaches the utility vector

$$\mathbf{U} = \sum_{l \in [k]} \mathbb{P}(u_{i_l} = 0, l' < l) \mathbb{E}[u_{i_l}] \mathbf{e}_i.$$

In particular,  $\mathbf{U} \in \mathcal{U}_0$ . We can easily check that it always allocates to the agent within  $\{i_l, l \in [k]\}$  with maximum value, which is sufficient since  $i_k \succeq i$  for all  $i \notin S$ . This ends the proof of  $2. \Rightarrow 3.$

**Proving 3.  $\Rightarrow$  1.** This is immediate because the vector  $\mathbf{U} \in \mathcal{U}_0$  is  $\alpha$ -optimal. ■

When  $\mathcal{U}_0$  does not contain an  $\alpha$ -optimal vector, the characterization from Lemma 2 does not hold: there exist agents  $i \in [n]$  that do not satisfy assumptions 2(a) nor 2(b). However, we can still give a simple characterization of the distributions  $\mathcal{D}_i$  for indices  $i \in [n]$  that satisfy one of these conditions, as detailed in the following lemma.

**Lemma 19.** *Let  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . Let  $I \subset [n]$  be the set defined in Eq. (12). Let  $J = \{i \in [n] \setminus I : \forall j \neq i, \mathbb{P}(\alpha_i u_i > \alpha_j u_j) > 0\}$  and  $K = \{i \in [n] \setminus (I \cup J) : \forall j \neq i, \mathbb{P}(\alpha_i u_i \geq \alpha_j u_j) > 0\}$ . Then, there is an ordering of  $J = \{j_1, \dots, j_{|J|}\}$  and values  $0 \leq m \leq m_{j_1} \leq M_{j_1} \leq m_{j_2} \leq M_{j_2} \leq \dots \leq m_{j_{|J|}} \leq M_{j_{|J|}}$  such that*

- For  $i \notin I \cup J \cup K$ ,  $\mathbb{P}(\alpha_i u_i = \max_{j \in [n]} \alpha_j u_j) = 0$ .
- For  $k \in K$ ,  $\mathbb{P}(\alpha_k u_k = m) = 1$ . Also,  $|K| \neq 1$

- For  $j \in J$ ,  $\mathbb{P}(\alpha_j u_j \in [m_j, M_j] \cup \{0\}) = 1$  and  $\mathbb{P}(\alpha_i u_i \in (m_i, M_i)) = 0$  for all  $i \neq j$ . Also, for  $j \in J \setminus \{j_1\}$ ,  $\mathbb{P}(u_j = 0) > 0$ .
- Either  $\mathbb{P}(u_{j_1} = 0) > 0$ ,  $K = \emptyset$ , or:  $m_{j_1} = m$  and  $\mathbb{P}(u_{j_1} = m) > 0$ .

**Proof** By construction of  $K$ , if  $i \notin I \cup J \cup K$  there exists  $j \in [n]$  such that  $\mathbb{P}(\alpha_j u_j > \alpha_i u_i) = 1$ . This proves the first claim.

We now suppose that  $K \neq \emptyset$ . For  $k \in K$  since  $k \notin (I \cup J)$ , there exists  $i \in [n]$  with  $j \neq k$  such that  $\mathbb{P}(\alpha_k u_k > \alpha_j u_j) = 0$ . Using the definition of  $K$  we obtain  $\mathbb{P}(\alpha_j u_j = \alpha_k u_k) > 0$ . In particular,  $j \notin (I \cup J)$ . On the other hand, we have  $j \in I \cup J \cup K$ , otherwise there is  $l \in [n]$  with  $\mathbb{P}(\alpha_l u_l > \alpha_k u_k) \geq \mathbb{P}(\alpha_l u_l > \alpha_j u_j \geq \alpha_k u_k) = \mathbb{P}(\alpha_l u_l > \alpha_j u_j) > 0$ . This shows that we also have  $j, k \in K$  so that  $|K| \geq 2$ . Now for any  $k' \in K$  with  $k \neq k'$ , by definition of  $K$ ,

$$\mathbb{P}(\alpha_k u_k \neq \alpha_{k'} u_{k'}) = \mathbb{P}(\alpha_k u_k > \alpha_{k'} u_{k'}) + \mathbb{P}(\alpha_k u_k < \alpha_{k'} u_{k'}) = 0$$

Now because  $u_k$  and  $u_{k'}$  are independent, this shows that  $\alpha_k u_k$  and  $\alpha_{k'} u_{k'}$  are both deterministic equal to some value  $m$ . This proves the second claim

Next, we turn to elements of  $J$ . For  $j \in J$ , let  $m_j := \alpha_j \inf \text{Supp}(\mathcal{D}_j) \setminus \{0\}$  and  $M_j := \alpha_j \sup \text{Supp}(\mathcal{D}_j)$ . By definition of  $J$ , we know that for all  $i \neq j$ ,  $\mathbb{P}(\alpha_i u_i \in (m_j, M_j)) = 0$ . Using the same arguments as in Lemma 2 we show that the intervals  $(m_j, M_j)$  for  $i \in J$  are all disjoint, which ensures that there is an ordering  $J = \{j_1, \dots, j_{|J|}\}$  satisfying the desired inequalities. The only remaining piece is to check that  $m_{j_1} \geq m$ , provided that  $K \neq \emptyset$ . Indeed, since  $j_1 \in J$  and  $\alpha_k u_k = m$  almost surely for  $k \in K$ , we have  $m \notin (m_{j_1}, M_{j_1})$ . By definition of  $J$ , we also have  $0 < \mathbb{P}(\alpha_{j_1} u_{j_1} > \alpha_k u_k) = \mathbb{P}(\alpha_{j_1} u_{j_1} > m)$ . Hence,  $M_{j_1} > m$  which implies  $M_{j_1} \geq m_{j_1} \geq m$ .

By definition of  $J$ , for  $l \in [|J| - 1]$  one has  $\mathbb{P}(\alpha_{j_l} u_{j_l} > \alpha_{j_{l+1}} u_{j_{l+1}}) > 0$ . From the previous characterization of the supports of  $\alpha_j u_j$  for  $j \in J$ , this implies that  $\mathbb{P}(\alpha_{j_{l+1}} u_{j_{l+1}} = 0) > 0$ . Since  $\alpha_j > 0$  for all  $j \in J$ , this ends the proof of the third claim.

Suppose that  $K \neq \emptyset$  and  $J \neq \emptyset$ . Then for  $k \in K$  by definition of  $K$  we also have  $0 < \mathbb{P}(\alpha_k u_k \geq \alpha_{j_1} u_{j_1}) = \mathbb{P}(\alpha_{j_1} u_{j_1} \leq m)$ . This implies either  $\mathbb{P}(u_{j_1} = 0) > 0$  or  $m_{j_1} = m$  and  $\mathbb{P}(u_{j_1} = m) > 0$ , which ends the proof. ■

#### 4.1 Tools to construct lower bounds on the achievable utility region

We now prove Lemma 3 which gives conditions for a ball to be achievable within  $\mathcal{U}_\gamma$ .

**Proof of Lemma 3** Fix parameters  $\gamma, \mathbf{x}, r, \delta$  satisfying the assumptions. To show that  $B(\mathbf{x}, r) \subset \mathcal{U}_\gamma$ , we construct an online allocation strategy that reaches these utilities. In fact, we give an allocation strategy to reach utilities

$$\mathcal{R} := \{\mathbf{y} : \mathbf{0} \leq \mathbf{y} \leq \mathbf{z}, \mathbf{z} \in \text{Conv}(\mathcal{U}^{NI}, B(\mathbf{x}, r))\},$$

where vector inequalities refer to the  $n$  inequalities coordinate by coordinate. To give some intuition about this set of utilities, we have  $\text{Conv}(\mathcal{U}^{NI}, B(\mathbf{x}, r)) = \text{Conv}(\mathbf{0}; \{\mathbb{E}[u_i] \mathbf{e}_i, i \in [n]\}; B(\mathbf{x}, r))$ , which is the convex hull of the target ball with the trivial allocations which always allocate the resource to one of the agents  $i \in [n]$ . Unfortunately, this set will not be sufficient, instead, we extend it in all directions  $-\mathbf{e}_i$  for  $i \in [n]$  so long as points stay within the positive orthant  $\mathbb{R}_+^n$ .

**Definition of the allocation function.** For convenience for any  $\mathbf{U} \in \mathcal{U}^*$ , we fix  $p(\cdot; \mathbf{U})$  an allocation function which reaches the utility vector  $\mathbf{U}$  when having access to the full information of agent's utilities.

We start by specifying the allocation strategy for extreme points  $\mathbf{U}$  of  $\mathcal{R}$  that still belong to the ball  $B(\mathbf{x}, r)$ . Following the promised utility definition above, we only need to specify an allocation function  $p(\cdot | \mathbf{U})$  and a direction  $\boldsymbol{\alpha}(\mathbf{U})$  for these utility vectors  $\mathbf{U}$ . In particular, these can be written as  $\mathbf{U} = \mathbf{x} + r\mathbf{y}$  where  $\mathbf{y} \in S_{n-1}$  is a unit vector. We define the allocation function via

$$p(\cdot | \mathbf{U}) := p(\cdot; \mathbf{x} + (r + \delta)\mathbf{y}) \quad \text{and} \quad \boldsymbol{\alpha}(\mathbf{U}) = \mathbf{x} - \mathbf{U} = -r\mathbf{y}.$$

We now extend the definition of the allocation function to the whole domain  $\mathcal{R}$ . To do so, note that any vector  $\mathbf{U} \in \mathcal{R}$  can be characterized by  $\mathbf{0} \leq \mathbf{U} \leq \mathbf{y}$  where  $\mathbf{y} \in \text{Conv}(\mathcal{U}^{NI}, B(\mathbf{x}, r))$  and is also an extreme point of  $\mathcal{R}$ . In particular,  $\mathbf{y}$  can be written as the convex combination of  $\mathbf{0}$ , vectors  $\mathbb{E}[u_i]\mathbf{e}_i$  and some extreme point  $\tilde{\mathbf{y}}$  of  $\mathcal{R}$  within the ball  $B(\mathbf{x}, r)$  (we only need one point from  $B(\mathbf{x}, r)$  at most because the ball is strictly convex). Such a decomposition can be obtained via standard Carathéodory constructions. As a summary, we obtain

$$\mathbf{y} = \sum_{i \in [n]} q_i \mathbb{E}[u_i] \mathbf{e}_i + q_0 \tilde{\mathbf{y}}, \quad \text{where} \quad \mathbf{q} \geq \mathbf{0}, \quad \sum_{i \leq n} q_i \leq 1,$$

and  $U_i = s_i y_i$  where  $s_i \in [0, 1]$  for all  $i \in [n]$ . We then define the allocation and promised utility function via

$$\mathbf{p}(\cdot | \mathbf{y}) = \sum_{i \in [n]} q_i \mathbf{e}_i + q_0 \mathbf{p}(\cdot | \tilde{\mathbf{y}}) \quad \text{and} \quad p_i(\cdot | \mathbf{U}) = s_i p_i(\cdot | \mathbf{y}), \quad i \in [n]. \quad (25)$$

$$\mathbf{W}(\cdot | \mathbf{y}) = \sum_{i \in [n]} q_i \mathbb{E}[u_i] \mathbf{e}_i + q_0 \mathbf{W}(\cdot | \tilde{\mathbf{y}}) \quad \text{and} \quad W_i(\cdot | \mathbf{U}) = s_i W_i(\cdot | \mathbf{y}), \quad i \in [n]. \quad (26)$$

Note in particular that for all the extreme points  $\mathbf{U} = \mathbb{E}[u_i]\mathbf{e}_i$ , the allocation function  $\mathbf{p}(\cdot | \mathbf{U}) = \mathbf{e}_i$  always gives the resource to agent  $i$ , as should be expected.

**Checking that this is a valid allocation strategy for region  $\mathcal{R}$ .** Using the promised utility framework, to show that this is a valid strategy for utilities within  $\mathcal{R}$ , we only need to show that Eqs. (3) and (7) are satisfied. That is, we check that the future promised utilities remain within  $\mathcal{R}$  and their expectation with respect to other agents' utility matches the interim future promise defined as in Eq. (6). We start by proving that this is the case for the extreme points  $\mathbf{U}$  of  $\mathcal{R}$  in the ball  $B(\mathbf{x}, r)$ . For these utility vectors, because we used the choice of allocation strategy from Eq. (11) we only need to check Eq. (3). We will in fact prove the stronger statement that these stay within the ball  $B(\mathbf{x}, r)$ .

As above, we write  $\mathbf{U} = \mathbf{x} + r\mathbf{y}$  with  $\mathbf{y} \in S_{n-1}$ . We note that  $\boldsymbol{\alpha}(\mathbf{U})$  is the normal vector to the boundary of  $\mathcal{R}$  at  $\mathbf{U}$ . Further,  $\boldsymbol{\alpha}(\mathbf{U}) \in \mathbb{R}_+^n$ . Indeed, fix  $i \in [n]$  and construct  $\mathbf{U}^{(i)}$  such that  $U_i^{(i)} = 0$  and  $U_j^{(i)} = U_j$  for all  $j \neq i$ . By construction of  $\mathcal{R}$  using the rectangles we still have  $\mathbf{U}^{(i)} \in \mathcal{R}$  so by convexity of  $\mathcal{R}$ ,

$$0 \leq \boldsymbol{\alpha}(\mathbf{U})^\top (\mathbf{U} - \mathbf{U}^{(i)}) = \alpha_i(\mathbf{U}) U_i.$$

Now because  $B(\mathbf{x}, r + \delta) \subset \mathcal{U}^* \subset \mathbb{R}_+^n$ , we must have  $U_i > 0$ . This proves that  $\alpha_i(\mathbf{U}) \geq 0$ .

Intuitively, the choice of  $\alpha(\mathbf{U}) = \mathbf{x} - \mathbf{U}$  pushes the promised utilities towards the center of the ball  $B(\mathbf{x}, r)$ . Further, we can also characterize the expectation of the promised utility vector which we denote  $\bar{\mathbf{W}} := \mathbb{E}[\mathbf{W}(v_i | \mathbf{U})]$ . By the promise keeping equality Eq. (2),

$$\gamma \bar{\mathbf{W}} = \gamma \mathbb{E}[\mathbf{W}(\mathbf{v} | \mathbf{U})] = \mathbf{U} - (1 - \gamma)(\mathbb{E}[u_i p_i(\mathbf{v} | \mathbf{U})])_{i \in [n]}.$$

Now because the incentive-compatibility constraint is enforced and by definition of the allocation function, we have  $\mathbb{E}[u_i p_i(\mathbf{v} | \mathbf{U})] = \mathbb{E}[u_i p_i(\mathbf{u}; \mathbf{x} + (r + \delta)\mathbf{y})] = x_i + (r + \delta)y_i$  for all  $i \in [n]$ . Combining this with the previous equation gives,

$$\bar{\mathbf{W}} = \mathbf{U} - \frac{1 - \gamma}{\gamma} \delta \mathbf{y} = \mathbf{x} + \left( r - \frac{1 - \gamma}{\gamma} \delta \right) \mathbf{y}. \quad (27)$$

Hence, the term in  $\delta$  from Eq. (27) also pushes the future promise vectors towards the center of the ball.

Using the definition of the promised utilities given the interim promise utility from Eq. (6), for any  $i \in [n]$  since  $W_i(v_i | \mathbf{U})$  is non-increasing in  $v_i$ ,

$$\begin{aligned} |W_i(v_i | \mathbf{U}) - \bar{W}_i| &\leq W_i(0 | \mathbf{U}) - W_i(\bar{v} | \mathbf{U}) \\ &= \frac{1 - \gamma}{\gamma} \int_0^{\bar{v}} (P_i(\bar{v} | \mathbf{U}) - P_i(v | \mathbf{U})) dv \\ &\leq \frac{1 - \gamma}{\gamma} \bar{v}. \end{aligned}$$

Now let  $i_1, i_2$  be the indices of largest and second-largest value of  $|\alpha_i(\mathbf{U})|$  respectively. By the coupling formula from Eq. (11) and because  $\alpha(\mathbf{U}) \geq \mathbf{0}$ , we have

$$\|\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}\|_1 \leq \frac{1 - \gamma}{\gamma} \bar{v} \left( n + \frac{\sum_{j \neq i_1} \alpha_j(\mathbf{U})}{\alpha_{i_1}(\mathbf{U})} + \frac{\alpha_{i_1}(\mathbf{U})}{\alpha_{i_2}(\mathbf{U})} \right) \leq \frac{1 - \gamma}{\gamma} \bar{v} \left( 2n + \frac{\alpha_{i_1}(\mathbf{U})}{\alpha_{i_2}(\mathbf{U})} \right). \quad (28)$$

We now give an upper bound on  $\alpha_{i_1}(\mathbf{U})/\alpha_{i_2}(\mathbf{U})$ . We recall that the region  $\mathcal{R}$  contains all extreme points  $\mathbb{E}[u_i] \mathbf{e}_i$  for  $i \in [n]$ . Hence, since  $\alpha(\mathbf{U})$  is a supporting hyperplane of  $\mathcal{R}$  at  $\mathbf{U}$ , we have,

$$0 \leq \alpha(\mathbf{U})^\top (\mathbf{U} - \mathbb{E}[u_{i_1}] \mathbf{e}_{i_1}) \leq \alpha_{i_1}(\mathbf{U})(U_{i_1} - \mathbb{E}[u_{i_1}]) + \alpha_{i_2}(\mathbf{U}) \sum_{j \neq i_1} U_j$$

In the second inequality, we used the fact that for all  $j \neq i_1$ , we have  $0 \leq \alpha_j(\mathbf{U}) \leq \alpha_{i_2}(\mathbf{U})$ . Because we can write  $\mathbf{U} = \mathbf{x} + r\mathbf{y}$  for  $\mathbf{y} \in S_{n-1}$ , we have  $U_{i_1} \leq x_{i_1} + r$ . Similarly, for any  $i \in [n]$ , we have  $U_i \leq x_i + r \leq 2x_i$ , where in the last inequality, we used the fact that  $\mathbf{x} - r\mathbf{e}_i \in \mathcal{R} \subset \mathbb{R}_+^n$ . As a result, we obtained

$$\alpha_{i_1}(\mathbf{U})(\mathbb{E}[u_{i_1}] - x_{i_1} - r) \leq 2\alpha_{i_2}(\mathbf{U}) \sum_{j \neq i_1} x_j \leq 2n\alpha_{i_2}(\mathbf{U}) \max_{i \in [n]} x_i,$$

which implies

$$\frac{\alpha_{i_1}(\mathbf{U})}{\alpha_{i_2}(\mathbf{U})} \leq \frac{2n \max_{i \in [n]} x_i}{\min_{i \in [n]} (\mathbb{E}[u_i] - x_i - r)}. \quad (29)$$

Putting this together with Eq. (28) gives

$$\|\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}\|_2 \leq \|\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}\|_1 \leq \frac{1 - \gamma}{\gamma} \cdot 2n\bar{v} \left( 1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{i \in [n]} (\mathbb{E}[u_i] - x_i - r)} \right).$$

For convenience, we will write  $V := 2n\bar{v} \left(1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{i \in [n]} (\mathbb{E}[u_i] - x_i - r)}\right)$ . We recall that by construction of the promised utilities, Lemma 1 shows that  $\mathbf{W}(\mathbf{v} \mid \mathbf{U})$  lies in the hyperplane of equation  $\boldsymbol{\alpha}(\mathbf{U})^\top (\mathbf{x} - \bar{\mathbf{W}}) = 0$ , that is  $\mathbf{y}^\top (\mathbf{x} - \bar{\mathbf{W}}) = 0$ . Given Eq. (27), to show that for all  $\mathbf{v}$ , we have  $\mathbf{W}(\mathbf{v} \mid \mathbf{U}) \in B(\mathbf{x}, r)$ , it suffices to show that

$$\left(r - \frac{1-\gamma}{\gamma}\delta, \sup_{\mathbf{v}} \|\mathbf{W}(\mathbf{v} \mid \mathbf{U}) - \mathbf{W}\|_2\right) \in B(0, r).$$

To prove this, we compute

$$r^2 - \left(r - \frac{1-\gamma}{\gamma}\delta\right)^2 - \left(\frac{1-\gamma}{\gamma}V\right)^2 = \frac{1-\gamma}{\gamma^2} [2\gamma\delta r - (1-\gamma)\delta^2 - (1-\gamma)V^2].$$

Therefore, using  $\delta \leq r\gamma/(1-\gamma)$ , we have

$$r^2 - \left(r - \frac{1-\gamma}{\gamma}\delta\right)^2 - \left(\frac{1-\gamma}{\gamma}V\right)^2 \geq \frac{1-\gamma}{\gamma^2} [\gamma\delta r - (1-\gamma)V^2] \geq 0. \quad (30)$$

This ends the proof that whenever  $\mathbf{U}$  is an extreme point of  $\mathcal{R}$  within the ball  $B(\mathbf{x}, r)$ , the future promised utilities  $\mathbf{W}(\mathbf{v} \mid \mathbf{U})$  stay within the ball  $B(\mathbf{x}, r) \subset \mathcal{R}$ .

It remains to check that Eqs. (3) and (7) hold for all other utility vectors  $\mathbf{U} \in \mathcal{R}$ . As in the definition of the allocation function, we write  $U_i = s_i y_i$  where  $s_i \in [0, 1]$  for  $i \in [n]$ , and  $\mathbf{y} = \sum_{i \in [n]} q_i \mathbb{E}[u_i] \mathbf{e}_i + q_0 \tilde{\mathbf{y}}$  where  $\mathbf{q} \geq \mathbf{0}$ ,  $\sum_{i \leq n} q_i \leq 1$ , and  $\tilde{\mathbf{y}}$  is an extreme point of  $\mathcal{R}$  within the ball  $B(\mathbf{x}, r)$ . In the definition of the allocation function from Eq. (25), the only component that is non-constant for  $\mathbf{v} \in [0, \bar{v}]^n$  is the contribution of  $\mathbf{p}(\cdot \mid \tilde{\mathbf{y}})$ . By linearity, Eq. (7) is then a consequence of the definitions Eqs. (25) and (26). We turn to Eq. (3). In the previous part, we showed that  $\mathbf{W}(\cdot \mid \tilde{\mathbf{y}}) \in B(\mathbf{x}, r)$  for all possible reports. As a result, by linearity this directly shows that the future promised utilities stay in the the corresponding ellipsoid  $\mathcal{E}(\mathbf{U})$  defined below:

$$\forall \mathbf{v} \in [0, \bar{v}]^n, \quad \mathbf{W}(\mathbf{v} \mid \mathbf{U}) \in \left\{ (s_i(q_i \mathbb{E}[u_i] + q_0 x_i))_{i \in [n]} + \mathbf{z} : q_0^2 \sum_{i \in [n]} (s_i z_i)^2 \leq r^2 \right\} := \mathcal{E}(\mathbf{U}).$$

We can easily show that  $\mathcal{E}(\mathbf{U}) \subset \mathcal{R}$  by construction of the region  $\mathcal{R}$ . This ends the proof.  $\blacksquare$

In Lemma 5, we simplified the form of the constant  $C$  from Lemma 3 under Assumption 4 which asks that all valuations are lower bounded by some fixed value  $\underline{v} > 0$ . To give some intuitions, the term  $C$  in Lemma 3 essentially measures how close the tentative ball  $B(\mathbf{x}, r)$  is to the limiting hyperplanes  $\{\mathbf{x} \in \mathbb{R}^n : x_i = \mathbb{E}[u_i]\}$  for  $i \in [n]$ . Under Assumption 4, the following result gives geometrical properties of the optimal set  $\mathcal{U}^*$ , which intuitively imply that points in the optimal set  $\mathcal{U}^*$  are sufficiently far from the hyperplanes  $\{\mathbf{x} \in \mathbb{R}^n : x_i = \mathbb{E}[u_i]\}$  for  $i \in [n]$ .

**Lemma 20.** *Under Assumption 4, we have*

$$\mathcal{U}^* \subset \mathbb{R}_+^n \cap \bigcap_{i \in [n]} \left\{ \mathbf{x} : \underline{v} \sum_{j \neq i} x_j \leq \bar{v} (\mathbb{E}[u_i] - x_i) \right\}.$$

**Proof** We already characterized the extreme points of  $\mathcal{U}^*$  as  $\mathbf{0}$  together with the utilities reached by optimal allocation functions  $\mathbf{p}(\cdot)$  for  $\alpha$ -weighted objectives from Eq. (1) for  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ .

Now for  $i \in [n]$ , consider the weights  $\alpha$  such that  $\alpha_i = \bar{v}$ , and  $\alpha_j = \underline{v}$  for all  $j \neq i$ . From Assumption 4 the valuations always lie in  $[\underline{v}, \bar{v}]$ , hence, the allocation function that always allocates the resource to agent  $i$  is  $\alpha$ -optimal. This allocation reaches the utility vector  $\mathbb{E}[u_i]e_i$ . Because this allocation is  $\alpha$ -optimal, this proves the constraint  $\mathcal{U}^* \subset \{\mathbf{x} : \alpha^\top \mathbf{x} \leq \bar{v}\mathbb{E}[u_i]\}$ . ■

We are now ready to give the proof of Lemma 5.

**Proof of Lemma 5** Going back to the proof of Lemma 3, we aim to improve the upper bound of  $\alpha_{i_1}(\mathbf{U})/\alpha_{i_2}(\mathbf{U})$  from Eq. (29) using Lemma 20. Up to rescaling, we suppose without loss of generality that  $\alpha_{i_1}(\mathbf{U}) = \bar{v}$ . Now suppose by contradiction that  $\alpha_{i_2}(\mathbf{U}) < \underline{v}$ . In particular, for all  $j \neq i_1$ , we have  $\alpha_j(\mathbf{U}) \leq \alpha_{i_2}(\mathbf{U}) < \underline{v}$ . Further, Lemma 20, since  $\mathbf{U} \in \mathcal{R} \subset \mathcal{U}^*$ , we have

$$\underline{v} \sum_{j \neq i_1} U_j \leq \bar{v}(\mathbb{E}[u_{i_1}] - U_{i_1})$$

Combining the previous remarks gives,

$$\alpha(\mathbf{U})^\top (\mathbf{U} - \mathbb{E}[u_{i_1}]e_{i_1}) \leq (\alpha_{i_2} - \underline{v})U_{i_2} + \underline{v} \sum_{j \neq i_1} U_j - \bar{v}(\mathbb{E}[u_{i_1}] - U_{i_1}) \leq (\alpha_{i_2} - \underline{v})U_{i_2}.$$

We briefly argue that  $U_{i_2} > 0$ . We recall that  $\mathbf{U} = \mathbf{x} + r\mathbf{y}$  for some element  $\mathbf{y} \in S_{n-1}$ . Further, because  $\alpha(\mathbf{U}) \in \mathbb{R}_+^n$ , we must also have  $\mathbf{y} \in \mathbb{R}_+^n$  otherwise  $\mathbf{U}$  wouldn't be  $\alpha$ -optimal within  $\mathcal{R}$ . Because  $B(x, r + \delta) \subset \mathbb{R}_+^n$  and  $r + \delta > 0$ , we have  $U_{i_2} \geq x_{i_2} > 0$ . This proves that

$$\alpha(\mathbf{U})^\top (\mathbf{U} - \mathbb{E}[u_{i_1}]e_{i_1}) < 0.$$

This contradicts the fact that  $\alpha(\mathbf{U})$  is the normal of a supporting hyperplane at  $\mathbf{U}$  of  $\mathcal{R}$ , which contains in particular  $\mathbb{E}[u_{i_1}]e_{i_1}$ . In conclusion,  $\alpha_{i_1}(\mathbf{U})/\alpha_{i_2}(\mathbf{U}) \leq \bar{v}/\underline{v}$ . We can now use the same proof as for Lemma 3 with  $V := \bar{v}(2n + \bar{v}/\underline{v})$ . ■

To understand the optimal  $\alpha$  policy for some objective direction  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ , it may be useful to focus locally on this direction. In particular, it may be possible to ignore some irrelevant agents in  $[n]$ . This leads to a stronger local version than the statement given in Lemma 3.

**Lemma 21.** Fix  $\gamma \in [0, 1)$  and  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . Define  $I \subset [n]$  as in Lemma 19. Further define  $\tilde{I} = I \cup \{i \in [n] : \mathbb{P}(\alpha_i u_i = \max_{j \neq i} \alpha_j u_j > 0) > 0\}$ . For any  $S \subset [n]$ , denote by  $P_S : \mathbb{R}^n \rightarrow \mathbb{R}^S$  the projection onto the coordinates of  $S$ . For  $\mathbf{y} \in \mathbb{R}^S$ , and  $r > 0$ , we also denote by  $B_S(\mathbf{y}, r)$  the ball  $B(\mathbf{y}, r)$  in  $\mathbb{R}^S$ .

Let  $\mathbf{U}$  an  $\alpha$ -optimal vector and  $\mathbf{x} \in \mathcal{U}^*$  such that for all  $i \notin I$ , we have  $x_i = U_i$ . Let  $r, \delta > 0$  and suppose that there is a partition  $\tilde{I} = I_1 \sqcup \dots \sqcup I_q$  with  $|I_s| \geq 2$  for all  $s \in [q]$  such that for all  $s \neq s' \in [q]$ , Eq. (14) holds and for all  $s \in [q]$ ,

$$B_{I_s}((x_i)_{i \in I_s}, r + \delta) \subset P_{I_s}(\mathcal{U}^* \cap \{\mathbf{y} : \forall i \notin I_s, y_i = U_i\}). \quad (31)$$

We denote  $Z = \max_{i \in [n]} \alpha_i u_i$  and  $Z_{I_s} = \max_{i \in I_s} \alpha_i u_i$  for  $s \in [q]$ . Then, if Eq. (13) holds with  $C := 4n^2 \bar{v}^2 \left(1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{s \in [q], i \in I_s} (\mathbb{E}[u_i] \mathbb{P}(Z_{I_s} = Z > 0) - x_i - r)}\right)^2$ , we have

$$B(\mathbf{x}, r) \cap \{\mathbf{y} : \forall i \notin \tilde{I}, y_i = x_i\} \subseteq \mathbf{x} + \{0\}^{[n] \setminus \tilde{I}} \otimes \bigotimes_{s \in [q]} B_{I_s}(\mathbf{0}, r) \subseteq \mathcal{U}_\gamma.$$

As a remark, note that while Lemma 2 essentially shows that only agents in  $I$  need to be carefully treated (all others can be treated perfectly even with  $\gamma = 0$ ), Lemma 21 takes all  $\tilde{I}$  agents into consideration. Precisely, we added agents  $i \in [n]$  such that  $\mathbb{P}(\alpha_i u_i = \max_{j \neq i} \alpha_j u_j > 0) > 0$ . In practice, adding these agents can only help to satisfy the constraints from Lemma 21. Indeed, by construction there is some other agent  $j \neq i$  with  $j \in \tilde{I}$  such that  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z) > 0$  hence we can group  $i$  and  $j$  in the same set of the partition  $\tilde{I} = I_1 \sqcup \dots \sqcup I_q$ . Because of the property defining  $j$ , the set of  $\alpha$ -optimal vectors of  $\mathcal{U}^*$  is perfectly flat along the direction  $(\alpha_i \mathbb{1}_{k=j} - \alpha_j \mathbb{1}_{k=i})_{k \in [n]}$ , hence we can always fit a ball to satisfy Eq (31). Hence, taking agents  $\tilde{I} \setminus I$  into account within Lemma 21 was without loss. On the other hand, these  $\tilde{I} \setminus I$  agents bring extra flexibility that can be beneficial for agents  $I$  to satisfy Eq (31). In particular, this flexibility will be useful in Section 5.5 for discrete utility distributions.

**Proof** We define  $J$  and  $K$  as in Lemma 19. Using the characterizations in the lemma, for any  $i \notin \tilde{I}$  we have either  $i \notin I \cup J \cup K$  in which case  $x_i = 0$  or  $i \in J$ . We do not have elements in  $K$  since otherwise there are at least two elements  $k \neq k' \in K$  such that  $\mathbb{P}(\alpha_k u_k = \max_{j \neq k} \alpha_j u_j > 0) \geq \mathbb{P}(\alpha_k u_k = \alpha_{k'} u_{k'} = m) \mathbb{P}(\forall j \notin \{k, k'\}, \alpha_j u_j \leq m) > 0$ . Next, for any  $\mathbf{V} \in \{\mathbf{y} : \forall i \notin \tilde{I}, y_i = x_i\}$ , and any allocation  $\mathbf{p}(\cdot)$  that realizes  $\mathbf{V}$  in the full-information setting, we have

$$\forall i \notin (I \cup J \cup K), \quad p_i(\mathbf{u}) = 0 \text{ (a.s.)} \quad (32)$$

$$\forall i \in J \setminus \tilde{I}, \quad p_i(\mathbf{u}) = \mathbb{1}_{\forall j \neq i, u_j \leq m_i} \mathbb{1}_{u_i > 0} \text{ (a.s.)}. \quad (33)$$

For convenience, we pose for  $i \notin \tilde{I}$ ,

$$p_i^{(0)}(\mathbf{u}) = \begin{cases} 0 & i \notin (I \cup J \cup K) \\ \mathbb{1}_{\forall j \neq i, u_j \leq m_i} \mathbb{1}_{u_i > 0} & i \in J \setminus \tilde{I}. \end{cases}$$

It will be useful later to view  $[n] \setminus \tilde{I}$  as part of the partition as well hence we pose  $I_0 := [n] \setminus \tilde{I}$ . We can easily check that Eq. (14) is also satisfied for all  $r \neq r' \in \{0, \dots, q\}$  since

$$\mathbb{P}\left(\max_{i \in I_0} \alpha_i u_i = \max_{i \in \tilde{I}} \alpha_i u_i > 0\right) \leq \sum_{i \in I_0} \mathbb{P}\left(\alpha_i u_i = \max_{j \in [n]} \alpha_j u_j > 0\right) = 0.$$

We first focus on coordinates within  $I_s$  for some fixed  $s \in [q]$ . Let  $\mathbf{V} \in \mathcal{U}^* \cap \{\mathbf{y} : \forall i \notin I_s, y_i = U_i\}$  and  $\mathbf{p}(\cdot; \mathbf{V})$  an allocation that realizes  $\mathbf{V}$  in the full information setting. We also denote  $Z_{-I_s} := \max_{i \notin I_s} \alpha_i u_i$ . Because  $\mathbf{U}$  is  $\alpha$ -optimal, we have

$$\mathbb{E} [Z_{-I_s} \mathbb{1}_{Z_{-I_s} > Z_{I_s}}] \leq \sum_{i \notin I_s} \alpha_i U_i \leq \mathbb{E} [Z_{-I_s} \mathbb{1}_{Z_{-I_s} \geq Z_{I_s}}]$$

Because of Eq. (14), we have  $\mathbb{P}(Z_{-I_s} = Z_{I_s} > 0) = 0$ , which therefore implies

$$\sum_{i \notin I_s} \alpha_i V_i = \sum_{i \notin I_s} \alpha_i U_i = \mathbb{E} [Z_{-I_s} \mathbb{1}_{Z_{-I_s} \geq Z_{I_s}}].$$

As a result, on the event  $\mathcal{E}_s := \{Z_{-I_s} \geq Z_{I_s}\}$ , without loss of generality, we can suppose that  $\mathbf{p}$  allocates to agents in  $[n] \setminus I_s$ :

$$\forall i \in I_s, \quad \mathbb{P}(p_i(\mathbf{u}) > 0, \mathcal{E}_s) = 0. \quad (34)$$



Indeed, if  $Z_{-I_s} = Z_{I_s} = 0$  since  $\alpha_i > 0$  for all  $i \in I_s$ , we have  $u_i = 0$  for all  $i \in I_s$ . Thus allocating to agents in  $I_s$  in that case is useless. We suppose that Eq. (34) will always be satisfied by the allocations of the form  $\mathbf{p}(\cdot; \mathbf{V})$  from now. We also note that  $1 - \mathbb{P}(\mathcal{E}_s) = \mathbb{P}(Z_{I_s} > Z_{-I_s}) = \mathbb{P}(Z_{I_s} \geq Z_{-I_s}, Z_{I_s} > 0) = \mathbb{P}(Z_{I_s} = Z > 0) > 0$ . Otherwise, this would imply  $I_s \cap (I \cup J \cup K) = \emptyset$  and contradict  $I_s \subset \tilde{I}$ .

Eq. (34) is also a characterization in the following sense. Consider the resource allocation problem with only agents  $I_s$  but such that there is no allocation on the event  $\mathcal{E}_s$ . Denote by  $\mathcal{U}_s^* \in \mathbb{R}_+^{I_s}$  the utility region that can be achieved in this setting. Then,

$$\mathcal{U}_s^* = P_{I_s}(\mathcal{U}^* \cap \{\mathbf{y} : \forall i \notin I_s, y_i = U_i\}). \quad (35)$$

Indeed, Eq. (34) shows the inclusion  $\supseteq$  and for the inclusion  $\subseteq$ , any allocation from the game with agents  $I_s$  can be completed on  $\mathcal{E}_s$  by allocating the resource to any agent in  $\arg \max_{i \notin I_s} \alpha_i u_i$ . In particular, by assumption, we have

$$B_{I_s}((x_i)_{i \in I_s}, r + \delta) \subset \mathcal{U}_s^*.$$

In this setting with  $I_s$  agents, we can also define the region  $\mathcal{U}_s^{NI}$  that can be achieved without reports, that is  $\mathcal{U}_s^{NI} = \text{Conv}(\mathbb{E}[u_i](1 - \mathbb{P}(\mathcal{E}_s))\mathbf{e}_i, i \in I_s)$ . The only difference with a standard resource allocation problem between agents  $I_s$  is the constraint on  $\mathcal{E}_s$ . However, the proof of Lemma 3 still holds in this setting as specified below. We construct the utility region

$$\mathcal{R}_s := \{\mathbf{y} \in \mathbb{R}^{I_s} : \mathbf{0} \leq \mathbf{y} \leq \mathbf{z}, \mathbf{z} \in \text{Conv}(\mathcal{U}_s^{NI}, B_{I_s}((x_i)_{i \in I_s}, r))\}.$$

We then specify an allocation strategy for extreme points  $\mathbf{U}$  of  $\mathcal{R}_s$  that still belong to the ball  $B(\mathbf{x}, r)$ . We write  $\mathbf{U} = \mathbf{x} + r\mathbf{y}$  where  $\|\mathbf{y}\| = 1$  and pose

$$\mathbf{p}^{(s)}(\cdot | \mathbf{U}) := \mathbf{p}^{(s)}(\cdot; \mathbf{x} + (r + \delta)\mathbf{y}) \quad \text{and} \quad \boldsymbol{\alpha}(\mathbf{U}) = \mathbf{x} - \mathbf{U},$$

where  $\mathbf{p}^{(s)}(\cdot; \mathbf{z})$  is an allocation function that realizes  $\mathbf{z} \in \mathcal{U}_s^*$  in the full information setting. We then extend the definition of  $\mathbf{p}^{(s)}(\cdot | \mathbf{U})$  and  $\mathbf{W}^{(s)}(\cdot | \mathbf{U})$  to the complete region  $\mathcal{R}_s$  as in Lemma 3 in Eqs. (25) and (26). The proof of Lemma 3 shows that if

$$\frac{r\gamma}{1 - \gamma} \geq \delta \geq C_s \frac{1 - \gamma}{\gamma r},$$

where  $C_s = 4|I_s|^2 \bar{v}^2 \left(1 + \frac{\max_{i \in I_s} \mathbb{E}[u_i]}{\min_{i \in I_s} (\mathbb{E}[u_i](1 - \mathbb{P}(\mathcal{E}_s)) - x_i - r)}\right)^2$ , then the constructed allocation and promised utility functions are valid on  $\mathcal{R}_s$  (they satisfy Eqs. (3) and (7)). We can check that this is the case since with the choice of parameters,  $C \geq \max_{s \in [q]} C_s$ .

We now pose  $\mathcal{R}_0 := (x_i)_{i \in I_0}$  and construct an allocation mechanism to reach the utilities

$$\mathcal{R} := \bigotimes_{0 \leq s \leq q} \mathcal{R}_s \supset \mathbf{x} + \{0\}^{[n] \setminus \tilde{I}} \otimes \bigotimes_{s \in [q]} B_{I_s}(\mathbf{0}, r),$$

by treating each cluster  $I_s$  of agents completely separately. We pose for all  $\mathbf{U} \in \mathcal{R}$ ,  $i \in [n]$  and  $\mathbf{u} \in [0, \bar{v}]^n$ ,

$$\mathbf{p}(\mathbf{u} | \mathbf{U}) := \begin{cases} p_i^{(s)}(\mathbf{u}_{I_s} | \mathbf{U}_{I_s}) \frac{\mathbb{1}(Z_{I_s} > Z_{-I_s})}{1 - \mathbb{P}(\mathcal{E}_s)} & i \in I_s, s \in [q] \\ p_i^{(0)}(\mathbf{u}) & i \notin \tilde{I}. \end{cases}$$

$$\mathbf{W}(\mathbf{u} | \mathbf{U}) := \begin{cases} W_i^{(s)}(\mathbf{u}_{I_s} | \mathbf{U}_{I_s}) & i \in I_s, s \in [q] \\ x_i & i \notin \tilde{I}. \end{cases}$$

By design of the events  $\mathcal{E}_s$ , we still have  $\mathbf{p}(\mathbf{u} \mid \mathbf{U}) \in \Delta_n$  (there are no collisions between different partitions). By construction, Eq. (3) holds and Eq. (7) is satisfied for all  $i \in \tilde{I}$ . It only remains to check that Eq. (7) holds for  $i \notin \tilde{I}$ , which is equivalent to checking both Eq. (2) and the incentive-compatibility constraint Eq. (4). Eq. (2) is directly satisfied by Eqs. (32) and (33). For the incentive-compatibility, note that  $i \notin \tilde{I}$  the definition of  $p_i^{(0)}(\mathbf{u})$  only uses the information  $\mathbb{1}_{u_i > 0}$  about  $u_i$ . Hence the interim promise  $P_i^{(0)}(u_i)$  is constant on  $(0, \bar{v}]$ . As a result, the allocation is also incentive-compatible for agent  $i$  (this is always true if  $u_i = 0$ ).

Altogether, this shows that  $\mathcal{R} \subset \mathcal{U}_\gamma$ , which ends the proof of the theorem.  $\blacksquare$

## 4.2 Tools to construct upper bounds on the achievable utility region

We now prove Lemma 6 that gives tools to prove upper bounds on the achievable region  $\mathcal{U}_\gamma$ . To prove this result, we first need to derive some properties of valid mechanisms for the central planner.

Using the promised utility framework, we suppose that we are given a utility region  $\mathcal{R}$ , allocation functions  $\mathbf{p}(\cdot \mid \mathbf{U})$  and promised utility functions  $\mathbf{W}(\cdot \mid \mathbf{U})$  satisfying Eqs. (3) and (7). We start by showing that to achieve the utilities from a point  $\mathbf{U}$  in the boundary of  $\mathcal{U}^*$  in the full information setting, the allocation function needs to discriminate enough between low and high agent utilities. In practice, this will be helpful to derive a lower bound on the range of interim future promises if one aims to reach the boundary of  $\mathcal{U}^*$ .

**Lemma 22.** *Fix a direction  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ , and an index  $i \in I$ , where  $I$  is defined as in Eq. (12). Define  $q_i := \min\{v \in [0, \bar{v}] : \mathbb{P}(u_i \leq v) \geq 1/2\}$  the median of  $\mathcal{D}_i$ . Then, there exists  $\eta_i > 0$  such that the following holds. Fix any allocation function  $\mathbf{p} : [0, \bar{v}]^n \rightarrow \Delta_n$  that is non-decreasing (that is, for all  $i \in [n]$ ,  $P_i(u_i) = \mathbb{E}_{\mathbf{u}_{-i}}[p_i(\mathbf{u})]$  is non-decreasing) and such that*

$$\boldsymbol{\alpha}^\top (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]} = \max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x}.$$

Then,

$$\mathbb{E}_{u_i \sim \mathcal{D}_i} [|P_i(u_i) - P_i(q_i)| \min(u_i, q_i)] \geq \eta_i.$$

**Proof** We recall that  $I$  was defined in Eq. (12) as the set of indices that do not satisfy condition 2(a) nor 2(b) (see Lemma 2). Fix  $\boldsymbol{\alpha}$  and an index  $i$  satisfying the assumptions. We decompose  $[0, \bar{v}] \setminus \text{Supp}(\mathcal{D}_i)$  as a union of disjoint intervals  $(a_0, b_0 = \bar{v}) \cup [a_1 = 0, b_1) \cup \bigcup_{k \geq 2} (a_k, b_k)$ . We note in particular that  $\alpha_i > 0$ , otherwise choosing  $m_i = M_i = 0$  would satisfy 2(b). Similarly,  $u_i \sim \mathcal{D}_i$  is not a constant random variable otherwise posing  $m_i = M_i$  would satisfy 2(b), where  $u_i = m_i$  almost surely. This implies that  $b_1 < a_0$ .

Now fix an optimal allocation  $p$  for the  $\boldsymbol{\alpha}$ -weighted objective. Because it is  $\boldsymbol{\alpha}$ -optimal, with probability one on  $\mathbf{u}$ , we have  $p_i(\mathbf{u}) > 0$  only if  $i \in \arg \max_{j \in [n]} \alpha_j u_j$ . We also recall that  $P_i(\cdot)$  is non-decreasing by hypothesis. As a result, for all  $u \in \text{Supp}(\mathcal{D}_i) \setminus \{a_k, b_k, k \geq 0\}$  we have

$$\mathbb{P}(Z_i < \alpha_i u) \leq P_i(u) \leq \mathbb{P}(Z_i \leq \alpha_i u). \quad (36)$$

Next, because  $P_i$  is non-decreasing, we have  $P_i(\bar{v}) \geq \mathbb{P}(Z_i < \alpha_i a_0)$  and

$$\begin{cases} \forall u \in [a_k, b_k], & \mathbb{P}(Z_i < \alpha_i a_k) \leq P_i(u) \leq \mathbb{P}(Z_i \leq \alpha_i b_k), & k \geq 1 \\ \forall u \in [a_0, \bar{v}], & \mathbb{P}(Z_i < \alpha_i a_0) \leq P_i(u) \leq P_i(\bar{v}) \end{cases} \quad (37)$$

We now consider the problem of minimizing the desired quantity

$$\begin{aligned} \mathbb{E}_{u_i \sim \mathcal{D}_i} [ |P_i(u_i) - P_i(q_i)| \min(u_i, q_i) ] \\ = \mathbb{E}_{u_i} [ q_i P_i(u_i) \mathbb{1}_{u_i > q_i} ] - \mathbb{E}_{u_i} [ u_i P_i(u_i) \mathbb{1}_{u_i < q_i} ] + (\mathbb{E}[u_i \mathbb{1}_{u_i < q_i}] - q_i \mathbb{P}(u_i > q_i)) P_i(q_i), \end{aligned}$$

under the constraints Eqs. (36) and (37). We can directly check that the optimum is achieved by setting  $P_i^*(u)$  equal to the upper bounds from Eqs. (36) and (37) if  $u < q_i$ , and set  $P_i^*(u)$  equal to their lower bounds if  $u > q_i$ . For  $q_i$ , we have  $P_i^*(q_i) = \mathbb{P}(Z_i < \alpha_i q_i)$  if  $\mathbb{E}[u_i \mathbb{1}_{u_i < q_i}] \leq q_i \mathbb{P}(u_i > q_i)$  and  $P_i^*(q_i) = \mathbb{P}(Z_i \leq \alpha_i q_i)$  otherwise. We denote  $\eta_i := \mathbb{E}_{u_i \sim \mathcal{D}_i} [ |P_i(u_i) - P_i(q_i)| \min(u_i, q_i) ]$  the corresponding value of  $P_i^*$ . As a result, we obtained that for any  $\alpha$ -optimal allocation function,

$$\mathbb{E}[ |P_i(u_i) - P_i(q_i)| \min(u_i, q_i) ] \geq \eta_i.$$

Note that to prove the desired result, it would suffice to show that  $\eta_i > 0$ . The main point is that this lower bound  $\eta_i$  is tight since the interim function  $P_i^*$  can be achieved by an  $\alpha$ -optimal allocation function that either allocates to the agent with index  $\arg \max_{j \neq i} \alpha_j u_j$  or  $i$  (only if  $\alpha_i u_i = \max_{j \neq i} \alpha_j u_j$ ). Hence, to end the proof it suffices to show that for any allocation function  $p$  that is optimal for the  $\alpha$ -weighted objective, one has

$$\mathbb{E}[ |P_i(u_i) - P_i(q_i)| \min(u_i, q_i) ] > 0.$$

Suppose this is not the case by contradiction, hence  $P_i(u) = P_i(q_i)$  for  $\mathcal{D}_i$ -almost all  $u \in (0, \bar{v}]$ . We recall that  $P_i$  is non-decreasing. Therefore, letting  $m_i = \inf \text{Supp}(\alpha_i u_i) \setminus \{0\}$  and  $M_i = \sup \text{Supp}(\alpha_i u_i)$ , if  $M_i > q_i$  we obtain that  $P_i$  is constant on the interval  $[q_i, M_i)$ . Similarly, if  $m_i < q_i$  then  $P_i$  is constant on  $(m_i, q_i]$ . In all cases, we showed that  $P_i$  is constant on  $(m_i, M_i)$ . Hence, Eq. (36) implies that

$$\mathbb{P}(Z_i \leq m_i) \geq \lim_{x \rightarrow m_i^+} P_i(x) = \lim_{x \rightarrow M_i^-} P_i(x) \geq \mathbb{P}(Z_i < M_i).$$

Because  $m_i \leq M_i$ , the previous equation implies  $\mathbb{P}(Z_i \in (m_i, M_i)) = 0$ . On the other hand, by construction we have  $\mathbb{P}(\alpha_i u_i \in \{0\} \cup [m_i, M_i]) = 1$ . Since 2(a) does not hold, for all  $j \neq i$ , we have

$$\mathbb{P}(Z_i < M_i) = \prod_{j \neq i} \mathbb{P}(\alpha_j u_j < M_i) \geq \prod_{j \neq i} \mathbb{P}(\alpha_j u_j < \alpha_i u_i) > 0.$$

Combining this with the previous equation implies  $\mathbb{P}(Z_i \leq m_i) > 0$ . We now fix  $j \neq i$ . We have

$$\begin{aligned} 0 = \mathbb{P}(Z_i \in (m_i, M_i)) &\geq \mathbb{P} \left( \max_{k \notin \{i, j\}} \alpha_k u_k \leq m_i \right) \mathbb{P}(\alpha_j u_j \in (m_i, M_i)) \\ &\geq \mathbb{P}(Z_i \leq m_i) \mathbb{P}(\alpha_j u_j \in (m_i, M_i)). \end{aligned}$$

Therefore,  $\mathbb{P}(\alpha_j u_j \in (m_i, M_i)) = 0$ . This holds for all  $j \neq i$ , hence condition 2(b) holds which contradicts the definition of  $i \in I$ . This ends the proof that  $\eta_i > 0$  and the desired result. ■

We note that having  $i \in I$  is necessary to obtain such a constant  $\eta_i > 0$ . Indeed, suppose that condition 2(a) holds. In that case, there are  $\alpha$  optimal allocations that never allocate the resource to agent  $i$  (since there is an agent  $j$  that always has at least the same utility), which results in having  $P_i(u) = 0$  for all  $u \in [0, \bar{v}]$ . On the other hand, if condition 2(b) holds, there is an interval  $[m_i, M_i]$  such that  $\mathbb{P}(\alpha_i u_i \in [m_i, M_i] \cup \{0\}) = 1$  but  $\mathbb{P}(\alpha_j u_j \in (m_i, M_i)) = 0$  for

all  $j \neq i$ . Then, taking into consideration the reports of agent  $i$  is not really necessary: we can safely allocate the resource to agent  $i$  if and only if  $\max_{j \neq i} \alpha_j u_j \leq m_i$  and  $u_i > 0$  (if  $u_i = 0$ , then truthful reporting is always an equilibrium for agent  $i$ ). This  $\alpha$ -optimal allocation then yields a constant interim allocation probability  $P_i(u) = \mathbb{P}(\max_{j \neq i} \alpha_j u_j \leq m_i)$  for all  $u \in (0, \bar{v}]$ .

We now use Lemma 22 to derive a the lower bound on the range of the interim future promise  $W_i$  even when the realized utility is not on the boundary of  $\mathcal{U}^*$ , but somewhat close to it. Having the realized utility vector  $\mathbf{U}$  close to the boundary is necessary, otherwise, for instance if  $\mathbf{U} \in \mathcal{U}^{NI}$ , we can use a constant allocation function  $\mathbf{p}(\cdot)$  so that  $W_i(v)$  is constant for  $v \in [0, \bar{v}]$  for all  $i \in [n]$ .

**Lemma 23.** Fix  $\gamma \in (0, 1]$  a direction  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ , and an index  $i \in I$  where  $I$  is defined as in Eq. (12). Define  $q_i \in [0, \bar{v}]$  as in Lemma 22. Then, there exists  $\delta_i, \eta_i > 0$  ( $\eta_i$  is the same as in Lemma 22) such that the following holds. For any allocation function  $\mathbf{p} : [0, \bar{v}]^n \rightarrow \Delta_n$  that is non-decreasing (that is, for all  $i \in [n]$   $P_i : u_i \mapsto \mathbb{E}_{\mathbf{u}_{-i}}[p_i(\mathbf{u})]$  is non-decreasing) and such that

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \alpha^\top (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]} \leq \delta_i,$$

the interim promised utility function  $W_i(\cdot)$  for  $i \in [n]$  constructed via Eq. (6) using the interim allocation functions  $P_i$  is non-increasing and satisfies

$$\mathbb{E}_{u_i \sim \mathcal{D}_i} |W_i(u_i) - W_i(q_i)| \geq \frac{1-\gamma}{2\gamma} \eta_i.$$

**Proof** The fact that the interim promise function  $W_i(\cdot | \mathbf{U})$  is non-increasing is direct from the definition in Eq. (6). We omit the terms  $| \mathbf{U}$  in the rest of the proof for readability. From the same definition, for any  $0 \leq a \leq b \leq \bar{v}$ ,

$$\begin{aligned} W_i(a) - W_i(b) &= \frac{1-\gamma}{\gamma} \left( \int_0^b (P_i(b) - P_i(v)) dv - \int_0^a (P_i(a) - P_i(v)) dv \right) \\ &= \frac{1-\gamma}{\gamma} \left( (P_i(b) - P_i(a))a + \int_a^b (P_i(b) - P_i(v)) dv \right) \\ &\geq \frac{1-\gamma}{\gamma} (P_i(b) - P_i(a))a. \end{aligned}$$

As a result,

$$\mathbb{E}|W_i(u_i) - W_i(q_i)| = \mathbb{E}[(W_i(q_i) - W_i(u_i)) \mathbb{1}_{u_i \geq q_i}] + \mathbb{E}[(W_i(u_i) - W_i(q_i)) \mathbb{1}_{u_i \leq q_i}] \quad (38)$$

$$\geq \frac{1-\gamma}{\gamma} \mathbb{E}[|P_i(u_i) - P_i(q_i)| \min(u_i, q_i)]. \quad (39)$$

Hence, we aim to lower bound the expectation on the right-hand side. First, recall that

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} = \mathbb{E} \max_{j \in [n]} \alpha_j u_j = \mathbb{E} \max(\alpha_i u_i, Z_i).$$

and that the maximum is attained for any function that allocates the resource to some agent  $i \in \arg \max_{j \in [n]} \alpha_j u_j$ . Next, for  $u \in [0, \bar{v}]$ , we have

$$\alpha^\top (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]} \leq \mathbb{E}[\alpha_i u_i p_i(\mathbf{u}) + Z_i(1 - p_i(\mathbf{u}))] = \mathbb{E}[\alpha_i u_i P_i(u_i) + Z_i(1 - p_i(\mathbf{u}))].$$

Now for any value  $u \in [0, \bar{v}]$ , let  $z_i(u) = \inf\{v : \mathbb{P}(Z_i \geq v) \leq 1 - P_i(u)\}$ . In particular, since  $P_i(\cdot)$  is non-decreasing, so is  $z_i(\cdot)$ . We can check that

$$\mathbb{E}_{\mathbf{u}_{-i}}[Z_i(1 - p_i(\mathbf{u}))] \leq \mathbb{E}_{\mathbf{u}_{-i}}[Z_i \mathbb{1}_{Z_i > z_i(u_i)}] + z_i(u_i) [\mathbb{P}(Z_i \geq z_i(u_i)) - (1 - P_i(u_i))].$$

Because of the second term in the right-hand side, we define a variable  $B_i \sim \mathcal{B}(\mathbb{P}(Z_i \geq z_i(u_i)) - (1 - P_i(u_i)))$  which is independent from  $\mathbf{u}_{-i}$ . Putting the previous equations together yields

$$\begin{aligned} \max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \boldsymbol{\alpha}^\top (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]} &\geq \\ \mathbb{E}[\max(\alpha_i u_i, Z_i) - Z_i \mathbb{1}_{Z_i > z_i(u_i)} - \alpha_i u_i \mathbb{1}_{Z_i < z_i(u_i)} - \mathbb{1}_{Z_i = z_i(u_i)}(Z_i B_i + \alpha_i u_i(1 - B_i))] &=: A(P_i). \end{aligned} \quad (40)$$

The above inequality is tight in the following sense. For any non-decreasing function  $P_i$ , there is a non-decreasing allocation function  $\mathbf{p}(\cdot)$  that has interim allocation function  $P_i$  for agent  $i$ , and such that

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \boldsymbol{\alpha}^\top (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]} = A(P_i).$$

This allocation can in fact directly be obtained from the definition of  $A(P_i)$ : whenever  $Z_i < z_i(u_i)$  we allocate to agent  $i$ , whenever  $Z_i > z_i(u_i)$  we allocate to some agent in  $\arg \max_{j \neq i} \alpha_j u_j$  and if  $Z_i = z_i(u_i)$  we allocate to one of these two cases depending on the value of  $B_i$ .

In summary, we have for any  $\delta \geq 0$ ,

$$\begin{aligned} \left\{ P_i(\cdot) : \exists \text{ non-decreasing } \mathbf{p}(\cdot), \max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \boldsymbol{\alpha}^\top (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]} \leq \delta, P_i(\cdot) = \mathbb{E}_{\mathbf{u}_{-i}}[p_i(\cdot, \mathbf{u}_{-i})] \right\} \\ = \{ \text{non-decreasing } P_i : [0, \bar{v}] \rightarrow [0, 1], A(P_i) \leq \delta \}. \end{aligned} \quad (41)$$

As a result, we now only focus on non-decreasing interim allocation functions  $P_i$  satisfying  $A(P_i) \leq \delta$ . The set of non-decreasing functions on  $[0, \bar{v}] \rightarrow [0, 1]$  is closed for the product measure (point-wise convergence) and so is the set of functions satisfying the constraint  $A(P_i) \leq \eta$  by the dominated convergence theorem. Further, Tychonov's theorem implies that the set of functions  $[0, \bar{v}] \rightarrow [0, 1]$  is compact for the product measure. This therefore implies that the following optimization problem

$$\inf_{\substack{\text{non-decreasing } P_i : [0, \bar{v}] \rightarrow [0, 1] \\ A(P_i) \leq \delta}} \mathbb{E}_{u_i \sim \mathcal{D}_i} [|P_i(u_i) - P_i(q_i)| \min(u_i, q_i)] =: B(\delta)$$

achieves its minimum, and further that  $B(\delta) \rightarrow B(0)$  as  $\delta \rightarrow 0$ . Note that  $\delta = 0$  corresponds to the case when the allocation is exactly optimal for the  $\boldsymbol{\alpha}$ -weighted objective. This is the case that was covered by Lemma 22, which shows that  $B(0) \geq \eta_i$  for some constant  $\eta_i > 0$  (that only depends on  $\boldsymbol{\alpha}$ ,  $i$ , and the distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$ ). In particular, there exists  $\delta_i > 0$  such that  $B(\delta_i) \geq \eta_i/2$ . Through Eq. (41), this exactly shows that for any non-decreasing allocation function  $p$  satisfying the constraint  $\boldsymbol{\alpha}^\top \mathbf{x} - \boldsymbol{\alpha}^\top (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]} \leq \delta_i$ , we have

$$\mathbb{E} [|P_i(u_i) - P_i(q_i)| \min(u_i, q_i)] \geq \frac{\eta_i}{2}.$$

Together with Eq. (39) this ends the proof. ■

Lemma 23 gives a lower bound on the range required for the interim future promised utilities. In fact it shows a stronger statement about their dispersion which is necessary for our proofs. We

now make use of these range lower bounds to show that some regions of  $\mathcal{U}^*$  cannot be achieved by  $\mathcal{U}_\gamma$ . Intuitively, because future promised utilities need to remain within the achievable region  $\mathcal{U}_\gamma$  (see Eq. (3)) if interim promised utilities are somewhat scattered, this gives a local upper bound on the maximum curvature of  $\mathcal{U}_\gamma$ .

**Proof of Lemma 6** We start by specifying the value of  $c_1$  and  $c_2$ . Fix a direction  $\boldsymbol{\alpha}$  satisfying the assumptions. By the second characterization of Lemma 2, we have  $I \neq \emptyset$ . Fix such an index  $i \in I$ . Let  $\delta_i, \eta_i > 0$  be the constants for which Lemma 23 holds. We pose  $c_1 := \delta_i$  and  $c_2 := \eta_i^2/16$ .

Next, fix some parameters  $\gamma, \mathbf{x}, r$  and  $\delta$  satisfying the assumptions of the lemma. Since  $\mathcal{U}^* \setminus B^\circ(\mathbf{x}, r)$  is compact, so is the component  $\mathcal{C}$ . By contradiction, we suppose that  $\mathcal{U}_\gamma \cap (\mathcal{C} \setminus B^\circ(\mathbf{x}, r + \frac{1-\gamma}{\gamma}\delta)) \neq \emptyset$ . We consider a solution  $\mathbf{U}_0$  to the following optimization problem

$$\max_{\mathbf{y} \in \mathcal{C} \cap \mathcal{U}_\gamma} \|\mathbf{y} - \mathbf{x}\|. \quad (42)$$

The maximum is attained because both  $\mathcal{C}$  and  $\mathcal{U}_\gamma$  are compact. Further, by assumption,

$$\|\mathbf{U}_0 - \mathbf{x}\| \geq r + \frac{1-\gamma}{\gamma}\delta.$$

Because  $\mathbf{U}_0 \in \mathcal{U}_\gamma$ , there exists a mechanism for the central planner that reaches this utility. Using the promised utility framework, we fix a non-decreasing allocation function  $\mathbf{p}(\cdot | \mathbf{U})$  and a promised utility function  $\mathbf{W}(\cdot | \mathbf{U})$  for all  $\mathbf{U} \in \mathcal{U}_\gamma$  that satisfy Eq. (3) (promises stay within  $\mathcal{U}_\gamma$ ) and Eq. (7) (interim promises  $W_i(v_i | \mathbf{U})$  as defined in Eq. (6)).

**Step 1.** To simplify the notations, we let  $\mathbf{U}_1 := (\mathbb{E}[u_i p_i(\mathbf{v} | \mathbf{U}_0)])_{i \in [n]} \in \mathcal{U}^*$ . As a first step, we aim to show that

$$\boldsymbol{\alpha}^\top \mathbf{U}_1 \geq \max_{\mathbf{z} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{z} - \delta_i. \quad (43)$$

By Eq. (3) and the convexity of  $\mathcal{U}_\gamma$ , we have

$$\mathbf{U}_2 := \mathbb{E}[\mathbf{W}(\mathbf{u} | \mathbf{U}_0)] = \mathbf{U}_0 + \frac{1-\gamma}{\gamma}(\mathbf{U}_0 - \mathbf{U}_1) \in \mathcal{U}_\gamma. \quad (44)$$

In the equality, we used the form of the interim promises Eq. (6), or equivalently, we used Eq. (2). We next write  $\mathbf{U}_0 = \mathbf{x} + r_0 \mathbf{y}_0$  where  $r_0 \geq r + \frac{1-\gamma}{\gamma}\delta$  and  $\mathbf{y}_0 \in S_{n-1}$ . We argue that it suffices to show that  $\mathbf{y}_0^\top (\mathbf{U}_1 - \mathbf{U}_0) \geq 0$  to obtain Eq. (43). Indeed, if this is the case, then for any  $\mathbf{U} \in [\mathbf{U}_0, \mathbf{U}_1]$ , one has

$$\|\mathbf{U} - \mathbf{x}\| \geq \mathbf{y}_0^\top (\mathbf{U} - \mathbf{x}) = \mathbf{y}_0^\top (\mathbf{U} - \mathbf{U}_0) + r_0 \geq r_0 \geq r.$$

Because  $\mathcal{U}^*$  is convex, we also have  $[\mathbf{U}_0, \mathbf{U}_1] \subset \mathcal{U}^*$ . Combining this remark with the previous equation shows that  $[\mathbf{U}_0, \mathbf{U}_1] \in \mathcal{U}^* \setminus B^\circ(\mathbf{x}, r)$  so that  $\mathbf{U}_1$  belongs to the same component  $\mathcal{C}$  as  $\mathbf{U}_0$ . By assumption, we have  $\mathcal{C} \subset \{\mathbf{y} : \boldsymbol{\alpha}^\top \mathbf{y} \geq \max_{\mathbf{z} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{z} - c_1\}$  and we posed  $c_1 = \delta_i$ . This ends the proof of Eq. (43).

We now show that

$$\mathbf{y}_0^\top (\mathbf{U}_1 - \mathbf{U}_0) \geq 0. \quad (45)$$

By contradiction suppose that this does not hold, then Eq. (44) implies that  $\mathbf{y}_0^\top (\mathbf{U}_2 - \mathbf{U}_0) > 0$ . The same arguments as above then show that for any  $\mathbf{U} \in [\mathbf{U}_0, \mathbf{U}_2]$ ,

$$\|\mathbf{U} - \mathbf{x}\| \geq \mathbf{y}_0^\top (\mathbf{U} - \mathbf{U}_0) + r_0 > r_0.$$

The previous arguments further show that  $\mathbf{U}_2 \in \mathcal{C}$ . We also recall from Eq. (44) that  $\mathbf{U}_2 \in \mathcal{U}_\gamma$ . Hence, the previous inequality contradicts the definition of  $\mathbf{U}_0$  as a maximizer of the problem in Eq. (42).

**Step 2.** We are now ready to apply Lemma 23 to the allocation  $\mathbf{p}(\cdot | \mathbf{U}_0)$  since the resulting utility vector  $\mathbf{U}_1$  satisfies the necessary conditions as checked in Step 1. We obtain

$$\mathbb{E}_{u_i \sim \mathcal{D}_i} |W_i(u_i | \mathbf{U}_0) - W_i(q_i | \mathbf{U}_0)| \geq \frac{1-\gamma}{2\gamma} \eta_i, \quad (46)$$

where  $q_i := \min\{v \in [0, \bar{v}] : \mathbb{P}(u_i \leq v) \geq 1/2\}$ . Define  $p_B = \frac{1/2 - \mathbb{P}(u_i < q_i)}{\mathbb{P}(u_i = q_i)}$ , with the convention  $0/0 = 0$  and let  $B \sim \mathcal{B}(p_B)$  be a Bernoulli random variable independent from all other random variables. We then define

$$\begin{aligned} \mathbf{U}_+ &= 2\mathbb{E}_{\mathbf{u}}[\mathbf{W}(u_i, \mathbf{u}_{-i} | \mathbf{U}_0) \mathbb{1}_{u_i < q_i} + B\mathbf{W}(u_i, \mathbf{u}_{-i} | \mathbf{U}_0) \mathbb{1}_{u_i = q_i}] \\ \mathbf{U}_- &= 2\mathbb{E}_{\mathbf{u}}[\mathbf{W}(u_i, \mathbf{u}_{-i} | \mathbf{U}_0) \mathbb{1}_{u_i > q_i} + (1-B)\mathbf{W}(u_i, \mathbf{u}_{-i} | \mathbf{U}_0) \mathbb{1}_{u_i = q_i}]. \end{aligned}$$

From Eq. (3), by convexity of  $\mathcal{U}_\gamma$ , and because  $\mathbb{P}(u_i < q_i) + \mathbb{P}(u_i = q_i, B = 1) = 1/2$ , both utility vectors satisfy  $\mathbf{U}_+, \mathbf{U}_- \in \mathcal{U}_\gamma$ . The goal of this step is to show that either  $\mathbf{U}_+$  or  $\mathbf{U}_-$  have a better objective than  $\mathbf{U}_0$  for the optimization problem Eq. (42), to reach a contradiction. We first give a few properties about  $\mathbf{U}_-$  and  $\mathbf{U}_+$ . We note that

$$\mathbf{U}_2 = \mathbb{E}[\mathbf{W}(\mathbf{u} | \mathbf{U}_0)] = \frac{\mathbf{U}_+ + \mathbf{U}_-}{2}. \quad (47)$$

Next, recalling that the interim allocation function  $W_i(\cdot | \mathbf{U}_0)$  is non-increasing, we have

$$\begin{aligned} \|\mathbf{U}_+ - \mathbf{U}_-\| &\geq (\mathbf{U}_+)_i - (\mathbf{U}_-)_i \\ &= (\mathbf{U}_+)_i - W_i(q_i | \mathbf{U}_0) + W_i(q_i | \mathbf{U}_0) - (\mathbf{U}_-)_i \\ &= 2\mathbb{E}_{u_i}[(W_i(u_i | \mathbf{U}_0) - W_i(q_i | \mathbf{U}_0)) \mathbb{1}_{u_i > q_i}] \\ &\quad + 2\mathbb{E}_{u_i}[(W_i(q_i | \mathbf{U}_0) - W_i(u_i | \mathbf{U}_0)) \mathbb{1}_{u_i < q_i}] \\ &= 2\mathbb{E}_{u_i} |W_i(u_i | \mathbf{U}_0) - W_i(q_i | \mathbf{U}_0)|. \end{aligned}$$

Combining with Eq. (46) yields

$$\|\mathbf{U}_+ - \mathbf{U}_-\| \geq \frac{1-\gamma}{\gamma} \eta_i, \quad (48)$$

We now give a lower bound on  $\|\mathbf{U}_2 - \mathbf{x}\|$ . By Eq. (44), and recalling the notation  $\mathbf{U}_0 = \mathbf{x} + r_0 \mathbf{y}_0$ , we obtain

$$\mathbf{y}_0^\top (\mathbf{U}_2 - \mathbf{x}) = r_0 - \frac{1-\gamma}{\gamma} \mathbf{y}_0^\top (\mathbf{U}_1 - \mathbf{U}_0).$$

Within Step 1, we also showed that  $\mathbf{U}_1 \in \mathcal{C} \subset B(\mathbf{x}, r + \delta)$ , where in the last step we used one of the assumptions. Hence,

$$\mathbf{y}_0^\top (\mathbf{U}_1 - \mathbf{U}_0) = \mathbf{y}_0^\top (\mathbf{U}_1 - \mathbf{x}) - r_0 \leq (r + \delta) - r_0 \leq \delta.$$

Putting the two previous equations together yields

$$\|\mathbf{U}_2 - \mathbf{x}\| \geq \mathbf{y}_0^\top (\mathbf{U}_2 - \mathbf{x}) \geq r_0 - \frac{1-\gamma}{\gamma} \delta \geq r, \quad (49)$$

where in the last inequality we used the assumption on  $r_0$ . We now argue that  $\mathbf{U}_2 \in \mathcal{C}$ . Indeed, since  $\mathbf{U}_0, \mathbf{U}_2 \in \mathcal{U}^*$ , by convexity  $[\mathbf{U}_0, \mathbf{U}_2] \subset \mathcal{U}^*$ . Next, for any  $\mathbf{U} = (1 - \theta)\mathbf{U}_2 + \theta\mathbf{U}_0$  for  $\theta \in [0, 1]$ , we have

$$\mathbf{y}_0^\top(\mathbf{U} - \mathbf{x}) = \mathbf{y}_0^\top(\mathbf{U}_2 - \mathbf{x}) + \theta\mathbf{y}_0^\top(\mathbf{U}_0 - \mathbf{U}_2) \geq r + \frac{1-\gamma}{\gamma}\theta\mathbf{y}_0^\top(\mathbf{U}_1 - \mathbf{U}_0) \geq r.$$

In the first inequality, we used Eq. (49), and in the second we used Eq. (45). As a result, using similar arguments as before, we have  $[\mathbf{U}_0, \mathbf{U}_2] \in \mathcal{U}^* \setminus B^\circ(\mathbf{x}, r)$ , hence  $\mathbf{U}_0 \in \mathcal{C}$  implies  $\mathbf{U}_2 \in \mathcal{C}$  as well. Next, by Eq. (47) we can write  $\mathbf{U}_+ = \mathbf{U}_2 + \mathbf{\Delta}$  and  $\mathbf{U}_- = \mathbf{U}_2 - \mathbf{\Delta}$  where  $\mathbf{\Delta} := (\mathbf{U}_+ - \mathbf{U}_-)/2$ .

Suppose that  $(\mathbf{U}_2 - \mathbf{x})^\top \mathbf{\Delta} \geq 0$ . Then,

$$\begin{aligned} \|\mathbf{U}_+ - \mathbf{x}\|^2 &= \|\mathbf{U}_2 - \mathbf{x}\|^2 + \|\mathbf{\Delta}\|^2 + 2(\mathbf{U}_2 - \mathbf{x})^\top \mathbf{\Delta} \\ &\geq \|\mathbf{U}_2 - \mathbf{x}\|^2 + \left(\frac{1-\gamma}{2\gamma}\eta_i\right)^2 \\ &\geq r_0^2 + \frac{1-\gamma}{\gamma} \left(\frac{1-\gamma}{4\gamma}(\eta_i^2 + 4\delta^2) - 2r_0\delta\right) > r_0^2 + \frac{1-\gamma}{\gamma} \left(\frac{1-\gamma}{4\gamma}\eta_i^2 - 2r_0\delta\right) \end{aligned}$$

where in the second inequality we used Eq. (48) and in the third inequality we used Eq. (49). We now recall that by hypothesis  $\delta = \min(c_2 \frac{1-\gamma}{\gamma r}, r)$ . Hence, from the choice of  $c_2$ ,

$$2r_0\delta \leq 4r\delta \leq \frac{1-\gamma}{4\gamma}\eta_i^2.$$

Putting the two previous inequalities together, we obtain  $\|\mathbf{U}_+ - \mathbf{x}\| > \|\mathbf{U}_0 - \mathbf{x}\| = r_0$ . We recall that  $\mathbf{U}_2 \in \mathcal{C}$  and by assumption  $(\mathbf{U}_2 - \mathbf{x})^\top (\mathbf{U}_+ - \mathbf{U}_2) \geq 0$ . Since we also have  $\mathbf{U}_+ \in \mathcal{U}^*$ , the same arguments as in Step 1 show that  $\mathbf{U}_+ \in \mathcal{C}$ . This contradicts the definition of  $\mathbf{U}_0$  as the maximizer of the problem in Eq. (42).

In the other case when  $(\mathbf{U}_2 - \mathbf{x})^\top \mathbf{\Delta} \leq 0$ , the same arguments show that  $\|\mathbf{U}_- - \mathbf{x}\| > \|\mathbf{U}_0 - \mathbf{x}\|$  and  $\mathbf{U}_- \in \mathcal{C}$ , reaching the same contradiction. This ends the proof of the desired result:

$$\mathcal{U}_\gamma \cap \left(\mathcal{C} \setminus B^\circ\left(\mathbf{x}, r + \frac{1-\gamma}{\gamma}\delta\right)\right) = \emptyset. \quad (50)$$

**Proof of the second claim.** We now prove the second claim of the lemma as a simple consequence of Eq. (50). First, note that because  $\mathcal{U}_\gamma$  is compact, the maximum in  $\max_{\mathbf{y} \in \mathcal{U}_\gamma} \mathbf{d}^\top(\mathbf{y} - \mathbf{x})$  is well defined. Fix  $\mathbf{U} = \mathbf{x} + \|\mathbf{U} - \mathbf{x}\|\mathbf{d} \in \mathcal{C}$  such a maximizer. Consider any point  $\mathbf{y} \in \mathcal{U}_\gamma$  such that  $\mathbf{d}^\top(\mathbf{y} - \mathbf{x}) \geq \|\mathbf{U} - \mathbf{x}\|$ . Then, by convexity of  $\mathcal{U}^*$ , we have  $[\mathbf{U}, \mathbf{y}] \subset \mathcal{U}^*$ . Further,

$$(\mathbf{U} - \mathbf{x})^\top(\mathbf{y} - \mathbf{U}) = \|\mathbf{U} - \mathbf{x}\|\mathbf{d}^\top(\mathbf{y} - \mathbf{x}) - \|\mathbf{U} - \mathbf{x}\|^2 \geq 0.$$

Then, the same arguments as in Step 1 show that since  $\mathbf{U} \in \mathcal{C}$ , we must also have  $\mathbf{y} \in \mathcal{C}$ . Because we have  $\mathbf{y} \in \mathcal{U}_\gamma$ , Eq. (50) implies that  $\mathbf{y} \in B^\circ(\mathbf{x}, r + \frac{1-\gamma}{\gamma}\delta)$ . In particular,

$$\mathbf{d}^\top(\mathbf{y} - \mathbf{x}) \leq \|\mathbf{y} - \mathbf{x}\|\|\mathbf{d}\| < \left(r + \frac{1-\gamma}{\gamma}\delta\right)\|\mathbf{d}\|.$$

This ends the proof. ■



In some cases, it may be useful to design lower bounds based on a subproblem instead of the full resource allocation problem as stated in Lemma 6. Intuitively, we may want to focus on the impact of a single agent  $i$ , which corresponds to a 2-agent setting in which one has utility  $\alpha_i u_i$  and the other has utility  $\max_{j \neq i} \alpha_j u_j$ . In practice, this setup will be very useful, and we specify the corresponding lower bounds in the result below. This is a simple consequence of Lemma 6. First, for  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  and  $i \in [n]$  such that  $\alpha_i > 0$  and  $\boldsymbol{\alpha}_{-i} \neq \mathbf{0}$ , define  $\tilde{\boldsymbol{\alpha}}_{-i} := \boldsymbol{\alpha}_{-i} / \|\boldsymbol{\alpha}_{-i}\|$ . For any  $\mathbf{x} = (x_i, \mathbf{x}_{-i}) \in \mathbb{R}^2$  and  $r > 0$ , we define the sets

$$\begin{aligned} B(\mathbf{x}, r; \boldsymbol{\alpha}, i) &:= \{\mathbf{y} \in \mathbb{R}^n : (y_i - x_i)^2 + (\tilde{\boldsymbol{\alpha}}_{-i}^\top \mathbf{y}_{-i} - x_{-i})^2 \leq r^2\} \\ B^\circ(\mathbf{x}, r; \boldsymbol{\alpha}, i) &:= \{\mathbf{y} \in \mathbb{R}^n : (y_i - x_i)^2 + (\tilde{\boldsymbol{\alpha}}_{-i}^\top \mathbf{y}_{-i} - x_{-i})^2 < r^2\}. \end{aligned}$$

**Lemma 24.** *Fix a direction  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  and an index  $i \in I$  where  $I$  is defined as in Eq. (12). Suppose that  $\boldsymbol{\alpha}_{-i} \neq \mathbf{0}$ . There exist two constants  $c_1, c_2 > 0$  such that the following holds.*

*For  $\gamma \in (0, 1)$ ,  $r > 0$ , and  $(x_i, \mathbf{x}_{-i}) \in \mathbb{R}^2$ , suppose that  $\mathcal{U}^* \setminus B^\circ(\mathbf{x}, r; \boldsymbol{\alpha}, i)$  has a connected component  $\mathcal{C} \subset \{\mathbf{y} : \boldsymbol{\alpha}^\top \mathbf{y} \geq \max_{\mathbf{z} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{z} - c_1\}$ . If*

$$\mathcal{C} \subset B(\mathbf{x}, r + \delta; \boldsymbol{\alpha}, i), \quad \delta = \min\left(c_2 \frac{1 - \gamma}{\gamma r}, r\right),$$

*then we have*

$$\mathcal{U}_\gamma \cap \left(\mathcal{C} \setminus B^\circ\left(\mathbf{x}, r + \frac{1 - \gamma}{\gamma} \delta; \boldsymbol{\alpha}, i\right)\right) = \emptyset.$$

*In particular, for any  $\mathbf{U} \in \mathcal{C}$ , writing  $(U_i, \tilde{\boldsymbol{\alpha}}_{-i}^\top \mathbf{U}_{-i}) = \mathbf{x} + \|(U_i, \tilde{\boldsymbol{\alpha}}_{-i}^\top \mathbf{U}_{-i}) - \mathbf{x}\| (d_i, d_{-i})$ ,*

$$\max_{\mathbf{y} \in \mathcal{U}_\gamma} d_i (y_i - x_i) + d_{-i} (\tilde{\boldsymbol{\alpha}}_{-i}^\top \mathbf{y}_{-i} - x_{-i}) < \left(r + \frac{1 - \gamma}{\gamma} \delta\right) \|\mathbf{d}\|.$$

**Proof** We first prove that  $\boldsymbol{\alpha}_{-i} \neq \mathbf{0}$ . Otherwise, taking  $m_i = 0$  and  $M_i = \alpha_i \bar{v}$ , we have of course  $\mathbb{P}(\alpha_i u_i \in [m_i, M_i]) = 1$  but  $\mathbb{P}(\alpha_j u_j \in (m_i, M_i)) \leq \mathbb{P}(\alpha_j u_j > 0) = 0$  for all  $j \neq i$ , hence  $i \notin I$ , which is absurd. In the proof of Lemma 6, we first used the characterization from Lemma 2 to show that there exists some index  $i \in I$  ( $I \neq \emptyset$ ). From then, the only incentive-compatibility constraint that was used within the proof was that of agent  $i$ . Precisely, it only uses Eq. (7) for agent  $i$  to use the fact that the interim promised utility is given according to Eq. (6) for which Lemma 22 gives bounds.

As a result, the lower bounds also hold in a setting in which agents  $j \neq i$  are non-strategic, that is, the central planner observes all utilities  $u_j$  for  $j \notin i$ . If we denote by  $\tilde{\mathcal{U}}_{\gamma, i}$  the utility region that can be achieved in this setting, we clearly have  $\mathcal{U}_\gamma \subset \tilde{\mathcal{U}}_{\gamma, i}$  since there are fewer constraints. From the perspective of maximizing the  $\boldsymbol{\alpha}$  total utility, this corresponds to the setting in which there are effectively two agents, agent  $i$  with the same utility distribution  $u_i \sim \mathcal{D}_i$  and an agent  $-i$  that has utilities  $u_{-i} := \max_{j \neq i} \tilde{\alpha}_i u_j$ . We denote by  $\mathcal{D}_{-i}$  its distribution and introduce the following linear mapping,

$$\Phi : \mathbf{U} \in \mathbb{R}^n \mapsto (U_i, \tilde{\boldsymbol{\alpha}}_{-i}^\top \mathbf{U}_{-i}).$$

Let  $\mathcal{U}_i^*$  and  $\mathcal{U}_{\gamma, i}$  be the regions achievable with full information, and in the  $\gamma$ -discounted strategic setting for this 2-agent allocation problem. We first note that

$$\mathcal{U}_i^* = \Phi(\mathcal{U}^*). \tag{51}$$

Since these are both convex sets, it suffices to focus on extreme points. By definition of  $\mathcal{D}_{-i}$ , for any  $\mathbf{d} \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ , we indeed have

$$\max_{\mathbf{x} \in \mathcal{U}_{\gamma,i}} \mathbf{d}^\top \mathbf{x} = \mathbb{E} \max \left( d_i u_i, d_{-i} \max_{j \neq i} \tilde{\alpha}_j u_j \right) = \max_{\mathbf{x} \in \mathcal{U}^*} (d_i, d_{-i} \tilde{\boldsymbol{\alpha}}_{-i})^\top \mathbf{x} = \max_{\mathbf{x} \in \mathcal{U}^*} \mathbf{d}^\top \Phi(\mathbf{x}).$$

Importantly, we also have

$$\Phi(\mathcal{U}_\gamma) \subset \Phi(\tilde{\mathcal{U}}_{\gamma,i}) = \mathcal{U}_{\gamma,i}. \quad (52)$$

The first inequality is immediate from  $\mathcal{U}_\gamma \subset \tilde{\mathcal{U}}_{\gamma,i}$  and the equality comes from the definition of  $\mathcal{D}_{-i}$  and  $\Phi$ . Before applying the proof of Lemma 6, we only need to check that  $\mathcal{D}_i$  still does not satisfy the conditions 2(a) nor 2(b) from Lemma 2 in the new 2-agent setting for the direction  $\mathbf{d} := (\alpha_i, \boldsymbol{\alpha}_{-i})$ . Using the assumption  $i \in I$ , for all  $i \in [n]$  we have  $\mathbb{P}(\alpha_i u_i > \alpha_j u_j) > 0$  hence since  $u_i$  and  $u_j$  are independent, there exists  $m_j$  such that  $\mathbb{P}(\alpha_i u_i > m_i), \mathbb{P}(\alpha_j u_j \leq m_j) > 0$ . Then,

$$\begin{aligned} \mathbb{P}(d_i u_i > d_{-i} u_{-i}) &\geq \mathbb{P}(\alpha_i u_i > \max_{j \neq i} m_j, \forall j \neq i, \alpha_j u_j \leq m_j) \\ &= \prod_{j \neq i} \mathbb{P}(\alpha_j u_j \leq m_j) \cdot \min_{j \neq i} \mathbb{P}(\alpha_i u_i > m_j) > 0. \end{aligned}$$

Hence 2(a) does not hold. Now suppose by contradiction that 2(b) holds and let  $[m_i, M_i]$  the guaranteed interval such that  $\mathbb{P}(\alpha_i u_i \in [m_i, M_i] \cup \{0\}) = 1$  and

$$0 = \mathbb{P}(d_{-i} u_{-i} \in (m_i, M_i)) = \mathbb{P} \left( \max_{j \neq i} \alpha_j u_j \in (m_i, M_i) \right).$$

In the proof of Lemma 2 (within the proof of  $1. \Rightarrow 2.$ ), we showed that this implies  $\mathbb{P}(\alpha_j u_j \in (m_i, M_i)) = 0$  for all  $j \neq i$ , which contradicts the hypothesis  $i \in I$ .

Therefore, we are ready to apply the proof of Lemma 6 for the 2-agent game for direction  $\mathbf{d}$  and agent  $i$ . Indeed, we can check that in the 2-agent setting,  $B(\mathbf{x}, r) = \Phi(B(\mathbf{x}, r; \boldsymbol{\alpha}, i))$  and similarly for their open counterparts. Hence, using Eq. (51),

$$\mathcal{U}_i^* \setminus B^\circ(\mathbf{x}, r) = \Phi(\mathcal{U}^* \setminus B^\circ(\mathbf{x}, r; \boldsymbol{\alpha}, i)),$$

and as a result  $\Phi(\mathcal{C})$  is a connected component of  $\mathcal{U}_i^* \setminus B^\circ(\mathbf{x}, r)$ , which satisfies

$$\begin{aligned} \Phi(\mathcal{C}) &\subset \Phi(\{\mathbf{y} : \boldsymbol{\alpha}^\top \mathbf{y} \geq \max_{\mathbf{z} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{z} - c_1\}) = \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{d}^\top \mathbf{y} \geq \max_{\mathbf{z} \in \mathcal{U}_i^*} \mathbf{d}^\top \mathbf{z} - c_1\}, \\ \Phi(\mathcal{C}) &\subset \Phi(B(\mathbf{x}, r + \delta; \boldsymbol{\alpha}, i)) = B(\mathbf{x}, r + \delta). \end{aligned}$$

Thus, the proof of Lemma 6 gives

$$\Phi(\mathcal{U}_\gamma) \cap \left( \Phi(\mathcal{C}) \setminus B^\circ \left( \mathbf{x}, r + \frac{1-\gamma}{\gamma} \delta \right) \right) \subset \mathcal{U}_{\gamma,i} \cap \left( \Phi(\mathcal{C}) \setminus B^\circ \left( \mathbf{x}, r + \frac{1-\gamma}{\gamma} \delta \right) \right) = \emptyset,$$

where in the first inclusion we used Eq. (52). This exactly gives the desired result. The second claim is also a direct translation of the second claim from Lemma 6 that we obtained on this 2-agent game.  $\blacksquare$

To simplify the use of Lemmas 3 and 6, we give simple equivalent computational constraints to prove that a ball belongs to  $\mathcal{U}^*$ , directly in terms of the distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$ .

**Lemma 25.** Let  $\mathbf{x}$  and  $r > 0$ .  $B(\mathbf{x}, r) \subset \mathcal{U}^*$  if and only if  $\min_{i \in [n]} x_i \geq r$  and

$$\forall \boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}, \quad \mathbb{E} \left[ \max_{i \in [n]} \alpha_i u_i \right] \geq \boldsymbol{\alpha}^\top \mathbf{x} + \|\boldsymbol{\alpha}\| r.$$

**Proof** Any convex set  $C$  is the intersection of its supporting hyperplanes. Hence,

$$\begin{aligned} B(\mathbf{x}, r) \subset \mathcal{U}^* &\iff \forall \boldsymbol{\alpha} \neq \mathbf{0}, \quad \max_{\mathbf{y} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{y} \geq \max_{\mathbf{y} \in B(\mathbf{x}, r)} \boldsymbol{\alpha}^\top \mathbf{y} \\ &\iff B(\mathbf{x}, r) \subset \mathbb{R}_+^n \quad \text{and} \quad \forall \boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}, \quad \mathbb{E} \left[ \max_{i \in [n]} \alpha_i u_i \right] \geq \boldsymbol{\alpha}^\top \mathbf{x} + \|\boldsymbol{\alpha}\| r. \end{aligned}$$

This ends the proof. ■

## 5 Characterizations in the discounted infinite-horizon setting

In this section, we prove our main results concerning lower bounds for the reachable region  $\mathcal{U}_\gamma$  compared to the full-information reachable set  $\mathcal{U}^*$ . Before doing so, we need to introduce a pruning procedure that will help “pre-condition” the allocation problem.

### 5.1 A pruning procedure

For a given vector  $\mathbf{U} \in \mathcal{U}^*$ , we consider the following recursive pruning process. First let  $I(\mathbf{U}) = \{i \in [n] : U_i > 0\}$  and initialize  $\mathbf{U}^{(0)} = \mathbf{U}$ . For any  $k \geq 0$ , if there exists  $i \in I(\mathbf{U})$  with  $U_i^{(k)} \geq \mathbb{E}[u_i]$ , we let  $i_{k+1} := i$ . If  $\mathbb{P}(u_i = 0) = 0$ , pose  $\mathbf{U}^{(k+1)} := \mathbf{0}$ , otherwise

$$\mathbf{U}^{(k+1)} := \frac{\mathbf{U}^{(k)} - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}}{\mathbb{P}(u_{i_k} = 0)}.$$

If for all  $i \in I(\mathbf{U})$ ,  $U_i < \mathbb{E}[u_i]$ , the process stops. We denote by  $K$  the index of the last constructed index  $i_K$  and denote  $\tilde{\mathbf{U}} := \mathbf{U}^{(K+1)}$  the final constructed vector. We pose  $J(\mathbf{U}) = \{i \in I(\mathbf{U}) : \tilde{U}_i > 0\}$ .

Properties of this pruning process are given below in Lemma 26. The last claim shows that the pruning procedure preserves the regions  $\mathcal{U}_\gamma$  while deleting agents. Essentially, this shows that without loss of generality, we can assume that  $0 < U_i < \mathbb{E}[u_i]$  for all  $i \in [n]$  (those are properties satisfied by  $(\tilde{U}_j)_{j \in J(\mathbf{U})}$ ), which will be useful to derive general lower bounds.

**Lemma 26.** Let  $\mathbf{U} \in \mathcal{U}^*$  and construct  $i_1, \dots, i_K$ ,  $\tilde{\mathbf{U}}$  and  $J(\mathbf{U})$  according to the pruning process above.

1. For all  $k \in [K]$ ,  $\mathbf{U}^{(k)} \in \mathcal{U}^*$ . In particular,  $U_i^{(k)} \leq \mathbb{E}[u_i]$  for all  $i \in I(\mathbf{U})$ . Also, for  $k \in [K-1]$ ,  $\mathbb{P}(u_{i_k} = 0) > 0$ .

2.

$$\forall k \in [K], \quad U_{i_k} = \mathbb{E}[u_{i_k}] \prod_{l < k} \mathbb{P}(u_{i_l} = 0) \quad \text{and} \quad \mathbf{U} = (U_i \mathbb{1}_{i \notin J(\mathbf{U})})_{i \in [n]} + \prod_{k \in [K]} \mathbb{P}(u_{i_k} = 0) \cdot \tilde{\mathbf{U}}.$$

3.  $\tilde{\mathbf{U}} \in \mathcal{U}^*$ . Equivalently,  $(\tilde{U}_j)_{j \in J(\mathbf{U})}$  belongs to the achievable set  $\tilde{\mathcal{U}}^*$  for the full-information game in which only agents  $J(\mathbf{U})$  are present.

4. Fix  $\gamma \in [0, 1)$  and denote by  $\tilde{\mathcal{U}}_\gamma$  the achievable set for the  $\gamma$ -discounted game in which only agents  $J(\mathbf{U})$  are present. Then,  $\mathbf{U} \in \mathcal{U}_\gamma$  if and only if  $(\tilde{U}_j)_{j \in J(\mathbf{U})} \in \tilde{\mathcal{U}}_\gamma$ .

**Proof** Suppose that for some  $k \in \{0, 1, \dots, K\}$  we have  $\mathbf{U}^{(k)} \in \mathcal{U}^*$ . By construction,  $U_{i_k}^{(k)} \geq \mathbb{E}[u_{i_k}]$ . However, since  $\mathbf{U}^{(k)} \in \mathcal{U}^*$  we also have  $U_{i_k}^{(k)} \leq \mathbb{E}[u_{i_k}]$ . This proves  $U_{i_k}^{(k)} = \mathbb{E}[u_{i_k}]$ . Also,  $i_k \in I(\mathbf{U})$  hence  $\mathbb{E}[u_{i_k}] \geq U_{i_k} > 0$ . As a result,  $\mathbb{P}(u_{i_k} > 0) > 0$  and any allocation function  $\mathbf{p}(\cdot)$  that realizes  $\mathbf{U}^{(k)}$  allocates to agent  $i_k$  whenever  $u_{i_k} > 0$ ,  $\mathcal{D}_{i_k}$ -almost surely. This implies that for any  $i \neq i_k$ ,

$$U_i^{(k)} = \mathbb{E}[u_i p_i(\mathbf{u})] = \mathbb{E}[u_i p_i(\mathbf{u}) \mathbb{1}_{u_{i_k}=0}] = \mathbb{P}(u_{i_k} = 0) \mathbb{E}[u_i p_i(0, \mathbf{u}_{-i_k})].$$

If  $\mathbb{P}(u_{i_k} = 0) = 0$ , then we had  $\mathbf{U}^{(k)} = \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}$  and the process ends with  $k = K$ . Otherwise, the previous equation already shows that the allocation in which we force  $u_{i_k} = 0$ , that is

$$\mathbf{p}' : \mathbf{u} \in [0, \bar{v}]^n \mapsto (p_i(0, \mathbf{u}_{-i_k}) \mathbb{1}_{i \neq i_k})_{i \in [n]}, \quad (53)$$

realizes  $\mathbf{U}^{(k+1)}$ . Indeed, it only suffices to check  $\mathbb{E}[u_{i_k} p'_i(\mathbf{u})] = 0 = U_{i_k}^{(k+1)}$ . As a result,  $\mathbf{U}^{(k+1)} \in \mathcal{U}^*$ . This proves the first claim of the lemma as well as  $\tilde{\mathbf{U}} = \mathbf{U}^{(K+1)} \in \mathcal{U}^*$ , which is the third claim. The above arguments also show  $U_i^{(k)} = \mathbb{E}[u_{i_k}]$  for all  $k \in [K]$ , and that  $U_i^{(k)} = 0$  for all  $i \notin I(\mathbf{U})$  and  $i \in \{i_1, \dots, i_{k-1}\}$ . We can then prove the second claim by simple induction.

**Proving that  $\mathbf{U} \in \mathcal{U}_\gamma \Rightarrow \tilde{\mathbf{U}} \in \mathcal{U}_\gamma$ .** To prove the last claim, further suppose that  $\mathbf{U}^{(k)} \in \mathcal{U}_\gamma$  for  $\gamma \in [0, 1)$  and some  $k \in \{0, 1, \dots, K\}$ . If  $\mathbb{P}(u_{i_k} = 0) = 0$ , we know that  $\mathbf{U}^{(k+1)} = \mathbf{0} \in \mathcal{U}_0 \subset \mathcal{U}_\gamma$ . From now, we suppose that

$$P_k := \mathbb{P}(u_{i_k} = 0) > 0.$$

Let  $\mathbf{p}(\cdot | \mathbf{V})$  and  $\mathbf{W}(\cdot | \mathbf{V})$  be valid incentive-compatible allocation and promised utility functions for all  $\mathbf{V} \in \mathcal{U}_\gamma$  (that is, satisfying Eqs. (2) to (4)). Let  $I_k = \{i \in [n], U_i^{(k)} > 0\}$ . We denote  $\mathcal{R}_k = \mathcal{U}_\gamma \cap \{\mathbf{V} : V_{i_k} = \mathbb{E}[u_{i_k}], \forall i \notin I_k, V_i = 0\}$ . In particular,  $\mathbf{U}^{(k)} \in \mathcal{R}_k$ . For any  $\mathbf{V} \in \mathcal{R}_k$ ,

$$\begin{aligned} \mathbb{E}[u_{i_k}] &= V_{i_k} = (1 - \gamma) \mathbb{E}[u_{i_k} p_{i_k}(\mathbf{u} | \mathbf{V})] + \gamma \mathbb{E}[W_{i_k}(\mathbf{u} | \mathbf{V})] \\ &\leq (1 - \gamma) \mathbb{E}[u_{i_k}] + \gamma \mathbb{E}[u_{i_k}] = \mathbb{E}[u_{i_k}]. \end{aligned}$$

In the inequality, we used the fact that  $\mathbf{W}(\mathbf{v} | \mathbf{U}^{(k)}) \in \mathcal{U}_\gamma \subset \mathcal{U}^*$ , hence  $W_{i_k}(\mathbf{v} | \mathbf{U}^{(k)}) \leq \mathbb{E}[u_{i_k}]$ . The above computations show that before,  $\mathcal{D}_{i_k}$ -almost surely,  $\mathbf{p}(\cdot | \mathbf{U}^{(k)})$  allocates to  $i_k$  whenever  $u_{i_k} > 0$ . Also, almost-surely  $\mathbf{W}(\mathbf{u} | \mathbf{U}^{(k)}) \in \mathcal{R}_k$  as well. Without loss of generality, we can assume that this the case deterministically for all  $\mathbf{u} \in [0, \bar{v}]^n$  (up to modification of the promised utility function on zero-measure regions). We now define

$$\mathcal{R}'_k := \left\{ \frac{\mathbf{V} - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}}{P_k}, \mathbf{V} \in \mathcal{R}_k \right\}.$$

We construct the following allocation and promised utility functions similarly as in Eq. (53),

$$\begin{aligned} \mathbf{p}'(\mathbf{v} | \mathbf{V}') &:= (p_i(0, \mathbf{v}_{-i_k} | \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k} + P_k \mathbf{V}') \mathbb{1}_{i \neq i_k})_{i \in [n]}, \\ \mathbf{W}'(\mathbf{v} | \mathbf{V}') &:= \frac{1}{P_k} (\mathbb{E}_{u_{i_k} \sim \mathcal{D}_{i_k}} [W_i(u_{i_k}, \mathbf{v}_{-i_k} | \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k} + P_k \mathbf{V}')] \mathbb{1}_{i \neq i_k})_{i \in [n]}, \end{aligned}$$

for all  $\mathbf{V}' \in \mathcal{R}'_k$  and  $\mathbf{v} \in [0, \bar{v}]^n$ . We can check that

$$\mathbf{W}'(\mathbf{v} \mid \mathbf{V}') = \frac{\mathbf{V} - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}}{P_k}, \quad \text{where } \mathbf{V} = \mathbb{E}_{u_{i_k}}[\mathbf{W}(u_{i_k}, \mathbf{v}_{-i_k} \mid \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k} + P_k \mathbf{V}')] \in \mathcal{R}_k.$$

Here we used the fact that  $\mathcal{R}_k$  is convex since  $\mathcal{U}_\gamma$  is convex. Thus, Eq. (3) is satisfied, that is all promised utilities stay within  $\mathcal{R}'_k$ . Then, the same arguments as for Eq. (53) show that for any  $\mathbf{V}' \in \mathcal{R}'_k$ , letting  $\tilde{\mathbf{V}} := \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k} + P_k \mathbf{V}'$  we have

$$(\mathbb{E}[u_i p'_i(\mathbf{u} \mid \mathbf{V}')]_{i \in [n]}) = \frac{(\mathbb{E}[u_i p_i(\mathbf{u} \mid \tilde{\mathbf{V}})]_{i \in [n]} - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k})}{P_k}.$$

On the other hand,

$$\mathbb{E}[\mathbf{W}'(\mathbf{u} \mid \mathbf{V}')] = \frac{\mathbb{E}_{\mathbf{u}} \mathbb{E}_{u'_{i_k}}[\mathbf{W}(u'_{i_k}, \mathbf{u}_{-i_k} \mid \tilde{\mathbf{V}})] - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}}{P_k} = \frac{\mathbb{E}_{\mathbf{u}}[\mathbf{W}(\mathbf{u} \mid \tilde{\mathbf{V}})] - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}}{P_k}$$

Combining the two previous equations we obtain

$$\begin{aligned} (1 - \gamma)(\mathbb{E}[u_i p'_i(\mathbf{u} \mid \mathbf{V}')]_{i \in [n]}) + \gamma \mathbb{E}[\mathbf{W}'(\mathbf{u} \mid \mathbf{V}')] \\ &= \frac{(1 - \gamma)(\mathbb{E}[u_i p_i(\mathbf{u} \mid \tilde{\mathbf{V}})]_{i \in [n]} + \gamma \mathbb{E}[\mathbf{W}(\mathbf{u} \mid \tilde{\mathbf{V}})] - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}}{P_k} \\ &= \frac{\tilde{\mathbf{V}} - \mathbb{E}[u_{i_k}] \mathbf{e}_{i_k}}{P_k} = \mathbf{V}', \end{aligned}$$

where in the second equality we used the fact that  $\mathbf{p}$  and  $\mathbf{W}$  satisfy Eq. (2). The previous equation shows that  $\mathbf{p}'$  and  $\mathbf{W}'$  also satisfy Eq. (2). Last, for any  $i \neq i_k$  and  $u, v \in [0, \bar{v}]$  we have

$$(1 - \gamma)uP'_i(v \mid \mathbf{V}') + \gamma W'_i(v \mid \mathbf{V}') = \frac{(1 - \gamma)uP_i(v \mid \tilde{\mathbf{V}}) + \gamma W_i(v \mid \tilde{\mathbf{V}})}{P_k}.$$

Hence, because  $\mathbf{p}, \mathbf{W}$  is incentive-compatible (Eq. (4)), so is  $\mathbf{p}', \mathbf{W}'$  (we recall that for  $i_k$ , the utility of agent  $i_k$  is always 0). This ends the proof that  $\mathbf{U}^{(k+1)} \in \mathcal{R}'_k \subset \mathcal{U}_\gamma$ . The induction then shows that  $\tilde{\mathbf{U}} \in \mathcal{U}_\gamma$ , and hence  $(\tilde{U}_j)_{j \in J(\mathbf{U})} \in \tilde{\mathcal{U}}_\gamma$ .

**Proving that  $\tilde{\mathbf{U}} \in \mathcal{U}_\gamma \Rightarrow \mathbf{U} \in \mathcal{U}_\gamma$ .** Now suppose that  $(\tilde{U}_j)_{j \in J(\mathbf{U})} \in \tilde{\mathcal{U}}_\gamma$ , which implies in particular  $\tilde{\mathbf{U}} \in \mathcal{U}_\gamma$ . Let  $\mathcal{R}' = \mathcal{U}_\gamma \cap \{\mathbf{V} : \forall i \notin J(\mathbf{U}), V_i = 0\}$ . Let  $\mathbf{p}'$  and  $\mathbf{W}'$  be valid incentive-compatible allocation and promised utility functions for  $\mathbf{V} \in \mathcal{U}_\gamma$ . We can easily check that for any  $\mathbf{V}' \in \mathcal{R}'$ , then almost surely  $W'_i(\mathbf{u} \mid \mathbf{V}') = 0$  for all  $i \notin J(\mathbf{U})$ , which implies  $\mathbf{W}'(\mathbf{u} \mid \mathbf{V}') \in \mathcal{R}'$ . Without loss of generality, we suppose this is the case (up to modifying  $\mathbf{W}$  on zero-measure regions).

We next define  $P_k := \prod_{k \in [K]} \mathbb{P}(u_{i_k} = 0)$  and  $\mathbf{V}_0 := \sum_{k \in [K]} \mathbb{E}_{u_{i_k}} \prod_{l < k} \mathbb{P}(u_{i_l} = 0) \mathbf{e}_{i_l}$ , we pose

$$\mathcal{R} := \{\mathbf{V}_0 + P_k \mathbf{V}', \mathbf{V}' \in \mathcal{R}'\}.$$

We consider the allocation  $\mathbf{p}'$  which always tries to allocate to agents  $i_1, i_2, \dots, i_K$  with this priority order whenever they have non-zero utility, then resorts to using the allocation  $\mathbf{p}$ . Formally, for all  $\mathbf{v} \in [0, \bar{v}]^n$  and  $\mathbf{V} \in \mathcal{R}$ , letting  $\mathbf{V}' = (\mathbf{V} - \mathbf{V}_0)/P_k \in \mathcal{R}'$  we pose

$$\begin{aligned} \mathbf{p}(\mathbf{v} \mid \mathbf{V}) &:= \begin{cases} \mathbf{e}_{i_l} & \text{if } \max_{k \in [K]} v_{i_k}^{(t)} > 0, \quad l = \min\{k \in [K] : v_{i_l}^{(t)} > 0\} \\ \mathbf{p}'(\mathbf{v} \mid \mathbf{V}') & \text{otherwise,} \end{cases} \\ \mathbf{W}(\mathbf{v} \mid \mathbf{V}) &:= \mathbf{V}_0 + P_k \mathbf{W}'(\mathbf{v} \mid \mathbf{V}'). \end{aligned}$$

We now check that  $\mathbf{p}, \mathbf{W}$  are valid for utility vectors in  $\mathcal{R}$ . By construction of  $\mathbf{W}$ , it directly satisfies Eq. (3). Next, for any  $\mathbf{V} \in \mathcal{R}$ , with the same notation  $\mathbf{V}' = (\mathbf{V} - \mathbf{V}_0)/P_k \in \mathcal{R}'$  we have

$$\begin{aligned} & (1 - \gamma)(\mathbb{E}[u_i p_i(\mathbf{u} \mid \mathbf{V})])_{i \in [n]} + \gamma \mathbb{E}[\mathbf{W}(\mathbf{v} \mid \mathbf{V})] \\ &= (1 - \gamma)(\mathbf{V}_0 + P_k(\mathbb{E}[u_i p'_i(\mathbf{u} \mid \mathbf{V}')]_{i \in [n]})) + \gamma(\mathbf{V}_0 + P_k \mathbb{E}[\mathbf{W}'(\mathbf{v} \mid \mathbf{V}')]) \\ &= \mathbf{V}_0 + P_k \mathbf{V}' = \mathbf{V}. \end{aligned}$$

In the second equality, we used the fact that  $\mathbf{p}', \mathbf{W}'$  satisfy Eq. (2). This shows that  $\mathbf{p}, \mathbf{W}$  also satisfy it. All that remains to check is the incentive-compatibility constraint Eq. (4). For any  $k \in [K]$ , assuming all other agents are truthful, agent  $i_k$  is allocated the good whenever  $u_{i_l} = 0$  for all  $l < k$  and its report satisfies  $v_{i_k} > 0$ . Hence we only need to check that incentive-compatibility when  $u_{i_k} = 0$ , which is immediate. For any  $i \notin \{i_k, k \in [K]\}$  and  $u, v \in [0, \bar{v}]$ , we directly have

$$(1 - \gamma)uP_i(v \mid \mathbf{V}) + \gamma W_i(v \mid \mathbf{V}) = P_k((1 - \gamma)uP'_i(v \mid \mathbf{V}') + \gamma W'_i(v \mid \mathbf{V}')),$$

hence the incentive-compatibility is guaranteed by that of  $\mathbf{p}', \mathbf{W}'$ . This ends the proof.  $\blacksquare$

## 5.2 A universal $\mathcal{O}(\sqrt{1 - \gamma})$ convergence rate

With the pruning procedure and Lemma 3 at hand, we now show that without any assumptions, the distance between the boundary of  $\mathcal{U}_\gamma$  and the optimal region  $\mathcal{U}_*$  is at most  $\mathcal{O}(\sqrt{1 - \gamma})$ . This implies Theorem 7.

**Theorem 27.** *For any  $\mathbf{U} \in \mathcal{U}^*$ , construct  $i_1, \dots, i_K, \tilde{\mathbf{U}}$  and  $J(\mathbf{U})$  according to the pruning process. If  $|J(\mathbf{U})| \leq 1$ , we have  $\mathbf{U} \in \mathcal{U}_0$ . Otherwise, there exist  $\gamma_0 \in [1/2, 1)$  and  $C > 0$  such that*

$$\forall \gamma \in [\gamma_0, 1), \quad d(\mathbf{U}; \mathcal{U}_\gamma) \leq C\sqrt{1 - \gamma}.$$

Precisely, introducing the constant  $\tilde{C} = 4|J(\mathbf{U})|^2 \bar{v}^2 \left(1 + \frac{\max_{j \in J(\mathbf{U})} \mathbb{E}[u_j]}{\min_{j \in J(\mathbf{U})} (\mathbb{E}[u_j] - \tilde{U}_j)}\right)^2 < \infty$ , we have  $C = 2\tilde{C}\sqrt{|J(\mathbf{U})|} \prod_{k \in [K]} \mathbb{P}(u_{i_k} = 0)$  and  $\gamma_0 = \max\left(1/2, 1 - \min_{j \in J(\mathbf{U})} \tilde{U}_j^2 / (32\tilde{C})\right) < 1$ .

**Proof** Fix  $\mathbf{U} \in \mathcal{U}^*$  and define  $i_1, \dots, i_K, \tilde{\mathbf{U}}$ , and  $J(\mathbf{U})$  according to the pruning process. If  $|J(\mathbf{U})| \leq 1$ , it is clear that  $(\tilde{U}_j)_{j \in J(\mathbf{U})} \in \tilde{\mathcal{U}}_0$ , and Lemma 26 ensures that  $\mathbf{U} \in \mathcal{U}_0$ . We suppose that this is not the case from now. By construction of the pruning process, for all  $j \in J(\mathbf{U})$ , we have  $\tilde{U}_j \in (0, \mathbb{E}[u_j])$ , and  $\tilde{U}_i = 0$  for all  $i \notin J(\mathbf{U})$ . We therefore consider the reduced problem in which only agents in  $J(\mathbf{U})$  are present. By Lemma 26,  $\tilde{\mathbf{U}} \in \mathcal{U}^*$ , hence the utility vector  $(\tilde{U}_j)_{j \in J(\mathbf{U})}$  can still be reached in this simplified setting with full information. We denote by  $\tilde{\mathcal{U}}^* = \{(x_j)_{j \in J(\mathbf{U})} : \mathbf{x} \in \mathcal{U}^*\}$  the reachable utilities in this setting. By slight abuse of notation, we will still denote by  $\tilde{\mathbf{U}}$  the vector  $(\tilde{U}_j)_{j \in J(\mathbf{U})}$ .

Note that  $\mathcal{R} := \prod_{j \in J(\mathbf{U})} [0, \tilde{U}_j] \subset \tilde{\mathcal{U}}^*$ . Let  $\tilde{C} = 4|J(\mathbf{U})|^2 \bar{v}^2 \left(1 + \frac{\max_{j \in J(\mathbf{U})} \mathbb{E}[u_j]}{\min_{j \in J(\mathbf{U})} (\mathbb{E}[u_j] - \tilde{U}_j)}\right)^2$  and pose

$$r := \delta := \sqrt{\tilde{C} \frac{1 - \gamma}{\gamma}}.$$

Next, using  $\gamma \geq \gamma_0$ , we have

$$2(r + \delta) \leq 4\sqrt{2\tilde{C}(1 - \gamma)} \leq \min_{j \in J(\mathbf{U})} \tilde{U}_j.$$

As a result, with  $\mathbf{x} = \tilde{\mathbf{U}} - (r + \delta)\mathbf{1}$ , we have  $B(\mathbf{x}, r + \delta) \subset \mathcal{R} \subset \tilde{\mathcal{U}}^*$ . Since  $\gamma \geq 1/2$ , the choice of parameters  $\mathbf{x}, r, \delta$  satisfies the assumptions of Lemma 3. Thus,  $B(\mathbf{x}, r) \subset \tilde{\mathcal{U}}_\gamma$ . In turn, Lemma 26 implies that with  $P := \prod_{k \in [K]} \mathbb{P}(u_{i_k} = 0)$ , we have

$$(U_i \mathbb{1}_{i \notin J(\mathbf{U})})_{i \in [n]} + P \cdot B_{J(\mathbf{U})}(\mathbf{x}, r) \subset \mathcal{U}^*,$$

where  $B_{J(\mathbf{U})}(\mathbf{x}, r) = \{\mathbf{y} : (y_j)_{j \in J(\mathbf{U})} \in B(\mathbf{x}, r), \forall i \notin J(\mathbf{U}), y_i = 0\}$ . Hence,

$$d(\mathbf{U}, \mathcal{U}^*) \leq P \cdot d(\tilde{\mathbf{U}}, \tilde{\mathcal{U}}^*) \leq P \cdot d(\tilde{\mathbf{U}}, B(\mathbf{x}, r)) = P(\sqrt{|J(\mathbf{U})|}(r + \delta) - r) \leq 2P\sqrt{|J(\mathbf{U})|}r.$$

This ends the proof of the theorem.  $\blacksquare$

### 5.3 Improved rates for smooth full-information regions $\mathcal{U}^*$

We now prove Theorem 8 which gives conditions for which faster rates  $\mathcal{O}(1 - \gamma)$  can be achieved, in terms of the utility distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$ .

**Proof of Theorem 8** We recall the notation  $P_S : \mathbb{R}^n \rightarrow \mathbb{R}^S$  for the projection on coordinates  $S$ . For  $s \in [q]$ , we define the function  $\phi : \mathbb{R}^{I_s} \rightarrow \mathbb{R}$  via

$$\phi^{(s)}(\boldsymbol{\beta}) := \max_{\mathbf{y} \in P_{I_s}(\mathcal{U}^* \cap \{\mathbf{y} : \forall i \notin I_s, y_i = U_i\})} \boldsymbol{\beta}^\top \mathbf{y} = \mathbb{E} \left[ \mathbb{1}_{Z_{I_s} = Z} \cdot \max_{i \in I_s} \beta_i u_i \right], \quad \boldsymbol{\beta} \in \mathbb{R}^{I_s}. \quad (54)$$

In the equality, we use the characterizations from Lemma 21. We also recall that by Eq. (14), we have  $\mathbb{P}(Z_{I_s} = Z_{[n] \setminus I_s}) = 0$ . Note that  $\phi$  is convex as the maximum of linear functions.

**Proving that the (SC1) is sufficient.** Suppose that (SC1) holds and fix parameters satisfying the assumptions. Lemma 21 implies that for  $1 - \gamma$  sufficiently small and  $\delta = 2C(1 - \gamma)/(\gamma r)$  for some constant  $C > 0$ ,

$$B(\mathbf{x}, r - \delta) \cap \{\mathbf{y} : \forall i \notin \tilde{I}, y_i = U_i\} \subset \mathcal{U}_\gamma.$$

In turn, we obtain the desired result

$$\max_{\mathbf{y} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{y} - \max_{\mathbf{y} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{y} \leq \delta \|\boldsymbol{\alpha}\| = \mathcal{O}(1 - \gamma).$$

**Proving that (SC3)  $\Rightarrow$  (SC2)  $\Rightarrow$  (SC1).** Suppose that (SC3) is satisfied for some direction  $\boldsymbol{\alpha}$  and denote  $\tilde{I} = \tilde{I}_1 \sqcup \tilde{I}_2 \sqcup \dots \sqcup \tilde{I}_q$  the guaranteed partition. To prove (SC2) and (SC1) we will use a slightly different partition by merging some sets. Precisely, consider the undirected graph  $\mathcal{F}$  on  $[q]$  such that  $s \neq s' \in [q]$  are connected if and only if  $\mathbb{P}(Z_{\tilde{I}_s} = Z_{\tilde{I}_{s'}} = Z > 0) > 0$ . We let  $q$  be the number of connected components  $C_1, \dots, C_q$  of  $\mathcal{F}$ . For  $s \in [q]$  we let  $I_s := \bigcup_{t \in C_s} \tilde{I}_t$ , that is we merge the partition according to the graph  $\mathcal{F}$ . Note that if  $s \neq s'$  are connected within  $\mathcal{F}$  then

$$0 < \mathbb{P}(Z_{\tilde{I}_s} = Z_{\tilde{I}_{s'}} = Z > 0) \leq \sum_{i \in \tilde{I}_s, j \in \tilde{I}_{s'}} \mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0).$$

Hence, there exists  $i \in \tilde{I}_s$  and  $j \in \tilde{I}_{s'}$  such that  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0)$ . This implies that  $i \rightarrow j$  and  $j \rightarrow i$  in the graph  $\mathcal{G}$ , and hence  $\tilde{I}_s$  is strongly connected to  $\tilde{I}_{s'}$  via this double edge. This proves that the sets of the constructed partition  $\tilde{I} = I_1 \sqcup I_2 \sqcup \dots \sqcup I_q$  are still strongly connected within  $\mathcal{G}$ .

We can directly check that Eq. (14) is satisfied by construction of the sets  $I_1, \dots, I_q$ : for any  $s \neq s' \in [q]$

$$\mathbb{P}(Z_{I_s} = Z_{I_{s'}} = Z > 0) \leq \sum_{t \in C_s, t' \in C_{s'}} \mathbb{P}(Z_{\tilde{I}_t} = Z_{\tilde{I}_{t'}} = Z > 0) = 0.$$

In the last equality we use the fact that  $C_s$  and  $C_{s'}$  are disconnected in  $\mathcal{F}$ .

We next define an allocation function such that  $\mathbf{p}(\mathbf{u})$  is the uniform distribution on  $\arg \max_{j \in [n]} \alpha_j u_j$ . We then define  $\mathbf{U} = (\mathbb{E}[u_i p_i(\mathbf{u})])_{i \in [n]}$  as the utility vector realized by  $\mathbf{p}$ . Note that by construction,  $\mathbf{U}$  is  $\alpha$ -optimal and we have  $U_i > 0$  for all  $i \in \tilde{I}$ . In particular, for all  $s \in [q]$  we have  $\phi^{(s)}(\alpha_{I_s}) = \alpha_{I_s}^\top \mathbf{U}_{I_s}$ .

Suppose that there exists  $c_1, c_2 > 0$  such that for any  $s \in [q]$  and  $\beta \in \mathcal{H}_s := \{\delta \in \mathbb{R}^{I_s} : \alpha_{I_s}^\top \delta = 0\}$ , with  $\|\beta\| \leq c_1$ , we have

$$\phi^{(s)}(\alpha_{I_s} + \beta) - \phi^{(s)}(\alpha_{I_s}) - \beta^\top \mathbf{U}_{I_s} \geq c_2 \|\beta\|^2. \quad (55)$$

Because the function  $\beta \mapsto \phi^{(s)}(\alpha_{I_s} + \beta) - \phi^{(s)}(\alpha_{I_s}) - \beta^\top \mathbf{U}_{I_s}$  is convex in  $\gamma$  and has value 0 for  $\beta = \mathbf{0}$ , this implies that for all  $\beta \in \mathcal{H}_s$ ,

$$\phi^{(s)}(\alpha_{I_s} + \beta) - \phi^{(s)}(\alpha_{I_s}) - \beta^\top \mathbf{U}_{I_s} \geq c_2 \min(\|\beta\|^2, c_1 \|\beta\|) \geq c_3 \min(\|\beta\|^2, \|\alpha_{I_s}\| \|\beta\|), \quad (56)$$

for  $c_3 = \min(c_2, c_1 c_2 / \|\alpha_{I_s}\|)$ . Let  $r = \min(2c_3 \min_{s \in [q]} \|\alpha_{I_s}\|, \min_{i \in \tilde{I}} U_i)$ . We note that  $\mathbf{x}_{I_s} = \mathbf{U}_{I_s} - r \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|}$ . Since  $\beta^\top \alpha_{I_s} = 0$ , we have

$$\begin{aligned} (\alpha_{I_s} + \beta)^\top \mathbf{x}_{I_s} + \|\alpha_{I_s} + \beta\| r &= (\alpha_{I_s} + \beta)^\top \mathbf{U}_{I_s} + r(\|\alpha_{I_s} + \beta\| - \|\alpha_{I_s}\|) \\ &= \phi^{(s)}(\alpha_{I_s}) + \beta^\top \mathbf{U}_{I_s} + r \|\alpha_{I_s}\| \left( \sqrt{1 + (\|\beta\| / \|\alpha_{I_s}\|)^2} - 1 \right) \\ &\leq \phi^{(s)}(\alpha_{I_s}) + \beta^\top \mathbf{U}_{I_s} + \frac{2r}{\|\alpha_{I_s}\|} \min(\|\beta\|^2, \|\alpha_{I_s}\| \|\beta\|) \\ &\leq \phi^{(s)}(\alpha_{I_s} + \beta). \end{aligned}$$

In the last inequality, we used Eq. (56). As a result, Lemma 25 shows that  $B_{I_s}(\mathbf{x}_{I_s}, r) \subset P_{I_s}(\mathcal{U}^* \cap \{\mathbf{y} : \forall i \notin I_s, y_i = U_i\})$ . We can also easily check that it contains  $\mathbf{U}$ . Hence, this proves (SC1).

We now show that Eq. (55) holds. We proceed independently for each  $s \in [q]$ . Fix  $\beta \in \mathcal{H}_s \setminus \{\mathbf{0}\}$ . We first show that there is a subset  $\emptyset \subsetneq B \subsetneq I_s$  such that

$$\min_{i \in B} \frac{\beta_i}{\alpha_i} - \max_{i \in I_s \setminus B} \frac{\beta_i}{\alpha_i} \geq \frac{\min_{i \in I_s} \alpha_i}{2 \|\alpha\|_\infty^2 n^2} \|\beta\|. \quad (57)$$

Let  $\epsilon = \frac{\min_{i \in I_s} \alpha_i}{2 \|\alpha\|_\infty^2 n^2} \|\beta\|$ . We consider the partition of  $\mathbb{R}$  into  $(-\infty, 0]$ ,  $(n\epsilon, \infty)$  and  $n$  length- $\epsilon$  intervals partitioning  $[0, n\epsilon)$ . To prove Eq. (57) it suffices to show that there is one of these intervals of length  $\epsilon$  that does not contain an element of the form  $\beta_i / \alpha_i$  for  $i \in I_s$ , but has elements below and above it. Suppose that this is not the case. Then, the elements  $\{\beta_i / \alpha_i \geq 0, i \in I_s\}$  occupy consecutive intervals of the partition. Since  $\alpha_{I_s}^\top \beta = 0$ ,  $\beta$  has both positive



and negative entries. This shows that the consecutive intervals occupied by  $\{\beta_i/\alpha_i, i \in I_s\}$  must start at  $[0, \epsilon)$ . However, there are at most  $|I_s| - 1 \leq n - 1$  such intervals (one point must belong to  $(-\infty, 0)$ ), hence we proved that all elements from  $\{\beta_i/\alpha_i \geq 0, i \in I_s\}$  lie consecutively within the  $n$  intervals within  $[0, (n - 1)\epsilon)$ . Then, since  $\alpha_{I_s}^\top \beta = 0$ , we have

$$\sum_{i \in I_s: \beta_i < 0} \alpha_i |\beta_i| = \sum_{i \in I_s: \beta_i \geq 0} \alpha_i \beta_i \leq \|\alpha\|_\infty^2 \sum_{i \in I_s: \beta_i \geq 0} \frac{\beta_i}{\alpha_i} \leq \|\alpha\|_\infty^2 |I_s| (n - 1) \epsilon < \|\alpha\|_\infty^2 n^2 \epsilon.$$

Therefore,

$$\|\beta\| \leq \|\beta\|_1 \leq \frac{1}{\min_{i \in I_s} \alpha_i} \sum_{i \in I_s} \alpha_i |\beta_i| < \frac{2\|\alpha\|_\infty^2 n^2}{\min_{i \in I_s} \alpha_i} \epsilon = \|\beta\|$$

and we reach a contradiction.

In the rest of the proof, we fix a subset  $\emptyset \subsetneq B \subsetneq I_s$  satisfying Eq. (57). We let

$$\gamma_1 := \min_{i \in B} \frac{\beta_i}{\alpha_i}, \quad \gamma_2 := \max_{i \in I_s \setminus B} \frac{\beta_i}{\alpha_i}.$$

By assumption, we have  $\gamma_1 \geq \gamma_2 + \epsilon$ . Provided that  $\|\beta\| \leq \min_{i \in I_s} \alpha_i / 2$ , we have  $|\gamma_2| \leq 1/2$ . We suppose that this is the case from now. Recall that  $\mathbf{p}$  is the allocation that realizes  $\mathbf{U}$  in the full information setting. We also introduce the notation  $\hat{i}$  for the agent that effectively receives the allocation (that is,  $\hat{i} \sim \mathbf{p}(\mathbf{u})$ ). Using Eq. (54), we have

$$\begin{aligned} \phi^{(s)}(\alpha_{I_s} + \beta) - \phi^{(s)}(\alpha_{I_s}) - \beta^\top \mathbf{U}_{I_s} &= \mathbb{E} \left[ \left( \max_{i \in I_s} (\alpha_i + \beta_i) u_i - (\alpha_i + \beta_i) u_i \mathbb{1}_{\hat{i} \in I_s} \right) \mathbb{1}_{Z=Z_{I_s}} \right] \\ &\geq \mathbb{E} \left[ \left( \max_{i \in I_s} (\alpha_i + \beta_i) u_i - (\alpha_i + \beta_i) u_i \mathbb{1}_{\hat{i} \in I_s} \right) \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq \frac{1+\gamma_1}{1+\gamma_2} Z_B} \right] \\ &\stackrel{(i)}{\geq} \frac{1}{n} \mathbb{E} \left[ \left( (1 + \gamma_1) Z_B - (1 + \gamma_2) Z_{I_s \setminus B} \right) \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq \frac{1+\gamma_1}{1+\gamma_2} Z_B} \right] \\ &\stackrel{(ii)}{\geq} \frac{1 + \gamma_2}{n} \mathbb{E} \left[ \left( (1 + \epsilon/2) Z_B - Z_{I_s \setminus B} \right) \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\epsilon/4) Z_B} \right] \\ &\stackrel{(iii)}{\geq} \frac{\epsilon}{8n} \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\epsilon/4) Z_B} \right]. \end{aligned} \quad (58)$$

In (i), the additional factor  $1/n$  comes from the fact that if we have a tie  $Z = Z_{I_s \setminus B}$ , there is at least a probability  $1/n$  that  $\hat{i} \in I_s \setminus B$  by definition of  $\mathbf{p}$ . In (ii) and (iii) we used  $|\gamma_2| \leq 1/2$  so that  $(1 + \gamma_1) \geq (1 + \gamma_2)(1 + \epsilon/2)$  and  $1 + \gamma_2 \geq 1/2$ . We compute for any  $\eta \geq 0$ ,

$$\begin{aligned} \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\eta) Z_B} \right] &\geq \max_{j \in I_s \setminus B} \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} \leq (1+\eta) Z_B} \mathbb{1}_{\alpha_j u_j = Z_{I_s \setminus B} = Z} \right] \\ &\geq \min_{i \in B} \alpha_i \cdot \max_{i \in B, j \in I_s \setminus B} \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_j u_j = Z} \mathbb{1}_{\alpha_j u_j \in [\alpha_i u_i, (1+\eta) \alpha_i u_i]} \right]. \end{aligned}$$

By assumption, because  $\mathcal{G}$  is strongly connected, there exists an outgoing edge from  $B$ . Using the previous inequality exactly gives

$$\mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\eta) Z_B} \right] \underset{\eta \rightarrow 0}{=} \Omega(\eta).$$

This proves the condition (SC2). Because there are only a finite number of subsets  $\emptyset \subsetneq B \subsetneq I_s$  and  $s \in [q]$ , this shows that there exists  $c_3, c_4 > 0$  such that for all  $s \in [q]$  and  $\emptyset \subsetneq B \subsetneq I_s$ ,

$$\mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\eta)Z_B} \right] \geq c_3 \eta, \quad \eta \in [0, c_4].$$

Together with the previous estimates, this shows that for  $s \in [q]$ , and  $\beta \in \mathcal{H}_s$  with  $\|\beta\| \leq \min(\min_{i \in \bar{I}} \alpha_i / 2, c_4 \|\alpha\|_\infty n^{3/2})$ , we have  $\epsilon \leq c_4$  hence

$$\phi^{(s)}(\alpha_{I_s} + \beta) - \phi^{(s)}(\alpha_{I_s}) - \beta^\top \mathbf{U}_{I_s} \geq \frac{c_3}{32n} \epsilon^2 = \frac{c_3}{32 \|\alpha\|_\infty^2 n^4} \|\beta\|^2.$$

This ends the proof of Eq. (55).

**Proving that (SC1)  $\Rightarrow$  (SC2).** Suppose that we are given  $\mathbf{U}$  and a radius  $r > 0$  satisfying (SC1). We use similar notations as before: let  $\mathbf{p}(\cdot)$  be an allocation function that realizes  $\mathbf{U}$  and denote by  $\hat{i} \sim \mathbf{p}(\mathbf{u})$  the agent that receives the resource with this allocation. By Lemma 25, and using the same arguments as before, for  $s \in [q]$  and every  $\beta \in \mathcal{H}_s$ , we have

$$\begin{aligned} \phi^{(s)}(\alpha_{I_s} + \beta) &\geq (\alpha_{I_s} + \beta)^\top \mathbf{x}_{I_s} + \|\alpha_{I_s} + \beta\| r \\ &= \phi^{(I_s)}(\alpha_{I_s}) + \beta^\top \mathbf{U}_{I_s} + r \|\alpha_{I_s}\| \left( \sqrt{1 + (\|\beta\| / \|\alpha_{I_s}\|)^2} - 1 \right). \end{aligned}$$

Using the inequality  $\sqrt{1+x} - x \geq (\sqrt{2}-1)x$  for  $x \in [0, 1]$ , this shows that for  $\beta \in \mathcal{H}_s$  with  $\|\beta\| \leq \|\alpha_{I_s}\|$ ,

$$\phi^{(s)}(\alpha_{I_s} + \beta) - \phi^{(s)}(\alpha_{I_s}) - \beta^\top \mathbf{U}_{I_s} \geq \frac{(\sqrt{2}-1)r}{\|\alpha_{I_s}\|} \|\beta\|^2.$$

Next fix any  $\emptyset \subsetneq B \subsetneq I_s$ . We consider the value  $\beta_B = (\alpha_i(\eta \mathbb{1}_{i \in B} - \eta' \mathbb{1}_{i \in I_s \setminus B}))_{i \in I_s}$  where  $\eta \geq 0$  and we posed  $\eta' := \eta(\sum_{i \in B} \alpha_i^2) / (\sum_{i \in I_s \setminus B} \alpha_i^2)$ , so that  $\alpha_{I_s}^\top \beta = 0$ . For sufficiently small  $\eta \geq 0$ , we have  $\|\beta\| = \eta \sqrt{2 \sum_{i \in B} \alpha_i^2} \leq \|\alpha_{I_s}\|$ . Then,

$$\begin{aligned} \phi^{(s)}(\alpha + \beta) - \phi^{(s)}(\alpha) - \beta^\top \mathbf{U} &= \mathbb{E} \left[ \left( \max_{i \in I_s} (\alpha_i + \beta_i) u_i - (\alpha_i + \beta_i) u_i \mathbb{1}_{i \in I_s} \right) \mathbb{1}_{Z=Z_{I_s}} \right] \\ &\leq \mathbb{E} \left[ \left( \max((1+\eta)Z_B, (1-\eta')Z_{I_s \setminus B}) - (1+\eta)Z_B \mathbb{1}_{\hat{i} \in B} - (1-\eta')Z_{I_s \setminus B} \mathbb{1}_{\hat{i} \in I_s \setminus B} \right) \mathbb{1}_{Z=Z_{I_s}} \right] \\ &\leq \mathbb{E} \left[ \left( (1+\eta)Z_B - (1-\eta')Z_{I_s \setminus B} \right) \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq \frac{1+\eta}{1-\eta'} Z_B} \right] \\ &\leq (\eta + \eta') \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq \frac{1+\eta}{1-\eta'} Z_B} \right]. \end{aligned}$$

For  $\eta \geq 0$  sufficiently small, we have  $\frac{1+\eta}{1-\eta'} \leq 1 + 2(\eta + \eta')$ . Therefore, for  $\zeta \geq 0$  sufficiently small, taking  $\eta \geq \zeta$  such that  $\zeta = \eta + \eta'$ , we obtained

$$\mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+2\zeta)Z_B} \right] \geq \frac{(\sqrt{2}-1)r \sum_{i \in B} \alpha_i^2}{2 \|\alpha_{I_s}\| \left( 1 + (\sum_{i \in B} \alpha_i^2) / (\sum_{i \in I_s \setminus B} \alpha_i^2) \right)^2} \zeta.$$

This proves (SC2). ■

We now prove Corollary 9 which gives simpler conditions for reaching the fast rates  $\mathcal{O}(1-\gamma)$ .

**Proof of Corollary 9** We denote  $Z = \max_{i \in [n]} \alpha_i u_i$ . We show that condition (SC3) from Theorem 8 is satisfied. For any  $i \in \tilde{I}$ , let  $j \in \tilde{I}$  be the corresponding index satisfying the assumption for  $i$ . In the first case, we obtain directly

$$\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z = m_i) \geq \mathbb{P}(\alpha_i u_i = m_i) \mathbb{P}(\alpha_j u_j = m_i) \prod_{k \notin \{i, j\}} \mathbb{P}(\alpha_k u_k \leq m_i) > 0.$$

Hence  $m_i > 0$  implies  $i \rightarrow j$  and  $j \rightarrow i$ .

In the second case, let  $f_i$  be the lower bound on the density of the absolutely-continuous part of  $\alpha_i u_i$  and  $\alpha_j u_j$  on  $[a_i, b_i]$ . Since  $a_i > m$ , we have  $\mathbb{P}(Z \leq a_i) > 0$ . This shows that for all  $k \in [n]$ ,  $\mathbb{P}(\alpha_k u_k \leq a_i) > 0$ . Then,

$$\begin{aligned} & \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_j u_j = Z} \mathbb{1}_{\alpha_j u_j \in [\alpha_i u_i, (1+\eta)\alpha_i u_i]} \right] \\ & \geq a_i \mathbb{P}(\forall k \notin \{i, j\}, \alpha_k u_k \leq a_i) \mathbb{P}(a_i \leq \alpha_i u_i \leq \alpha_j u_j \leq \alpha_i u_i + \eta a_i) \\ & \geq a_i \prod_{k \notin \{i, j\}} \mathbb{P}(\alpha_k u_k \leq a_i) \cdot f_i^2(b_i - a_i - \eta a_i) \eta a_i = \Omega(\eta). \end{aligned}$$

This shows that  $i \rightarrow j$ . The same arguments also show that  $j \rightarrow i$ .

In summary, we showed that  $\mathcal{G}$  contains a subgraph  $\mathcal{G}'$  in which  $i \rightarrow j$  implies  $j \rightarrow i$  and such that every node has an outgoing edge. We then use the partition of  $\tilde{I}$  given by the connected components of  $\mathcal{G}'$ , that all have at least 2 elements and are strongly connected. Hence (SC3) is satisfied. This ends the proof.  $\blacksquare$

## 5.4 General rates for arbitrary utility distributions

As discussed in Section 3.2, the general tools Lemmas 3 and 21, and Lemmas 6 and 24 can be used more generally to prove any convergence rates between  $\sqrt{1-\gamma}$  and  $1-\gamma$ . We now prove Theorem 11 which gives such characterizations.

**Proof of Theorem 11** We prove the upper and lower bound separately.

**Proof of the first claim.** We aim to apply Lemma 21 with the given partition  $\tilde{I} = I_1 \sqcup I_2 \sqcup \dots \sqcup I_q$ . By hypothesis, it already satisfies Eq. (14). We follow similar arguments as in the proof of Theorem 8. In particular, we define the function  $\phi^{(s)} : \mathbb{R}^{I_s} \rightarrow \mathbb{R}$  as in Eq. (54). We consider the allocation function  $\mathbf{p}(\mathbf{u})$  that allocates uniformly on  $\arg \max_{j \in [n]} \alpha_j u_j$  and let  $\mathbf{U}$  be the utility vector realized by  $\mathbf{p}$ .

We next introduce the parameters used. Let  $C_2 := 16n^2 \bar{v}^2 \left( 1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{s \in [q], i \in I_s} (\mathbb{E}[u_i] \mathbb{P}(Z_{I_s} = Z > 0) - U_i)} \right)^2$  and  $r_0 = \min_{s \in [q], i \in I_s} (\mathbb{E}[u_i] \mathbb{P}(Z_{I_s} = Z > 0) - U_i) / 2$ . Next, for convenience, we pose  $c_0 = \frac{\min_{i \in \tilde{I}} \alpha_i}{16 \|\alpha\|_\infty^2 n^3} > 0$  and  $c' := \min \left( 1, \frac{\min_{i \in \tilde{I}} \alpha_i}{8 \|\alpha\|_\infty^2 n^2} \right)$ . Let  $C_r := \sqrt{\frac{C_2}{2}} c' \min_{i \in \tilde{I}} \alpha_i$ ,  $C_\eta = \frac{4c_0 c' C_r}{\min_{i \in \tilde{I}} \alpha_i}$ , and  $C_\delta = \frac{4C_r}{c'^2 \min_{i \in \tilde{I}} \alpha_i^2} = \frac{2C_2}{C_r}$ . Last, we pose  $f_0 = f(c' \min_{i \in \tilde{I}} \alpha_i / 2)$ . We define

$$\eta(\gamma) := \inf \{1\} \cup \left\{ \eta \in \left[ \frac{C_r}{\min(r_0, c_0 f_0 / 4)} \sqrt{1-\gamma}, 1 \right] : \forall \eta' \in [\eta, 1], f(\eta') \geq C_\eta \sqrt{1-\gamma} \frac{\eta'}{\eta} \right\}.$$

First, if  $\eta(\gamma) \geq \frac{c' \min_{i \in \bar{I}} \alpha_i}{2\sqrt{2}} > 0$ , by Theorem 27 we have

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{x} \leq C_1 \sqrt{1 - \gamma}.$$

Hence ensuring that  $C \geq \frac{2\sqrt{2}C_1}{c' \min_{i \in \bar{I}} \alpha_i}$  gives the desired result. We now suppose that  $\eta(\gamma) < c' \min_{i \in \bar{I}} \alpha_i / (2\sqrt{2})$ . Note that  $f$  is right-continuous, hence we have  $f(\eta') \geq C_\eta \sqrt{1 - \gamma} \frac{\eta'}{\eta(\gamma)}$  for all  $\eta' \in [\eta(\gamma), 1]$ .

In the following, we check that the assumptions of Lemma 21 are satisfied for  $\gamma \in [1/2, 1)$  with the parameters:

$$r = C_r \frac{\sqrt{1 - \gamma}}{\eta(\gamma)}, \quad \delta = C_\delta \sqrt{1 - \gamma} \eta(\gamma), \quad \text{and} \quad \mathbf{x} = \left( U_i - (r + 2\delta) \sum_{s \in [q]} \frac{\alpha_i \mathbb{1}_{i \in I_s}}{\|\boldsymbol{\alpha}_{I_s}\|} \right)_{i \in [n]}.$$

First note that by assumption we have  $\eta(\delta) \leq \frac{c' \min_{i \in \bar{I}} \alpha_i}{2\sqrt{2}} = \sqrt{\frac{C_r}{2C_\delta}}$  so that  $r \geq 2\delta$ . In particular, since  $\gamma \geq 1/2$  the constraint  $\frac{r\gamma}{1-\gamma} \geq \delta$  from Eq. (13) is satisfied.

We now fix  $s \in [q]$  and let  $\mathcal{H}_s := \{\boldsymbol{\delta} \in \mathbb{R}^{I_s} : \boldsymbol{\alpha}_{I_s}^\top \boldsymbol{\delta} = 0\}$ . From the proof of Theorem 8 (see Eq. (58)) for any  $\boldsymbol{\beta} \in \mathcal{H}_s$  with  $\|\boldsymbol{\beta}\| \leq \min_{i \in I_s} \alpha_i / 2$ , we have

$$\phi^{(s)}(\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta}) - \phi^{(s)}(\boldsymbol{\alpha}_{I_s}) - \boldsymbol{\beta}^\top \mathbf{U}_{I_s} \geq c_0 \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1 + c' \|\boldsymbol{\beta}\|) Z_B} \right] \|\boldsymbol{\beta}\|$$

This implies

$$\phi^{(s)}(\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta}) - \phi^{(s)}(\boldsymbol{\alpha}_{I_s}) - \boldsymbol{\beta}^\top \mathbf{U}_{I_s} \geq c_0 f(c' \|\boldsymbol{\beta}\|) \|\boldsymbol{\beta}\|.$$

Now let  $f_s := f(c' \min_{i \in I_s} \alpha_i / 2) \geq f_0$ . Because the function  $\boldsymbol{\beta} \mapsto \phi^{(s)}(\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta}) - \phi^{(s)}(\boldsymbol{\alpha}_{I_s}) - \boldsymbol{\beta}^\top \mathbf{U}_{I_s}$  is convex in  $\boldsymbol{\beta}$  and has value 0 for  $\boldsymbol{\beta} = \mathbf{0}$ , this implies that for all  $\boldsymbol{\beta} \in \mathcal{H}_s$ ,

$$\phi^{(s)}(\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta}) - \phi^{(s)}(\boldsymbol{\alpha}_{I_s}) - \boldsymbol{\beta}^\top \mathbf{U}_{I_s} \geq c_0 \min(f(c' \|\boldsymbol{\beta}\|), f_s) \|\boldsymbol{\beta}\|.$$

As a result, for any  $\boldsymbol{\beta} \in \mathcal{H}_s$ ,

$$\begin{aligned} A_s(\boldsymbol{\beta}) &:= \phi^{(s)}(\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta}) - (\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta})^\top \mathbf{x}_{I_s} - (r + \delta) \|\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta}\| \\ &\stackrel{(i)}{\geq} c_0 \min(f(c' \|\boldsymbol{\beta}\|), f_s) \|\boldsymbol{\beta}\| - (r + \delta) (\|\boldsymbol{\alpha}_{I_s} + \boldsymbol{\beta}\| - \|\boldsymbol{\alpha}_{I_s}\|) + \delta \|\boldsymbol{\alpha}_{I_s}\| \\ &\stackrel{(ii)}{\geq} c_0 \min(f(c' \|\boldsymbol{\beta}\|), f_s) \|\boldsymbol{\beta}\| - \frac{4r}{\|\boldsymbol{\alpha}_{I_s}\|} \min(\|\boldsymbol{\beta}\|^2, \|\boldsymbol{\alpha}_{I_s}\| \|\boldsymbol{\beta}\|) + \delta \|\boldsymbol{\alpha}_{I_s}\|. \end{aligned}$$

In (i) we used  $\phi^{(s)}(\boldsymbol{\alpha}_{I_s}) = \boldsymbol{\alpha}_{I_s}^\top \mathbf{U}_{I_s}$ ,  $\boldsymbol{\alpha}_{I_s}^\top \boldsymbol{\beta} = 0$ , and the previous inequality. In (ii) we used  $r \geq 2\delta$  and  $\boldsymbol{\alpha}_{I_s}^\top \boldsymbol{\beta} = 0$ . We now distinguish between three cases. First suppose that  $\|\boldsymbol{\beta}\| \leq \eta(\gamma) / c'$ . Then,

$$A_s(\boldsymbol{\beta}) \geq \delta \|\boldsymbol{\alpha}_{I_s}\| - \frac{4r}{c'^2 \|\boldsymbol{\alpha}_{I_s}\|} \eta(\gamma)^2 = \sqrt{1 - \gamma} \eta(\gamma) \left( C_\delta \|\boldsymbol{\alpha}_{I_s}\| - \frac{4C_r}{c'^2 \|\boldsymbol{\alpha}_{I_s}\|} \right) \geq 0.$$

Next suppose that  $\eta(\gamma) / c' \leq \|\boldsymbol{\beta}\| \leq \min_{i \in I_s} \alpha_i / 2$ . Note that by assumption, we have  $\eta(\gamma) / c' \leq \min_{i \in I_s} \alpha_i / 2$ . Therefore, since  $f$  is non-decreasing we have  $f(c' \|\boldsymbol{\beta}\|) \leq f_s$ . Further, since  $c' \|\boldsymbol{\beta}\| \geq \eta(\gamma)$ , we also have  $f(c' \|\boldsymbol{\beta}\|) \geq C_\eta \sqrt{1 - \gamma} c' \|\boldsymbol{\beta}\| / \eta(\gamma)$ . Hence,

$$A_s(\boldsymbol{\beta}) \geq c f(c' \|\boldsymbol{\beta}\|) \|\boldsymbol{\beta}\| - \frac{4r}{\|\boldsymbol{\alpha}_{I_s}\|} \|\boldsymbol{\beta}\|^2 \geq \left( c_0 c' C_\eta - \frac{4C_r}{\|\boldsymbol{\alpha}_{I_s}\|} \right) \frac{\sqrt{1 - \gamma}}{\eta(\gamma)} \|\boldsymbol{\beta}\|^2 \geq 0.$$

In the last inequality, we used the definition of  $C_\eta$ .

Last, for  $\|\beta\| \geq \min_{i \in I_s} \alpha_i/2$ , we obtain  $A_s(\beta) \geq (c_0 f_0 - 4r)\|\beta\|$ . Now by construction of  $\eta(\gamma)$ , we have

$$r = C_r \frac{\sqrt{1-\gamma}}{\eta(\gamma)} \leq \min(r_0, c f_0/4).$$

This therefore shows that for all  $\beta \in \mathcal{H}_s$  we have  $A_s(\beta) \geq 0$ . Finally, we check that the second constraint from Eq. (13) is satisfied. By construction and because  $\gamma \geq 1/2$ ,

$$\delta = 2C_2 \frac{1-\gamma}{r} \geq C_2 \frac{1-\gamma}{\gamma r}.$$

Further, since  $r \leq r_0$ , we can check that

$$4n^2 \bar{v}^2 \left( 1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{s \in [q], i \in I_s} (\mathbb{E}[u_i] \mathbb{P}(Z_{I_s} = Z > 0) - x_i - r)} \right)^2 \leq C_2.$$

Finally, we have checked that all assumptions to apply Lemma 21 are satisfied. Therefore,

$$\mathbf{x} + \{0\}^{[n] \setminus \tilde{I}} \otimes \bigotimes_{s \in [q]} r \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|} \in \mathcal{U}_\gamma.$$

As a result,

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \leq 2\delta \sum_{s \in [q]} \|\alpha_{I_s}\| \leq n \|\alpha\| C_\delta \sqrt{1-\gamma} \eta(\gamma).$$

Ensuring that  $C \geq n \|\alpha\| C_\delta \max\left(1, \frac{C_r}{\min(r_0, c_0 f_0/4)}\right)$ , this implies the desired result. Note that the term  $C(1-\gamma)$  takes care of the case when  $\eta(\gamma) = \frac{C_r}{\min(r_0, c_0 f_0/4)} \sqrt{1-\gamma}$ .

**Proof of the second claim.** For convenience, for  $i \in \tilde{I}$ , we define the functions  $g_i^+$  and  $g_i^-$  on  $\mathbb{R}_+$  via

$$g_i^+(\eta) := \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_i u_i \leq Z_i \leq (1+\eta)\alpha_i u_i} \right] \quad \text{and} \quad g_i^-(\eta) := \mathbb{E} \left[ u_i \mathbb{1}_{Z_i < \alpha_i u_i \leq (1+\eta)Z_i} \right].$$

Note that  $g_i(\eta) = g_i^-(\eta) + g_i^+(\eta)$  for all  $\eta \geq 0$ . Let  $\tilde{c}_\eta = \min\{1\} \cup \{\mathbb{E}[u_i \mathbb{1}_{\alpha_i u_i = Z_i}]/2, i \in \tilde{I}, \mathbb{P}(\alpha_i u_i = Z_i > 0) > 0\} > 0$ . Ensuring  $c_\eta \leq \tilde{c}_\eta$  implies that for any  $i \in \tilde{I}$  such that  $\mathbb{P}(\alpha_i u_i = Z_i > 0) > 0$ , we have (for any  $\eta_0 > 0$ )

$$\sup \{0\} \cup \{\eta \in [0, \eta_0] : g_i(\eta) \leq c_\eta \sqrt{1-\gamma}\} \leq \sup \{0\} \cup \{\eta \in [0, \eta_0] : g_i(\eta) \leq \tilde{c}_\eta\} = 0.$$

Hence, the desired inequality is trivial in that case. In the rest of the proof, we therefore focus on an index  $i \in \tilde{I}$  with  $\mathbb{P}(\alpha_i u_i = Z_i > 0) = 0$ . In particular, we necessarily have  $i \in I$ . Note that  $\alpha_{-i} \neq \mathbf{0}$ , otherwise  $\mathcal{U}_0$  would contain an  $\alpha$ -optimal utility vector by always allocating to  $i$ . Hence, the initial conditions for Lemma 24 are satisfied with  $\alpha$  and  $i$ .

We now introduce the function

$$\phi(\mathbf{y}) := \max_{\mathbf{x} \in \mathcal{U}^*} \mathbf{y}^\top (x_i, \tilde{\alpha}_{-i}^\top \mathbf{x}_{-i}) = \mathbb{E} \left[ \max \left( y_i u_i, y_{-i} \max_{j \neq i} \tilde{\alpha}_j u_j \right) \right], \quad \mathbf{y} \in \mathbb{R}^2 \setminus \{0\}.$$

We fix  $\mathbf{d} := (\alpha_i, \|\alpha_{-i}\|)$  and  $\mathbf{f} := (\|\alpha_{-i}\|, -\alpha_i)$  so that  $\mathbf{d}^\top \mathbf{f} = 0$ . Let  $\mathbf{U}$  be an  $\alpha$ -optimal vector and let  $\mathbf{p}(\cdot)$  be an optimal allocation that realizes  $\mathbf{U}$ . In the 2-agent game it induced the

utility vector  $\mathbf{V} = (U_i, \tilde{\alpha}_{-i} \mathbf{U}_{-i})$ . We denote  $\hat{i} \sim \mathbf{p}(\mathbf{u})$  the agent that receives the allocation. For convenience, we also denote  $Z_i := \max_{j \neq i} \alpha_j u_j$  and  $\tilde{Z}_i = Z_i / \|\alpha_{-i}\| = \max_{j \neq i} \tilde{\alpha}_j u_j$ . Then, for any  $\eta \geq 0$ , using the fact that  $\phi(\mathbf{d}) = \mathbf{d}^\top \mathbf{V}$  and the same arguments as in the proof of Theorem 8, we have

$$\begin{aligned} & \phi(\mathbf{d} + \eta \mathbf{f}) - \phi(\mathbf{d}) - \eta \mathbf{f}^\perp \mathbf{V} \\ &= \mathbb{E} \left[ \max \left( (d_i + \eta f_i) u_i, (d_{-i} + \eta f_{-i}) \tilde{Z}_i \right) - (d_i + \eta f_i) u_i \mathbb{1}_{\hat{i}=i} - (d_{-i} + \eta f_{-i}) \tilde{Z}_i \mathbb{1}_{\hat{i} \neq i} \right] \\ &= \mathbb{E} \left[ \left( (d_i + \eta f_i) u_i - (d_{-i} + \eta f_{-i}) \tilde{Z}_i \right) \mathbb{1}_{\hat{i} \neq i} \mathbb{1}_{(d_i + \eta f_i) u_i \geq (d_{-i} + \eta f_{-i}) \tilde{Z}_i} \right] \\ &\stackrel{(i)}{=} \mathbb{E} \left[ \left( (1 + \eta C_i) \alpha_i u_i - (1 - \eta / C_i) Z_i \right) \mathbb{1}_{\alpha_i u_i < Z_i \leq \frac{1 + \eta C_i}{1 - \eta / C_i} \alpha_i u_i} \right], \end{aligned}$$

where  $C_i = \|\alpha_{-i}\| / \alpha_i$ . In (i) we used the fact that  $\mathbb{P}(\alpha_i u_i = Z) = 0$  hence almost surely,  $\mathbf{p}$  only allocates to  $i$  if  $\alpha_i u_i > Z_i$  or if  $\alpha_i u_i = Z_i = 0$  in which case  $u_i = 0$  and the contribution is null. For  $\eta \geq 0$  sufficiently small (independently of  $\gamma$ ), with  $D_i = C_i + 1/C_i$ , we have  $\frac{1 + \eta C_i}{1 - \eta / C_i} \leq 1 + 2D_i \eta$ . Then, we obtain

$$\phi(\mathbf{d} + \eta \mathbf{f}) - \phi(\mathbf{d}) - \eta \mathbf{f}^\perp \mathbf{V} \leq D_i \alpha_i \eta \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_i u_i < Z_i \leq (1 + 2D_i \eta) \alpha_i u_i} \right] = D_i \alpha_i \eta g_i^+(2D_i \eta).$$

In the last inequality we used the fact that  $\mathbb{P}(\alpha_i u_i = Z_i > 0) = 0$ .

Similarly, for  $\eta \leq 0$ , with  $|\eta|$  sufficiently small, we have

$$\begin{aligned} \phi(\mathbf{d} + \eta \mathbf{f}) - \phi(\mathbf{d}) - \eta \mathbf{f}^\perp \mathbf{V} &= \mathbb{E} \left[ \left( (d_{-i} + \eta f_{-i}) \tilde{Z}_i - (d_i + \eta f_i) u_i \right) \mathbb{1}_{\hat{i}=i} \mathbb{1}_{(d_i + \eta f_i) u_i \leq (d_{-i} + \eta f_{-i}) \tilde{Z}_i} \right] \\ &= \mathbb{E} \left[ \left( (1 - \eta / C_i) Z_i - (1 + \eta C_i) \alpha_i u_i \right) \mathbb{1}_{Z_i < \alpha_i u_i \leq \frac{1 - \eta / C_i}{1 + \eta C_i} Z_i} \right] \\ &\leq D_i \alpha_i |\eta| \mathbb{E} \left[ u_i \mathbb{1}_{Z_i < \alpha_i u_i \leq (1 + 2D_i |\eta|) Z_i} \right] = D_i \alpha_i |\eta| g_i^-(2D_i |\eta|). \end{aligned}$$

We next introduce  $c_1, c_2 > 0$  the constants guaranteed from Lemma 24. We now pose  $c_\eta = \min \left( \tilde{c}_\eta, \frac{\sqrt{3c_2}}{24\alpha_i D_i} \|\mathbf{d}\| \right)$ ,  $c_r = 2D_i \sqrt{3c_2}$ , and fix  $0 < \eta_0 \leq \min(1, \sqrt{2c_r^2 / (3c_2)}, c_1 / (\frac{c_2 \|\mathbf{d}\|}{2c_r} + \frac{c_r \|\mathbf{d}\|}{8D_i^2}))$  sufficiently small such that all the previous estimates hold for  $|\eta| \leq \eta_0$ . Now let  $\gamma \in [9/10, 1)$  and define

$$\eta'(\gamma) := \sup \{0\} \cup \left\{ \eta \in [0, \eta_0] : g_i^+(\eta) + g_i^-(\eta) \leq c_\eta \sqrt{1 - \gamma} \right\}.$$

If  $\eta'(\gamma) = 0$ , the desired result is immediate. We suppose that is not the case and now fix  $\lambda(\gamma) \in (0, \eta'(\gamma))$ . By construction, since  $g_i^+$  and  $g_i^-$  are non-decreasing, we have  $g_i(\lambda(\gamma)) \leq c_\eta \sqrt{1 - \gamma}$ . We then pose

$$r := c_r \frac{\sqrt{1 - \gamma}}{\lambda(\gamma)}, \quad \xi := \frac{c_2}{2c_r} \sqrt{1 - \gamma} \lambda(\gamma), \quad \text{and} \quad \mathbf{x} := \mathbf{V} - (r + \xi) \frac{\mathbf{d}}{\|\mathbf{d}\|}.$$

For convenience, we introduce the function

$$\psi(\mathbf{y}) := \max_{\mathbf{z} \in B(\mathbf{x}, r)} \mathbf{y}^\top \mathbf{z} = \mathbf{y}^\top \mathbf{x} + r \|\mathbf{y}\|$$

We then compute for any  $\eta$ ,

$$\psi(\mathbf{d} + \eta \mathbf{f}) - \phi(\mathbf{d}) - \eta \mathbf{f}^\top \mathbf{V} = r(\|\mathbf{d} + \eta \mathbf{f}\| - \|\mathbf{d}\|) - \xi \|\mathbf{d}\| = r \|\mathbf{d}\| \left( \sqrt{1 + \eta^2} - 1 \right) - \xi \|\mathbf{d}\|$$

In the first equality, we used the fact that  $\mathbf{U}$  is  $\alpha$ -optimal, hence  $\mathbf{d}^\top \mathbf{V} = \phi(\mathbf{d})$ . Next, using  $1 + x/2 \geq \sqrt{1+x} \geq 1 + x/3$  for all  $x \in [0, 1]$ , we obtain that for  $\eta$  with  $|\eta| \leq 1$ ,

$$\frac{1}{3}r\|\mathbf{d}\|\eta^2 - \xi\|\mathbf{d}\| \leq \psi(\mathbf{d} + \eta\mathbf{f}) - \phi(\mathbf{d}) - \eta\mathbf{f}^\top \mathbf{V} \leq \frac{1}{2}r\|\mathbf{d}\|\eta^2 - \xi\|\mathbf{d}\|.$$

Putting this with the previous equations, for any  $\eta$  with  $|\eta| \leq \eta_0$ , we obtained

$$\phi(\mathbf{d} + \eta\mathbf{f}) - \psi(\mathbf{d} + \eta\mathbf{f}) \leq |\eta| \left( D_i \alpha_i g_i(2D_i|\eta|) - \frac{\|\mathbf{d}\|}{3}|\eta|r \right) + \xi\|\mathbf{d}\|.$$

In particular, for  $\eta \in \{\pm\lambda(\gamma)/(2D_i)\}$ , we have

$$\begin{aligned} \phi(\mathbf{d} + \eta\mathbf{f}) - \psi(\mathbf{d} + \eta\mathbf{f}) &\leq \frac{\lambda(\gamma)}{2D_i} \left( D_i \alpha_i c_\eta \sqrt{1-\gamma} - \frac{c_r \|\mathbf{d}\|}{6D_i} \sqrt{1-\gamma} \right) + \frac{c_2}{2c_r} \|\mathbf{d}\| \lambda(\gamma) \sqrt{1-\gamma} \\ &\stackrel{(i)}{=} \lambda(\gamma) \sqrt{1-\gamma} \left( \frac{\alpha_i c_\eta}{2} - \frac{\sqrt{3c_2} \|\mathbf{d}\|}{12D_i} \right) \\ &\stackrel{(ii)}{=} -\lambda(\gamma) \sqrt{1-\gamma} \frac{\alpha_i c_\eta}{2} < 0. \end{aligned}$$

In (i) we used the definition of  $c_r$  and in (ii) we used the definition of  $c_\eta$ . As a result,  $\mathbf{x} + r \frac{\mathbf{d} + \eta\mathbf{f}}{\|\mathbf{d} + \eta\mathbf{f}\|} \notin \{(u_i, \tilde{\alpha}_{-i}^\top \mathbf{u}_{-i}), \mathbf{u} \in \mathcal{U}^*\}$  for  $\eta \in \{\pm\lambda(\gamma)/(2D_i)\}$ . We can then define  $\mathcal{C}'$  as the union of all connected components of  $\mathcal{U}^* \setminus B^\circ(\mathbf{x}, r; \alpha, i)$  that are included within

$$\left\{ \mathbf{u} : \mathbf{d}^\top (u_i, \tilde{\alpha}_{-i}^\top \mathbf{u}_{-i}) > \mathbf{d}^\top \left( \mathbf{x} + r \frac{\mathbf{d} + \eta\mathbf{f}}{\|\mathbf{d} + \eta\mathbf{f}\|} \right), \eta = \frac{\lambda(\gamma)}{2D_i} \right\}.$$

Note that for  $|\eta|$  sufficiently small, we have

$$\mathbf{d}^\top \left( \mathbf{x} + r \frac{\mathbf{d} + \eta\mathbf{f}}{\|\mathbf{d} + \eta\mathbf{f}\|} \right) = \max_{\mathbf{z} \in \mathcal{U}^*} \alpha^\top \mathbf{z} - \xi\|\mathbf{d}\| - r\|\mathbf{d}\| \left( 1 - \frac{1}{\sqrt{1+\eta^2}} \right).$$

Now by construction of  $\eta_0$ , for  $|\eta| = \lambda(\gamma)/(2D_i)$ , since  $\lambda(\gamma) < \eta(\gamma) \leq \eta_0$ , we have

$$\xi\|\mathbf{d}\| + r\|\mathbf{d}\| \left( 1 - \frac{1}{\sqrt{1+\eta^2}} \right) \leq \xi\|\mathbf{d}\| + \frac{1}{2}r\|\mathbf{d}\|\eta^2 \leq \frac{c_2}{2c_r} \|\mathbf{d}\| \lambda(\gamma) + \frac{c_r}{8D_i^2} \|\mathbf{d}\| \lambda(\gamma) \leq c_1.$$

This proves that  $\mathcal{C}' \subset \{\mathbf{u} : \alpha^\top \mathbf{u} \geq \max_{\mathbf{z} \in \mathcal{U}^*} \alpha^\top \mathbf{z} - c_1\}$ .

We next pose  $\delta = c_2 \frac{1-\gamma}{\gamma r} = \frac{2}{\gamma} \xi \geq 2\xi$ . Since  $\gamma \geq 9/10$ ,

$$\frac{\delta}{r} \leq \frac{3\xi}{r} = \frac{3c_2}{2c_r^2} \lambda(\gamma)^2 \leq \frac{3c_2}{2c_r^2} \eta_0^2 \leq 1.$$

That is,  $\delta \leq r$ . We next prove  $\mathcal{C}' \subset B(\mathbf{x}, r + \delta; \alpha, i)$ . To do so, it suffices to check that

$$\phi(\mathbf{d} + \eta\mathbf{f}) \leq (\mathbf{d} + \eta\mathbf{f})^\top \mathbf{x} + (r + \delta)\|\mathbf{d} + \eta\mathbf{f}\|, \quad \eta \in \left[ -\frac{\lambda(\gamma)}{2D_i}, \frac{\lambda(\gamma)}{2D_i} \right].$$

Using the previous computations, for any  $\eta$  such that  $|\eta| \leq \lambda(\gamma)/(2D_i)$ , we have

$$\begin{aligned} &\phi(\mathbf{d} + \eta\mathbf{f}) - (\mathbf{d} + \eta\mathbf{f})^\top \mathbf{x} - (r + \delta)\|\mathbf{d} + \eta\mathbf{f}\| \\ &\leq D_i \alpha_i |\eta| g_i(2D_i|\eta|) - \frac{1}{3}(r + \delta)\|\mathbf{d}\|\eta^2 - (\delta - \xi)\|\mathbf{d}\| \\ &\stackrel{(i)}{\leq} \frac{\alpha_i c_\eta \lambda(\gamma)}{2} \sqrt{1-\gamma} - \xi\|\mathbf{d}\| \stackrel{(ii)}{\leq} 0, \end{aligned}$$

where in (i) we used  $|\eta| \leq \lambda(\gamma)/(2D_i)$  and  $\delta \geq 2\xi$ , and in (ii) we used the definition of  $c_\eta$  and  $\xi$ . This ends the proof that  $\mathcal{C}' \subset B(\mathbf{x}, r + \delta; \boldsymbol{\alpha}, i)$ .

We have now checked all the assumptions needed to use Lemma 24 which implies

$$\mathcal{U}_\gamma \cap \left( \mathcal{C}' \cap B^\circ \left( \mathbf{x}, r + \frac{1-\gamma}{\gamma} \delta; \boldsymbol{\alpha}, i \right) \right) = \emptyset.$$

Next, note that by construction,  $\mathbf{U} \in \mathcal{C}$  and  $(U_i, \tilde{\boldsymbol{\alpha}}_{-i}^\top \mathbf{U}_{-i}) = \mathbf{x} + (r + \xi) \frac{\mathbf{d}}{\|\mathbf{d}\|}$ . Hence, Lemma 24 implies that

$$\max_{\mathbf{y} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{y} < \mathbf{d}^\top \mathbf{x} - \left( r + \frac{1-\gamma}{\gamma} \delta \right) \|\mathbf{d}\| = \max_{\mathbf{y} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{y} - \left( \xi - \frac{1-\gamma}{\gamma} \delta \right) \|\mathbf{d}\|$$

Hence, for  $1 - \gamma \leq 1/10$ , we obtain

$$\max_{\mathbf{y} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{y} - \max_{\mathbf{y} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{y} \geq \frac{\xi}{2} \|\mathbf{d}\| = \frac{c_2 \|\mathbf{d}\|}{2c_r} \sqrt{1-\gamma} \lambda(\gamma).$$

Because this holds for all  $\lambda(\gamma) \in (0, \eta'(\gamma))$ , this also holds for  $\eta'(\gamma)$ , which ends the proof of the theorem.  $\blacksquare$

We can now prove Theorem 10 that gives necessary conditions for having rates  $\mathcal{O}(1 - \gamma)$ , as a consequence of Theorem 11.

**Proof of Theorem 10** If  $g_i(\eta) \underset{\eta \rightarrow 0}{=} o(\eta)$ , then

$$\sup \{0\} \cup \{\eta \in [0, \eta_0] : g_i(\eta) \leq c_\eta \sqrt{1-\gamma}\} = \omega(\sqrt{1-\gamma}),$$

which gives the desired lower bound  $\omega(1 - \gamma)$  on the convergence rate.

We now show that the (NC2) implies  $g_i(\eta) \underset{\eta \rightarrow 0}{=} o(\eta)$  (NC1). Using the same notations as in Theorem 11, for any  $\eta \geq 0$ ,

$$\begin{aligned} g_i(\eta) &= g_i^+(\eta) + g_i^-(\eta) \\ &\leq \sum_{j \neq i} \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_i u_i \leq Z_i = \alpha_j u_j \leq (1+\eta) \alpha_i u_i} \right] + \mathbb{E} \left[ u_i \mathbb{1}_{Z_i = \alpha_j u_j \leq \alpha_i u_i \leq (1+\eta) \alpha_j u_j} \right] \\ &\leq \sum_{j \neq i, \alpha_j > 0} \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_j u_j = Z} \mathbb{1}_{\alpha_j u_j \in [\alpha_i u_i, (1+\eta) \alpha_i u_i]} \right] + \frac{\alpha_j}{\alpha_i} (1 + \eta) \mathbb{E} \left[ u_j \mathbb{1}_{Z_i = \alpha_j u_j \leq \alpha_i u_i \leq (1+\eta) \alpha_j u_j} \right] \\ &\leq \sum_{j \neq i, \alpha_j > 0} \mathbb{E} \left[ u_i \mathbb{1}_{\alpha_j u_j = Z} \mathbb{1}_{\alpha_j u_j \in [\alpha_i u_i, (1+\eta) \alpha_i u_i]} \right] + \frac{\alpha_j}{\alpha_i} (1 + \eta) \mathbb{E} \left[ u_j \mathbb{1}_{\alpha_i u_i = Z} \mathbb{1}_{\alpha_i u_i \in [\alpha_j u_j, (1+\eta) \alpha_j u_j]} \right]. \end{aligned}$$

Hence, (NC2) implies (NC1), which ends the proof of the result.  $\blacksquare$

As discussed in Section 3.2, we can replace the function  $f(\cdot)$  defined in Eq. (15) with the function  $\tilde{f}(\cdot)$  defined in Eq. (18) within the statement of Theorem 11. The function  $\tilde{f}$  can be more easily computed as it only involves solving at most  $n^2$  max-flow problems. We check that the two functions are equivalent below.



**Lemma 28.** Fix  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  and a partition  $\tilde{I} = I_1 \sqcup \dots \sqcup I_q$  with  $|I_s| \geq 2$  for all  $s \in [q]$ . For any  $\eta \geq 0$ , we have

$$\frac{1}{\|\alpha\|_\infty} f(\eta) \leq \tilde{f}(\eta) \leq \frac{n}{\min_{i \in \tilde{I}} \alpha_i} f(\eta).$$

Thus, Eq. (16) from Theorem 11 holds by replacing  $f$  by  $\tilde{f}$  up to changing the constant  $C$ .

**Proof** Fix  $\eta \geq 0$  and  $s \in [q]$ . Using the min-cut/max-flow duality theorem, we have

$$\begin{aligned} \min_{i \neq j \in I_s} (\max \text{ flow from } i \text{ to } j \text{ on } \mathcal{G}_s(\eta)) &= \min_{i \neq j \in I_s} \min_{\{i\} \subset B \subset I_s \setminus \{j\}} \sum_{k \in B, l \in I_s \setminus B} f(\eta; k, l) \\ &= \min_{\emptyset \subsetneq B \subsetneq I_s} \sum_{i \in B, j \in I_s \setminus B} f(\eta; i, j). \end{aligned}$$

We next note that for any  $\emptyset \subsetneq B \subsetneq I_s$ , we have

$$\mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\eta)Z_B} \right] = \mathbb{E} \left[ \max_{i \in B, j \in I_s \setminus B} \alpha_i u_i \mathbb{1}_{\alpha_i u_i = Z_B} \mathbb{1}_{\alpha_j u_j = Z} \mathbb{1}_{\alpha_j u_j \in [\alpha_i u_i, (1+\eta)\alpha_i u_i]} \right].$$

Therefore,

$$\begin{aligned} \frac{\min_{i \in \tilde{I}} \alpha_i}{n^2} \sum_{i \in B, j \in I_s \setminus B} f(\eta; i, j) &\leq \max_{i \in B, j \in I_s \setminus B} \alpha_i f(\eta; i, j) \\ &\leq \mathbb{E} \left[ Z_B \mathbb{1}_{Z_B \leq Z_{I_s \setminus B} = Z \leq (1+\eta)Z_B} \right] \\ &\leq \sum_{i \in B, j \in I_s \setminus B} \alpha_i f(\eta; i, j) \leq \|\alpha\|_\infty \sum_{i \in B, j \in I_s \setminus B} f(\eta; i, j). \end{aligned}$$

Taking the minimum over all  $s \in [q]$  and  $\emptyset \subsetneq B \subsetneq I_s$  then gives

$$\frac{\min_{i \in \tilde{I}} \alpha_i}{n^2} \tilde{f}(\eta) \leq f(\eta) \leq \|\alpha\|_\infty \tilde{f}(\eta).$$

which implies the desired result. ■

## 5.5 Remaining proofs for discrete distributions

In this section, we give the remaining proof of Theorem 17 from Section 3.4. The tools developed until now already imply the following.

**Corollary 29.** Suppose that all utility distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  are discrete. Let  $\alpha \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  such that  $\mathcal{U}_0$  does not already contain an  $\alpha$ -optimal vector (see Lemma 2 for a characterization). Let  $\tilde{I} \subset [n]$  be defined as in Theorem 8.

If there exists  $i \in \tilde{I}$  such that for all  $j \neq i$ ,  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0) = 0$ , then

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \underset{\gamma \rightarrow 1}{=} \Theta(\sqrt{1-\gamma}).$$

Otherwise,

$$\max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \alpha^\top \mathbf{x} \underset{\gamma \rightarrow 1}{=} \mathcal{O}(1-\gamma).$$

**Proof** We let  $m := \max_{i \in [n]} \min(\alpha_i \text{Supp}(\mathcal{D}_i))$  be the minimum of the support of  $\max_{i \in [n]} \alpha_i u_i$ .

We start with the first case. We denote  $Z_i = \max_{j \neq i} \alpha_j u_j$ . The assumption implies that  $\mathbb{P}(\alpha_i u_i = Z_i > 0) = 0$ . We recall that since  $i \in \tilde{I}$ , we have  $\alpha_i > 0$ . As a result, there exists  $\eta_1 > 0$  such that

$$\mathbb{P}(u_i > 0, Z_i \in [\alpha_i u_i / (1 + \eta_1), (1 + \eta_1) \alpha_i u_i]) = 0.$$

This implies that for all  $\eta \in [0, \eta_1]$ , we have  $g_i(\eta) = 0$ . Then, Theorem 11 implies

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{x} \geq c \sqrt{1 - \gamma} \min(\eta_0, \eta_1).$$

The other inequality is already guaranteed by Theorem 27, which ends the first claim.

We now consider the second alternative. In this case, the first bullet point from Corollary 9 is satisfied for all  $i \in \tilde{I}$ . This implies

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{x} = \mathcal{O}(1 - \gamma).$$

This ends the proof. ■

We now use a gluing method to improve the convergence rate in the second case of Corollary 29. In this case, the frontier of  $\mathcal{U}^*$  at  $\boldsymbol{\alpha}$ -optimal points is flat, which will allow to fit local caps from larger radius balls close to this flat surface. Here, the form of the resulting frontier can be explicitly constructed, which allows for a simpler analysis. The intuition is however the same as that from the gluing approach described in Section 3.4 and gives the same convergence rates up to constants (potentially in exponents).

**Theorem 30.** Fix  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  such that  $\mathcal{U}_0$  does not already contain an  $\boldsymbol{\alpha}$ -optimal vector (see Lemma 2 for a characterization). Define  $\tilde{I}$  as in Theorem 8.

If for all  $i \in \tilde{I}$  there exists  $j \neq i$  with  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0) > 0$ , then

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{x} \underset{\gamma \rightarrow 1}{=} \mathcal{O}\left((1 - \gamma) e^{-c(\boldsymbol{\alpha}) / \sqrt{1 - \gamma}}\right),$$

for some constant  $c(\boldsymbol{\alpha})$  that only depends on  $\boldsymbol{\alpha}$  and the utility distributions.

**Proof** Suppose that the distributions and  $\boldsymbol{\alpha}$  satisfy the assumptions. We denote the set of  $\boldsymbol{\alpha}$ -optimal utility vectors by

$$\mathcal{U}^*(\boldsymbol{\alpha}) := \left\{ \mathbf{x} \in \mathcal{U}^*, \boldsymbol{\alpha}^\top \mathbf{x} = \max_{\mathbf{y} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{y} \right\}.$$

We next construct the (undirected) graph  $\mathcal{G}$  on  $\tilde{I}$  such that for any  $i \neq j \in \tilde{I}$ ,  $i$  is connected to  $j$  if and only if  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0) > 0$ . By assumption, for all  $i \in \tilde{I}$  there exists  $j \neq i$  such that  $\mathbb{P}(\alpha_i u_i = \alpha_j u_j = Z > 0) > 0$ . In particular, this implies  $j \in \tilde{I}$ . This shows that  $\mathcal{G}$  does not have any isolated node. We then consider the partition  $\tilde{I} = I_1 \sqcup \dots \sqcup I_q$  by connected components of  $\mathcal{G}$ . Since there is no isolated node, we have  $|I_s| \geq 2$  for all  $s \in [q]$ . By construction, since two components are not connected, Eq. (14) is satisfied.

Following the same arguments as in Lemma 21, it suffices to focus on each set  $I_s$  independently for all  $s \in [q]$ . Precisely, we focus on the game where only agents  $I_s$  are present but the central planner cannot allocate on the event  $\mathcal{E}_s := \{Z_{-I_s} \geq Z_{I_s}\}$ . Suppose we showed that the

region  $\mathcal{R}_s \subset \mathbb{R}_+^{I_s}$  is achievable in this setting for all  $s \in [q]$ , then the proof of Lemma 6 implies that for any  $\alpha$ -optimal vector  $\mathbf{U}$ , the region

$$\mathcal{R} := \mathbf{U}_{[n] \setminus \tilde{I}} \otimes \bigotimes_{s \in [q]} \mathcal{R}_s \quad (59)$$

is achievable in the original setting with  $[n]$  agents:  $\mathcal{R} \subset \mathcal{U}_\gamma$ .

We therefore focus on a single connected component  $I_s$  for  $s \in [n]$  from now. We fix any  $\alpha$ -optimal vector  $\mathbf{U}$  and let  $\mathcal{H}_s = \{\mathbf{y} : \forall i \notin I_s, y_i = U_i\}$ . For any  $\mathbf{x} \in \mathbb{R}^{I_s}$  and  $r > 0$ , we also denote  $B_{I_s}(\mathbf{x}, r)$  the corresponding ball on the space  $\mathbb{R}^{I_s}$ . We first show that  $\mathcal{U}^*(\alpha) \cap \mathcal{H}_s$  has (full) dimension  $|I_s| - 1$ . Note that since  $I_s$  is connected, there exist values  $m_1 > \dots > m_T > 0$  such that letting

$$J_t := \{i \in I_s : \mathbb{P}(\alpha_i u_i = m_t = Z) > 0\}, \quad t \in [T],$$

we have (1)  $|J_t| \geq 2$  for  $t \in [T]$  and (2) no set  $J_t$  is disjoint from the others, that is  $J_t \cap \bigcup_{t' \neq t} J_{t'} \neq \emptyset$  for all  $t \in [T]$ . We now define the event  $\mathcal{F}_t = \{\forall i \in J_t, \alpha_i u_i = m_t = Z\}$ . By construction,  $\mathbb{P}(\mathcal{F}_t) > 0$  and all events  $\mathcal{F}_1, \dots, \mathcal{F}_T$  are disjoint. On each event  $\mathcal{F}_t$ , an  $\alpha$ -optimal allocation can choose to allocate to any agent in  $J_t$ . Let  $\mathbf{p}^{(0)}(\cdot)$  an  $\alpha$ -optimal allocation such that  $\mathbb{E}[u_i p_i^{(0)}(\mathbf{u})] = U_i$  for all  $i \notin I_s$  and that for any  $t \in [T]$ , allocates uniformly among agents  $J_t$  on the event  $\mathcal{F}_t$ . Denote  $\mathbf{U}^{(0)}$  the utility vector realized by  $\mathbf{p}^{(0)}$ . The previous arguments imply

$$\mathcal{S} := \mathbf{U}^{(0)} + \sum_{t \in [T]} \mathbb{P}(\mathcal{F}_t) \left\{ \left( \frac{m_t}{\alpha_i} \left( q_i - \frac{1}{|J_t|} \right) \mathbb{1}_{i \in J_t} \right)_{i \in [n]}, \mathbf{q} \in \Delta_{J_t} \right\} \subset \mathcal{U}^*(\alpha),$$

where the sum between sets are Minkowski sums. Here each sum corresponds to the freedom that an  $\alpha$ -optimal allocation has on the event  $\mathcal{F}_t$  to use any allocation within  $\Delta_{J_t}$ . From the assumptions (1) and (2) on the sets  $J_1, \dots, J_T$ , with  $\tilde{r}_0 = \min_{t \in [T]} \mathbb{P}(\mathcal{F}_t) m_t / n$ , we have

$$\mathbf{U}^{(0)} + \left\{ \left( \frac{y_i}{\alpha_i} \mathbb{1}_{i \in I_s} \right)_{i \in [n]}, \mathbf{y} \in B(0, \tilde{r}_0), \alpha^\top \mathbf{y} = 0 \right\} \subset \mathcal{U}^*(\alpha) \cap \mathcal{H}_s. \quad (60)$$

We recall that for any  $\mathbf{x} \in \mathcal{U}^*$ , any  $\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}$  satisfies  $\mathbf{y} \in \mathcal{U}^*$ . This also holds in the setting where only agents  $I_s$  are present and there is no allocation on the event  $\mathcal{E}_s$ . We denote by  $\mathcal{U}_s^*$  the corresponding achievable region. In the proof of Lemma 21, we showed (Eq. (35)) that

$$\mathcal{U}_s^* = P_{I_s}(\mathcal{U}^* \cap \mathcal{H}_s),$$

where  $P_{I_s} : \mathbb{R}^n \rightarrow \mathbb{R}^{I_s}$  is the projection on the  $I_s$  coordinates. For convenience, we also write  $\mathbf{U}_s := \mathbf{U}_{I_s}^{(0)}$ . As a result, from Eq. (60) we can check that with  $r_0 = \tilde{r}_0 / (4\|\alpha\|)$ , for any  $\mathbf{z} \in B_{I_s}(\mathbf{U}_s, r_0) \cap \{\mathbf{y} : \alpha_{I_s}^\top (\mathbf{y} - \mathbf{U}_s) = 0\} := S_s$ ,

$$B_{I_s} \left( \mathbf{z} - r_0 \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|}, r_0 \right) \subset P_{I_s}(\mathcal{U}^* \cap \mathcal{H}_s) = \mathcal{U}_s^*.$$

Note that  $\min_{i \in I_s} \mathbb{E}[u_i] \mathbb{P}(Z_{I_s} = Z > 0) - (\mathbf{z} - r_0 \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|})_i - r_0 \geq \mathbb{P}(\mathcal{F}_t) \frac{m_t}{\alpha_i} (1 - 1/|J_t|) - r_0 - r_0 \geq 4r_0 - 2r_0 = 2r_0$ , where  $t$  is such that  $i \in J_t$ . Then, with  $C = 4n^2 \bar{v}^2 \left( 1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i] + r_0}{r_0} \right)^2$  and  $\delta = 4C \frac{1-\gamma}{r_0}$ , provided that  $\delta \leq r_0/2$  then Eq. (13) is satisfied with  $r = r_0 - \delta$  and  $\delta$ :

$$\frac{r\gamma}{1-\gamma} \geq r \geq \frac{r_0}{2} \geq \delta = 4C \frac{1-\gamma}{r_0} \geq C \frac{1-\gamma}{\gamma r}.$$

Here, we used  $\gamma \geq 1/2$  and  $r_0 - \delta \geq r_0/2$ . Let  $C_1 := \bar{v} \left( 2n + 3 \frac{\|\alpha\|}{\min_{i \in \bar{I}} \alpha_i} \right)$ . The inequality  $\delta \leq r_0/4$  holds whenever  $\gamma \geq \gamma_1 := \max(1 - \min(\frac{r_0^2}{16C}, (\frac{r_0 \min_{i \in \bar{I}} \alpha_i}{8C \|\alpha\|})^2, \frac{r_0}{4C_1}), 1/2)$ . Then, for  $\gamma \in [\gamma_1, 1)$ , the proof of Lemma 21 shows that

$$B_{I_s} \left( \mathbf{z} - r_0 \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|}, r_0 - \delta \right) \subset \mathcal{U}_{s,\gamma}, \quad (61)$$

where  $\mathcal{U}_{s,\gamma}$  is the achievable region for the setting with only agents  $I_s$  and in which the central planner cannot allocate on the event  $\mathcal{E}_s$ , with discount factor  $\gamma$ . We define the following function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(r) := \delta \frac{\cosh\left(\frac{c_f r}{\sqrt{1-\gamma}}\right)}{\cosh\left(\frac{c_f r_0}{\sqrt{1-\gamma}}\right)} = 4C \frac{1-\gamma}{r_0} \frac{\cosh\left(\frac{c_f r}{\sqrt{1-\gamma}}\right)}{\cosh\left(\frac{c_f r_0}{\sqrt{1-\gamma}}\right)}, \quad r \geq 0.$$

where  $c_f = \min\left(\frac{1}{4C_1}, 1\right)$ . We then pose  $\gamma_0 = \max(\gamma_1, 1 - \min(r_0^2 c_f^2, (\frac{c_f C}{C_1})^2))$  and consider  $\gamma \in [\gamma_0, 1)$ . We then consider the following region

$$\mathcal{R}_s = \left\{ \mathbf{y} \in \mathbb{R}^{I_s} : \mathbf{0} \leq \mathbf{y} \leq \mathbf{z} - f(r) \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|}, \|\mathbf{z} - \mathbf{U}_{I_s}\| = r, \mathbf{z} \in S_s \right\}$$

and now construct an allocation and promised utility function on  $\mathcal{R}_s$ . For any  $\mathbf{z} \in S_s$ , we denote by  $\mathbf{p}(\cdot; \mathbf{z})$  an allocation that realizes  $\mathbf{z}$  in the full information setting (within the game when only agents  $I_s$  are present). We start by specifying the strategy on the boundary points of  $\mathcal{R}_s$ . For  $\mathbf{z} \in S_s$ , let  $r = \|\mathbf{z} - \mathbf{U}_s\|$  and  $\mathbf{U} = \mathbf{z} - f(r) \alpha_{I_s} / \|\alpha_{I_s}\|$ . We pose

$$\mathbf{p}(\cdot | \mathbf{U}) := \mathbf{p}(\cdot; \mathbf{z}) \quad \text{and} \quad \alpha(\mathbf{U}) := -\frac{\alpha_{I_s}}{\|\alpha_{I_s}\|} - f'(r) \frac{\mathbf{z} - \mathbf{U}_s}{\|\mathbf{z} - \mathbf{U}_s\|}$$

Note that  $\alpha(\mathbf{U})$  was selected (up to the sign) as the normal to the frontier manifold  $\mathcal{M}_s := \{\mathbf{z}' - f(\|\mathbf{z}' - \mathbf{U}_s\|) \alpha_{I_s} / \|\alpha_{I_s}\|, \mathbf{z}' \in S_s\}$ . We then extend the definition of the allocation and promised utility functions to all  $\mathbf{U} \in \mathcal{R}_s$  as follows. Let  $\mathbf{V} \in \mathcal{M}_s$  such that  $\mathbf{0} \leq \mathbf{U} \leq \mathbf{V}$ . We let

$$p_i(\cdot | \mathbf{U}) := \frac{U_i}{V_i} p_i(\cdot | \mathbf{V}) \quad \text{and} \quad W_i(\cdot | \mathbf{U}) := \frac{U_i}{V_i} W_i(\cdot | \mathbf{V}), \quad i \in I_s,$$

with the convention  $0/0 = 0$  when  $U_i = V_i = 0$ . To prove that  $\mathcal{R}_s \subset \mathcal{U}_{s,\gamma}$  it suffices to show that for all  $\mathbf{U} \in \mathcal{R}_s$  and  $\mathbf{v} \in [0, \bar{v}]^{I_s}$ ,

$$\mathbf{W}(\mathbf{v} | \mathbf{U}) \in \mathcal{R}_s \cup \bigcup_{\mathbf{z} \in S_s} \left\{ \mathbf{y} : \mathbf{0} \leq \mathbf{y} \leq \mathbf{x}, \mathbf{x} \in B_{I_s} \left( \mathbf{z} - r_0 \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|}, r_0 - \delta \right) \right\}, \quad (62)$$

and Eq. (7) is satisfied for all  $\mathbf{U} \in \mathcal{R}_s$ . Here, we used Eq. (61) that already guarantees that the second term in the right-hand side of Eq. (62) is contained within  $\mathcal{U}_\gamma$ . By the linearity of the definitions of the allocation and promised utility function, it suffices to check that for all boundary utility vectors  $\mathbf{U} \in \mathcal{M}_s$ , Eq. (7) is satisfied (this is guaranteed by construction via Eq. (11)) and to check that the following holds:

$$\mathbf{W}(\mathbf{v} | \mathbf{U}) \in \mathcal{R}_s \cup \bigcup_{\mathbf{z} \in S_s} B_{I_s} \left( \mathbf{z} - r_0 \frac{\alpha_{I_s}}{\|\alpha_{I_s}\|}, r_0 - \delta \right), \quad \mathbf{U} \in \mathcal{M}_s, \mathbf{v} \in [0, \bar{v}]^{I_s}, \quad (63)$$

Fix such a vector  $\mathbf{U} \in \mathcal{M}_s$  and write  $\mathbf{U} = \mathbf{z} - f(r) \frac{\boldsymbol{\alpha}_{I_s}}{\|\boldsymbol{\alpha}_{I_s}\|}$  where  $r = \|\mathbf{z} - \mathbf{U}_s\|$ . By the promise-keeping equality Eq. (2) (which is a consequence of the construction),

$$\bar{\mathbf{W}} := \mathbb{E}[\mathbf{W}(\mathbf{u} | \mathbf{U})] = \mathbf{U} - \frac{1-\gamma}{\gamma} f(r) \frac{\boldsymbol{\alpha}_{I_s}}{\|\boldsymbol{\alpha}_{I_s}\|}. \quad (64)$$

By construction, we also have  $\boldsymbol{\alpha}(\mathbf{U})^\top (\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}) = 0$  for  $v \in [0, \bar{v}]^{I_s}$ . Last, the proof of Lemma 3 (Eq. (28)) shows that

$$\|\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}\| \leq \frac{1-\gamma}{\gamma} \bar{v} \left( 2n + \frac{\alpha_{i_1}(\mathbf{U})}{\alpha_{i_2}(\mathbf{U})} \right), \quad \mathbf{v} \in [0, \bar{v}]^{I_s},$$

where  $\alpha_{i_1}(\mathbf{U})$  and  $\alpha_{i_2}(\mathbf{U})$  are the first and second largest components in absolute value of  $\boldsymbol{\alpha}(\mathbf{U})$ . Note that from  $\gamma \geq \gamma_1$  and the definition of  $\gamma_1$ , we have

$$\left\| \boldsymbol{\alpha}(\mathbf{U}) - \frac{\boldsymbol{\alpha}_{I_s}}{\|\boldsymbol{\alpha}_{I_s}\|} \right\| = |f'(r)| = c_f \frac{\delta}{\sqrt{1-\gamma}} \frac{|\sinh(c_f r / \sqrt{1-\gamma})|}{\cosh(c_f r_0 / \sqrt{1-\gamma})} \leq \frac{\min_{i \in \bar{I}} \alpha_i}{2\|\boldsymbol{\alpha}\|}.$$

As a result, this shows that for all  $i \in I_s$  we have  $\alpha_i(\mathbf{U}) \|\boldsymbol{\alpha}_{I_s}\| \in [\alpha_i/2, 3\alpha_i/2]$ . Therefore, we obtained

$$\|\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}\| \leq \frac{1-\gamma}{\gamma} \bar{v} \left( 2n + 3 \frac{\|\boldsymbol{\alpha}\|}{\min_{i \in \bar{I}} \alpha_i} \right) \leq 2C_1(1-\gamma), \quad \mathbf{v} \in [0, \bar{v}]^{I_s}. \quad (65)$$

Now let  $d(\mathbf{v}) := \|P_{\boldsymbol{\alpha}_{I_s}^\perp}(\mathbf{W}(\mathbf{v} | \mathbf{U}) - \mathbf{U}_s)\| - \|P_{\boldsymbol{\alpha}_{I_s}^\perp}(\bar{\mathbf{W}} - \mathbf{U}_s)\|$ , where  $P_{\boldsymbol{\alpha}_{I_s}^\perp}$  denotes the projection onto the orthogonal space to  $\boldsymbol{\alpha}_{I_s} \mathbb{R}$ . Because  $\boldsymbol{\alpha}(\mathbf{U})^\top (\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}) = 0$ , we can write

$$\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}} = d(\mathbf{v}) \frac{\mathbf{z} - \mathbf{U}_s}{\|\mathbf{z} - \mathbf{U}_s\|} + d(\mathbf{v}) f'(r) \frac{(-\boldsymbol{\alpha}_{I_s})}{\|\boldsymbol{\alpha}_{I_s}\|} + \mathbf{W}_0(\mathbf{v}),$$

where  $\mathbf{W}_0(\mathbf{v}) \in \text{Span}(\boldsymbol{\alpha}_{I_s}, \mathbf{z} - \mathbf{U}_s)^\perp$ . Using Eq. (64) we have

$$\begin{aligned} \frac{-\boldsymbol{\alpha}_{I_s}^\top}{\|\boldsymbol{\alpha}_{I_s}\|} (\mathbf{W}(\mathbf{v} | \mathbf{U}) - \mathbf{U}_s) &= f(r) + f(r) \frac{1-\gamma}{\gamma} + \frac{-\boldsymbol{\alpha}_{I_s}^\top}{\|\boldsymbol{\alpha}_{I_s}\|} (\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}) \\ &= f(r) + f(r) \frac{1-\gamma}{\gamma} + d(\mathbf{v}) f'(r). \end{aligned}$$

Next, using Taylor expansion's theorem, there exists  $\tilde{r} \in [\min(r, r + d(\mathbf{v})), \max(r, r + d(\mathbf{v}))]$  such that

$$\begin{aligned} f(r) \frac{1-\gamma}{\gamma} &\geq \frac{-\boldsymbol{\alpha}_{I_s}^\top}{\|\boldsymbol{\alpha}_{I_s}\|} (\mathbf{W}(\mathbf{v} | \mathbf{U}) - \mathbf{U}_s) - f(r + d(\mathbf{v})) = f(r) \frac{1-\gamma}{\gamma} - \frac{f''(\tilde{r})}{2} d(\mathbf{v})^2 \\ &\stackrel{(i)}{=} f(r)(1-\gamma) - \frac{c_f^2 f(\tilde{r})}{2(1-\gamma)} d(\mathbf{v})^2 \\ &\stackrel{(ii)}{\geq} (1-\gamma) (f(r) - 2c_f^2 C_1^2 f(\tilde{r})) \\ &\stackrel{(iii)}{\geq} (1-\gamma) \left( f(r) - \frac{f(r + |d(\mathbf{v})|)}{2} \right) \end{aligned}$$

In (i) we used the definition the fact that  $\cosh''(x) = \cosh(x)$  and in (ii) we used  $|d(\mathbf{v})| \leq \|\mathbf{W}(\mathbf{v} | \mathbf{U}) - \bar{\mathbf{W}}\|$  together with Eq. (65). Last, in (iii) we used the definition of  $c_f$  together with the fact that  $f(\tilde{r}) \leq f(r + |d(\mathbf{v})|)$ . Now note that

$$f(r + |d(\mathbf{v})|) \leq f(r)e^{\frac{c_f |d(\mathbf{v})|}{\sqrt{1-\gamma}}} \leq f(r)e^{2c_f C_1 \sqrt{1-\gamma}} \leq 2f(r). \quad (66)$$

In summary, since  $\gamma \geq 1/2$ , we showed that

$$f(r + d(\mathbf{v})) \leq \frac{-\boldsymbol{\alpha}_{I_s}^\top}{\|\boldsymbol{\alpha}_{I_s}\|} (\mathbf{W}(\mathbf{v} | \mathbf{U}) - \mathbf{U}_s) \leq f(r + d(\mathbf{v})) + 2f(r)(1 - \gamma). \quad (67)$$

First suppose that  $|r + d(\mathbf{v})| \leq r_0$ . We let  $\mathbf{z}(\mathbf{v}) := \mathbf{U}_s + P_{\boldsymbol{\alpha}_{I_s}^\top}(\mathbf{W}(\mathbf{v} | \mathbf{U}) - \mathbf{U}_s) \in S_s$ . The previous equation implies that  $\mathbf{W}(\mathbf{v} | \mathbf{U}) \leq \mathbf{z}(\mathbf{v}) - f(\|\mathbf{z}(\mathbf{v}) - \mathbf{U}_s\|) \frac{\boldsymbol{\alpha}_{I_s}}{\|\boldsymbol{\alpha}_{I_s}\|}$ . Also, we have  $2f(r)(1 - \gamma) \leq f(r_0) = \delta$ , hence, we also obtain  $\mathbf{0} \leq \mathbf{W}(\mathbf{v} | \mathbf{U})$ . Together with the previous inequalities, this implies  $\mathbf{W}(\mathbf{v} | \mathbf{U}) \in \mathcal{R}_s$ .

We now treat the remaining case when  $|r + d(\mathbf{v})| \geq r_0$ . We first show  $d(\mathbf{v}) \geq 0$ . Otherwise, we have  $2C_1(1 - \gamma) \geq d(\mathbf{v}) \geq r_0$ , which is absurd since  $\gamma \geq \gamma_1$ . For this boundary case, we let

$$\mathbf{z}(\mathbf{v}) := \mathbf{U}_s + r_0 \frac{P_{\boldsymbol{\alpha}_{I_s}^\top}(\mathbf{W}(\mathbf{v} | \mathbf{U}) - \mathbf{U}_s)}{\|P_{\boldsymbol{\alpha}_{I_s}^\top}(\mathbf{W}(\mathbf{v} | \mathbf{U}) - \mathbf{U}_s)\|},$$

and aim to show that

$$\mathbf{W}(\mathbf{v}, \mathbf{U}) \in B_{I_s} \left( \mathbf{z}(\mathbf{v}) - r_0 \frac{\boldsymbol{\alpha}_{I_s}}{\|\boldsymbol{\alpha}_{I_s}\|}, r_0 - \delta \right). \quad (68)$$

Note that  $f(r_0) = \delta$  and by Eq. (67) there exists  $\tilde{f} \in [0, 2f(r_0)(1 - \gamma)] \subset [0, f(r_0)]$  such that

$$\begin{aligned} R(\mathbf{v}) &:= \left\| \mathbf{W}(\mathbf{v}, \mathbf{U}) - \left( \mathbf{z}(\mathbf{v}) - r_0 \frac{\boldsymbol{\alpha}_{I_s}}{\|\boldsymbol{\alpha}_{I_s}\|} \right) \right\|^2 = (r + d(\mathbf{v}) - r_0)^2 + (f(r + d(\mathbf{v})) + \tilde{f} - r_0)^2 \\ &\stackrel{(i)}{\leq} (r + d(\mathbf{v}) - r_0)^2 + (r_0 - f(r + d(\mathbf{v})))^2 \\ &= (r_0 - \delta)^2 + (r + d(\mathbf{v}) - r_0)^2 - (2r_0 + f(r_0) - f(r + d(\mathbf{v}))) (f(r + d(\mathbf{v})) - f(r_0)) \\ &\stackrel{(ii)}{\leq} (r_0 - \delta)^2 + (r + d(\mathbf{v}) - r_0)^2 - r_0 (f(r + d(\mathbf{v})) - f(r_0)). \end{aligned}$$

In (i) we used Eq. (66) to have  $\delta = f(r_0) \leq f(r + d(\mathbf{v})) \leq 2f(r) \leq 2\delta$  which implies in particular  $r_0 - f(r + d(\mathbf{v})) - \tilde{f} \geq r_0 - 3\delta \geq r_0/4$  since  $\delta \leq r_0/4$ , and in (ii) we again used  $f(r + d(\mathbf{v})) \leq 2\delta$  and  $r_0 \geq \delta$ . Now because  $f$  is convex, we have  $f(r + d(\mathbf{v})) - f(r_0) \geq f'(r_0)(r + d(\mathbf{v}) - r_0)$ . Further, by definition of  $\gamma \geq \gamma_0$ , we have  $\frac{c_f r_0}{\sqrt{1-\gamma}} \geq 1$ . As a result, we obtain  $f'(r_0) \geq \frac{c_f \delta}{\sqrt{1-\gamma}} \cdot \frac{e-1/e}{e+1/e} \geq \frac{c_f \delta}{2\sqrt{1-\gamma}}$ . Altogether, this implies

$$\begin{aligned} R(\mathbf{v}) - (r_0 - \delta)^2 &\leq (r + d(\mathbf{v}) - r_0)(r + d(\mathbf{v}) - r_0 - r_0 f'(r_0)) \\ &\stackrel{(i)}{\leq} (r + d(\mathbf{v}) - r_0)(2C_1(1 - \gamma) - 2c_f C_1 \sqrt{1 - \gamma}) \stackrel{(ii)}{\leq} 0 \end{aligned}$$

where in (i) we used  $r + d(\mathbf{v}) - r_0 \leq d(\mathbf{v}) \leq 2C_1(1 - \gamma)$  and in (ii) we used the definition of  $\gamma_0$ . This ends the proof of Eq. (68).

In both cases, we proved that Eq. (63) holds. Therefore, we have  $\mathcal{R}_s \subset \mathcal{U}_{s,\gamma}$ . This holds for all  $s \in [q]$ , hence the tensored region  $\mathcal{R}$  defined in Eq. (59) is achievable in the original setting with  $[n]$  agents:  $\mathcal{R} \subset \mathcal{U}_\gamma$ . In particular, with  $\mathbf{U}_1 := \mathbf{U}_{[n] \setminus \tilde{I}} \otimes \bigotimes_{s \in [q]} \left( \mathbf{U}_s - f(0) \frac{\boldsymbol{\alpha}_{I_s}}{\|\boldsymbol{\alpha}_{I_s}\|} \right)$ , we obtain

$$\max_{\mathbf{x} \in \mathcal{U}^*} \boldsymbol{\alpha}^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_\gamma} \boldsymbol{\alpha}^\top \mathbf{x} \leq f(0) \sum_{s \in [q]} \|\boldsymbol{\alpha}_{I_s}\| \leq \frac{4n \|\boldsymbol{\alpha}\| (1 - \gamma)}{r_0 \cosh\left(\frac{c_f r_0}{\sqrt{1 - \gamma}}\right)} = \mathcal{O}\left((1 - \gamma) e^{-c_f r_0 / \sqrt{1 - \gamma}}\right),$$

which ends the proof of the theorem.  $\blacksquare$

Theorem 30 implies in particular the second claim of Theorem 17.

## 6 Extension to the finite-horizon setting

In this section, we show how the techniques developed in previous sections for the  $\gamma$ -discounted infinite-horizon setting can be used for the finite-horizon setting. We start by proving Lemma 12 which shows that for  $T \geq 2$ , we have  $\mathcal{V}_T \subset \mathcal{U}_{\gamma(T)}$ . We recall that  $\gamma(T) = 1 - 1/T$  by definition.

**Proof of Lemma 12** For  $t \geq 2$ , we let  $\mathbf{p}^{(t)}(\cdot | \mathbf{U})$  and  $\mathbf{W}^{(t)}(\cdot | \mathbf{U})$  be allocation and promised utility functions for  $\mathbf{U} \in \mathcal{V}_t$  satisfying Eqs. (2) and (4) for  $\gamma(t)$  and Eq. (19) for  $t$ .

For  $T = 1$ , we have directly  $\mathcal{V}_1 = \mathcal{U}_0 = \mathcal{U}_{\gamma(1)}$ . For  $T \geq 2$ , note that  $\mathbf{p}^{(T)}$  and  $\mathbf{W}^{(T)}$  already satisfy almost all the equations to show that the region  $\mathcal{V}_T$  belongs to  $\mathcal{U}_{\gamma(T)}$ . The only difference is in Eq. (19) which implies that the promised utilities stay in  $\mathcal{V}_{T-1}$ . Precisely, to show  $\mathcal{V}_T \subset \mathcal{U}_{\gamma(T)}$  it suffices to show that  $\mathcal{V}_{T-1} \subset \mathcal{V}_T$ . This is also immediate if  $T = 2$  since  $\mathcal{V}_1 = \mathcal{U}_0 \subset \mathcal{V}_t$  for all  $t \geq 1$ . We now suppose that  $T \geq 3$  and that we showed  $\mathcal{V}_{T-2} \subset \mathcal{V}_{T-1}$ .

We introduce an allocation function and promised utility function on  $\mathcal{V}_{T-1}$  via

$$\mathbf{p}(\cdot | \mathbf{U}) := \mathbf{p}^{(T-1)}(\cdot | \mathbf{U}) \quad \text{and} \quad \mathbf{W}(\cdot | \mathbf{U}) := \frac{T-2}{T-1} \mathbf{W}^{(T-1)}(\cdot | \mathbf{U}) + \frac{1}{T-1} \mathbf{U}, \quad \mathbf{U} \in \mathcal{V}_{T-1}.$$

We now check that this is a valid allocation strategy. Indeed, with  $\mathbf{U} \in \mathcal{V}_{T-1}$ , for all  $i \in [n]$ ,

$$\begin{aligned} & (1 - \gamma(T)) \mathbb{E}[u_i p_i(\mathbf{u})] + \gamma(T) \mathbb{E}[\mathbf{W}(\mathbf{u} | \mathbf{U})] \\ &= \frac{\mathbf{U}}{T} + \frac{T-1}{T} \left( (1 - \gamma(T-1)) \mathbb{E}[u_i p_i^{(T-1)}(\mathbf{u})] + \gamma(T-1) \mathbb{E}[\mathbf{W}^{(T-1)}(\mathbf{u} | \mathbf{U})] \right) = \mathbf{U}. \end{aligned}$$

Hence Eq. (2) holds. Here, in the last inequality we used the fact that  $\mathbf{p}^{(T-1)}$  and  $\mathbf{W}^{(T-1)}$  satisfy Eq. (2) for  $\gamma = \gamma(T-1)$ . Similarly, for any  $i \in [n]$  and  $u, v \in [0, \bar{v}]$ ,

$$\begin{aligned} & (1 - \gamma(T)) u P_i(v | \mathbf{U}) + \gamma(T) W_i(v | \mathbf{U}) \\ &= \frac{U_i}{T} + \frac{T-1}{T} \left( (1 - \gamma(T-1)) u P_i^{(T-1)}(v | \mathbf{U}) + \gamma(T-1) W_i^{(T-1)}(v | \mathbf{U}) \right), \end{aligned}$$

hence the incentive-compatibility Eq. (4) also holds from that of  $\mathbf{p}^{(T-1)}$  and  $\mathbf{W}^{(T-1)}$ . Last, for any  $\mathbf{u} \in [0, \bar{v}]^n$ , we have  $\mathbf{W}^{(T-1)}(\mathbf{u} | \mathbf{U}) \in \mathcal{V}_{T-2} \subset \mathcal{V}_{T-1}$  and  $\mathbf{U} \in \mathcal{V}_{T-1}$ . Because  $\mathcal{V}_{T-1}$  is convex, this implies  $\mathbf{W}(\mathbf{u} | \mathbf{U}) \in \mathcal{V}_{T-1}$ . That is, Eq. (19) also holds, which ends the inductive proof that  $\mathcal{V}_{T-1} \subset \mathcal{V}_T$ . This proves the desired result.  $\blacksquare$

As a consequence of Lemma 12, all techniques developed earlier to give upper bounds on  $\mathcal{U}_{\gamma(T)}$  also apply to  $\mathcal{V}_T$ . We next show the simple universal  $1/T$  lower bound for the finite-horizon setting.

**Proof of Lemma 13** Let  $\mathbf{U} \in \mathcal{V}_T$  and consider an allocation strategy that realizes  $\mathbf{U}$ . For any  $t \in [T]$ , we recall the notation  $\mathbf{p}(t) \in \Delta_n$  and  $\mathbf{u}(t)$  for the allocation distribution and the agent utilities respectively at time  $t$ . The main remark is that the  $\alpha$ -suboptimality of  $\mathbf{U}$  is at least that of the first allocation  $\mathbf{p}(1)$  and this corresponds to an allocation from  $\mathcal{V}_1$ :

$$\begin{aligned} \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \alpha^\top \mathbf{U} &= \mathbb{E} \left[ \frac{1}{T} \sum_{t \in [T]} \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \alpha^\top (p_i(t) u_i(t))_{i \in [n]} \right] \\ &\geq \frac{1}{T} \mathbb{E} \left[ \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \alpha^\top (p_i(1) u_i(1))_{i \in [n]} \right] \\ &\stackrel{(i)}{\geq} \frac{1}{T} \left( \max_{\mathbf{x} \in \mathcal{U}^*} \alpha^\top \mathbf{x} - \max_{\mathbf{x} \in \mathcal{U}_0} \alpha^\top \mathbf{x} \right). \end{aligned}$$

In (i) we used the fact that  $(\mathbb{E}[p_i(1)u_i(1)])_{i \in [n]} \in \mathcal{V}_1 = \mathcal{U}_0$ . By assumption, the term in the bracket in the last inequality is nonzero, otherwise  $\mathcal{U}_0$  would contain an  $\alpha$ -optimal vector. This ends the proof.  $\blacksquare$

We next turn to the lower bound tools on  $\mathcal{U}_\gamma$  and show that they still hold in the finite-horizon  $T$  setting up to logarithmic factors. We start by showing that Lemma 3 still holds. The proof is constructive and proceeds by giving a trajectory of achievable regions for all times in  $t \in [T]$ . This process is illustrated in Fig. 6, which shows how one can approximate balls within  $\mathcal{U}^*$  by constructing balls within  $\mathcal{V}_t$  converging to the original ball as  $t \in [T]$  grows.

**Lemma 31.** *There exists a constant  $c_0 \geq 8$  and  $r_0 > 0$  (depending on the utility distributions) such that the following holds. Fix  $T \geq 2$ ,  $\mathbf{x} \in \mathcal{U}^*$ , and  $r, \delta > 0$  such that  $B(\mathbf{x}, r + \delta) \subset \mathcal{U}^*$  and*

$$r_0 \geq r \geq \delta \geq c_0 C \frac{1 - \gamma(T)}{\gamma(T)r} \left( 1 + \ln \frac{r}{\delta} \right), \quad (69)$$

where  $C$  is as defined in Lemma 3. Then, we have  $B(\mathbf{x}, r) \subset \mathcal{V}_T$ .

Alternatively, suppose that the assumptions of Lemma 3 are satisfied with  $\gamma = \tilde{\gamma}(T) = 1 - \ln T/T$  and the constant  $C' = c_0 C$ , as well as  $r_0 \geq r \geq \delta$ . Then, we have  $B(\mathbf{x}, r) \subset \mathcal{V}_T$ .

**Proof** Let  $\mathbf{x}$  and  $r, \delta > 0$  satisfying the requirements. The assumptions imply that  $0 < x_i \leq \mathbb{E}[u_i]$  for all  $i \in [n]$ . Now let  $r_0 := \min_{i \in [n]} \mathbb{E}[u_i]/(6n)$  and let  $\mathbf{x}^{(0)} := 3r_0 \mathbf{1}$ . We can note that

$$B(\mathbf{x}^{(0)}, 3r_0) \subset [0, 6r_0]^n \subset \{\mathbf{y} : \mathbf{0} \leq \mathbf{y} \leq (\mathbb{E}[u_i]/n)_{i \in [n]}\} \subset \mathcal{U}_0.$$

Also, up to renaming the constant  $r_0$  to  $r_0/2$ , the assumption gives  $r + \delta \leq 2r \leq r_0$ . We next introduce the constant  $\tilde{c}_0 := (1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{r_0})^2$ .

**Step 1.** We start with the case in which the assumption that Eq. (13) holds with the same constant  $\tilde{C} := \tilde{c}_0 C$ , where  $C$  is the same constant as in the original statement of Lemma 3, and we have  $\delta \leq r \leq r_0/2$ . We will later see that from there we can easily extend to the general case.

We define the set  $\mathcal{T} = \left\{ t \in \{2, \dots, T\}, r \geq \sqrt{\frac{\tilde{C}(1-\gamma(t))}{\gamma(t)}} \right\}$ . Note that by hypothesis, we have  $2r \geq r + \delta \geq r + \tilde{C} \frac{1-\gamma(T)}{\gamma(T)r} \geq 2\sqrt{\frac{\tilde{C}(1-\gamma(T))}{\gamma(T)}}$ . Thus,  $T \in \mathcal{T}$ . Because  $\frac{1-\gamma(t)}{\gamma(t)}$  is decreasing in  $t$ , there



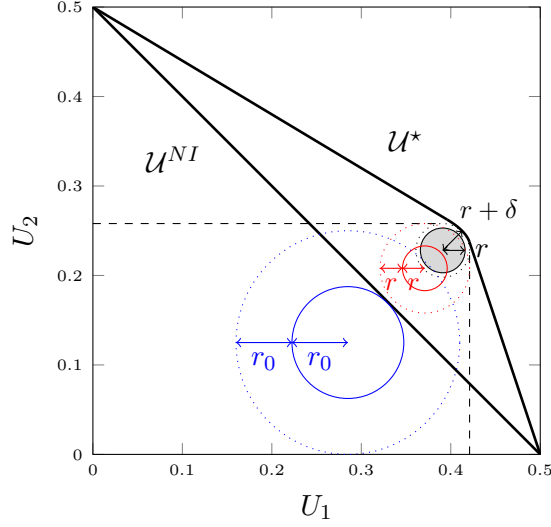


Figure 6: Illustration of the mechanism to approach the gray radius- $r$  ball in the finite-horizon  $T$  setting (see Lemma 31). For all  $t \in [T]$  we construct a ball within  $\mathcal{V}_t$ . The mechanism has two phases. For  $t = 1$ , we can take the blue radius- $r_0$  ball within  $\mathcal{U}^{NI} \subset \mathcal{V}_1$ . We first start by reducing the radius to  $r$  until we reach the red ball. As  $t$  grows, the radius reduces as  $r_t = \sqrt{C \frac{1-\gamma(t)}{\gamma(t)}} = \sqrt{\frac{C}{t-1}}$ , which requires conserving a margin  $\delta_t = r_t$  depicted in the blue and red dotted circles. The second phase is to decrease the margin from  $\delta_t = r$  (as shown in the red dotted circle) towards the desired margin  $\delta$ , all while staying within the dashed region which belongs to  $\mathcal{U}^*$ . In this phase, the margin decreases as  $\delta_t = C \frac{1-\gamma(t)}{\gamma(t)r}$ .

exists  $T_1 \in [T]$  such that  $\mathcal{T} = \{T_1, T_1 + 1, \dots, T\}$ . We next let

$$T_0 = \max\{1\} \cup \left\{ t \in [T], 2\sqrt{\frac{\tilde{C}(1-\gamma(t))}{\gamma(t)}} \geq r_0 \right\}.$$

We note that for  $T_1$ , we have

$$2\sqrt{\frac{\tilde{C}(1-\gamma(T_1))}{\gamma(T_1)}} < 2r \leq r_0,$$

Therefore,  $T_0 < T_1$  (by construction  $T_1 \geq 2$ ). We now specify our allocation mechanism.

**Definition of the allocation strategy.** We use a trivial allocation for times in  $[T_0]$ : we will only use the fact that  $B(\mathbf{x}^{(0)}, 3r_0) \subset \mathcal{U}_0 \subset \mathcal{V}_{T_0}$ . For times in  $(T_0, T]$ , we let

$$r_t := \begin{cases} \sqrt{\frac{\tilde{C}(1-\gamma(t))}{\gamma(t)}} = \sqrt{\frac{\tilde{C}}{t-1}} & t \in \{T_0, \dots, T_1 - 1\} \\ r & t \in \{T_1, \dots, T\}. \end{cases}$$

To prove the desired result, we inductively prove that  $B(\mathbf{x}^{(t)}, r_t) \subset \mathcal{V}_t$  for appropriately chosen parameters  $\mathbf{x}^{(t)}$ . Precisely, we first introduce the parameter

$$\delta_t := \begin{cases} r_t & t \in \{T_0 + 1, \dots, T_1 - 1\} \\ \max\left(\tilde{C} \frac{1-\gamma(t)}{\gamma(t)r_t}, \delta\right) & t \in \{T_1, \dots, T\}. \end{cases}$$

We pose  $\mathbf{x}^{(t)} := \mathbf{x}^{(0)}$  for all  $t \in [T_0]$  and for  $t > T_0$  we introduce

$$\mathbf{y}^{(t)} := \mathbf{x} + \frac{r_t + \delta_t - (r + \delta)}{2r_0 - (r + \delta)}(\mathbf{x}^{(0)} - \mathbf{x}).$$

We note that by construction,  $r_t + \delta_t \geq r + \delta$ : this is immediate for  $t \in [T_1, T]$  and for  $t \in (T_0, T_1)$  since  $t < T_1$  one has  $r_t + \delta_t = 2r_t > 2r \geq r + \delta$ . We then construct iteratively the sequence of vectors  $\mathbf{x}^{(t)}$  via

$$\mathbf{x}^{(t)} := \gamma(t)\mathbf{x}^{(t-1)} + (1 - \gamma(t))\mathbf{y}^{(t)}, \quad t \in \{T_0 + 1, \dots, T\}. \quad (70)$$

To prove that  $B(\mathbf{x}^{(t)}, r_t) \subset \mathcal{V}_t$ , we construct an allocation mechanism on the region

$$\mathcal{R}_t := \{\mathbf{y} : \mathbf{0} \leq \mathbf{y} \leq \mathbf{z}, \mathbf{z} \in \text{Conv}(\mathcal{U}^{NI}, B(\mathbf{x}^{(t)}, r_t))\}.$$

Next, by construction, note that

$$B(\mathbf{y}^{(t)}, r_t + \delta_t) = \frac{r_t + \delta_t - (r + \delta)}{2r_0 - (r + \delta)}B(\mathbf{x}^{(0)}, 2r_0) + \frac{2r_0 - (r_t + \delta_t)}{2r_0 - (r + \delta)}B(\mathbf{x}, r + \delta). \quad (71)$$

By convexity of  $\mathcal{U}^*$ , since we have  $B(\mathbf{x}^{(0)}, 2r_0), B(\mathbf{x}, r + \delta) \subset \mathcal{U}^*$  this also implies that  $B(\mathbf{y}^{(t)}, r_t + \delta_t) \subset \mathcal{U}^*$ . This is precisely the reason behind the definition of  $\mathbf{y}^{(t)}$ .

We now construct an allocation function  $\mathbf{p}^{(t)}(\cdot | \mathbf{U})$  and a promised utility function  $\mathbf{W}^{(t)}(\cdot | \mathbf{U})$  for  $\mathbf{U} \in \mathcal{R}_t$  in a similar fashion to those constructed in the proof of Lemma 3. We first specify the allocation strategy when  $\mathbf{U}$  is an extreme point of  $\mathcal{R}_t$  that still belongs to  $B(\mathbf{x}^{(t)}, r_t)$ . We write  $\mathbf{U} = \mathbf{x}^{(t)} + r_t\mathbf{y}$  where  $\mathbf{y} \in S_{n-1}$  and let

$$\mathbf{p}^{(t)}(\cdot | \mathbf{U}) := \mathbf{p}(\cdot; \mathbf{y}^{(t)} + (r_t + \delta_t)\mathbf{y}) \quad \text{and} \quad \boldsymbol{\alpha}(\mathbf{U}) = \mathbf{x}^{(t)} - \mathbf{U} = -r_t\mathbf{y}, \quad t \in (T_0, T], \quad (72)$$

where  $\mathbf{p}(\cdot; \mathbf{z})$  refers to an allocation that realizes  $\mathbf{z}$  in the full information setting (we checked earlier that we always used an allocation  $\mathbf{p}(\cdot; \mathbf{z})$  with  $\mathbf{z} \in \mathcal{U}^*$ ). We then extend the definition of the allocation mechanism to the complete region  $\mathcal{R}_t$  exactly as in the original proof using Eqs. (25) and (26).

**Checking that the allocation strategy is valid.** We now check that this is a valid allocation. Using the same linearity arguments as in the proof of Lemma 3, it suffices to show that for extreme points  $\mathbf{U}$  of  $\mathcal{R}_t$  within  $B(\mathbf{x}^{(t)}, r_t)$ , the promised utilities lie in  $B(\mathbf{x}^{(t-1)}, r_{t-1})$  to show that for all  $\mathbf{U} \in \mathcal{R}_t$ , Eq. (7) is satisfied, and

$$\forall \mathbf{v} \in [0, \bar{v}]^n, \quad \mathbf{W}^{(t)}(\mathbf{v} | \mathbf{U}) \in \mathcal{R}_{t-1}. \quad (73)$$

We now fix such an extreme point  $\mathbf{U} = \mathbf{x}^{(t)} + r_t\mathbf{y}$  where  $\mathbf{y} \in S_{n-1}$ . First, recall that  $B(\mathbf{y}^{(t)}, r_t + \delta_t) \subset \mathcal{U}^*$ . We now check that Eq. (13) is also satisfied. Note that by construction the sequences  $r_t$  and  $\delta_t$  are both non-increasing on  $(T_0, T]$ . This is clear by definition of  $T_1$  for

the sequence  $(r_t)_t$ . For  $\delta_t$ , we only need to check that  $\delta_{T_1-1} = r_{T_1-1} \geq \delta_{T_1}$ . To do so, we use  $\delta_{T_1} \leq r_{T_1} \leq r_{T_1-1}$ . Now by Eq. (71), we have

$$y_i^{(t)} + r_t + \delta_t \leq \max(x_i^{(0)} + 2r_0, x_i + r + \delta).$$

Now by hypothesis Eq. (13) is satisfied for the parameters  $\mathbf{x}, r, \delta$  and  $\gamma = \gamma(T)$ , and the constant  $\tilde{C}$  instead of  $C$ . Hence,  $\delta \geq \tilde{C} \frac{1-\gamma(T)}{\gamma(T)r}$  which implies  $\delta_T = \delta$ . Hence, because  $\delta_t$  is non-increasing in  $t$ , we have  $\delta_t \leq \delta$  for all  $t \in [T_0, T_1]$ , which implies

$$y_i^{(t)} + r_t \leq \max(x_i^{(0)} + 2r_0, x_i + r) \leq \max(\mathbb{E}[u_i] - r_0, x_i + r),$$

where in the last inequality we used the fact that  $B(\mathbf{x}^{(0)}, 3r_0) \subset \mathcal{U}_0$ . The previous remarks imply that

$$\tilde{C} = \tilde{c}_0 4n^2 \bar{v}^2 \left( 1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{i \in [n]} (\mathbb{E}[u_i] - x_i - r)} \right)^2 \geq 4n^2 \bar{v}^2 \left( 1 + \frac{\max_{i \in [n]} \mathbb{E}[u_i]}{\min_{i \in [n]} (\mathbb{E}[u_i] - y_i^{(t)} - r_t)} \right)^2.$$

Last, we check that

$$\frac{\gamma(t)r_t}{1-\gamma(t)} \stackrel{(i)}{\geq} r_t \geq \delta_t \geq \tilde{C} \frac{1-\gamma(t)}{\gamma(t)r_t}.$$

where in (i) we used the fact that  $t \geq T_0 + 1 \geq 2$ , hence  $\gamma(t) \geq 1/2$ . Altogether, the previous arguments show that Eq. (13) is satisfied for the parameters  $\mathbf{y}^{(t)}, r_t, \delta_t$  and  $\gamma = \gamma(t)$ .

Next, compared to the definition of Lemma 3, these would have been identical if the allocation was for  $\mathbf{U}' := \mathbf{y}^{(t)} + r_t \mathbf{y}$  instead of  $\mathbf{U} = \mathbf{x}^{(t)} + r_t \mathbf{y}$ . To clarify this comparison, we introduce the (fictive) allocation function  $\mathbf{p}'(\cdot | \mathbf{U}') := \mathbf{p}(\cdot; \mathbf{U})$  and denote by  $\mathbf{W}'(\cdot | \mathbf{U}')$  the promised utility function resulting from this choice of allocation function and the same direction choice  $\boldsymbol{\alpha}(\mathbf{U})$  as in Eq. (72). Having checked that all conditions apply, the proof of Lemma 3 implies that

$$\forall \mathbf{v} \in [0, \bar{v}]^n, \quad \mathbf{W}'(\mathbf{v} | \mathbf{U}') \in B(\mathbf{y}^{(t)}, r_t).$$

Now by definition of the promised utility functions (see Eqs. (6) and (11)), we have for any  $\mathbf{v} \in [0, \bar{v}]^n$ ,

$$\mathbf{W}^{(t)}(\mathbf{v} | \mathbf{U}) = \mathbf{W}'(\mathbf{v} | \mathbf{U}') + \frac{\mathbf{U} - \mathbf{U}'}{\gamma(t)} = \mathbf{W}'(\mathbf{v} | \mathbf{U}') - \frac{\mathbf{y}^{(t)} - \mathbf{x}^{(t)}}{\gamma(t)}.$$

As a result, using  $r_t \leq r_{t-1}$ , we obtain

$$\mathbf{W}^{(t)}(\mathbf{v} | \mathbf{U}) \in B\left(\mathbf{y}^{(t)} + \frac{\mathbf{x}^{(t)} - \mathbf{y}^{(t)}}{\gamma(t)}, r_t\right) = B(\mathbf{x}^{(t-1)}, r_t) \subset B(\mathbf{x}^{(t-1)}, r_{t-1}).$$

This ends the proof of Eq. (73) hence  $\mathcal{R}_t \subset \mathcal{V}_t$ . As a summary, we obtained the induction: if  $B(\mathbf{x}^{(t-1)}, r_{t-1}) \subset \mathcal{V}_{t-1}$ , then  $B(\mathbf{x}^{(t)}, r_t) \subset \mathcal{V}_t$  where  $\mathbf{x}^{(t)}$  is defined in Eq. (70). We recall that the initialization at  $t = T_0$  is immediate from  $B(\mathbf{x}^{(0)}, r_0) \subset \mathcal{U}_0 \subset \mathcal{V}_{T_0}$ .

**Putting the recursive construction together.** Using the induction formula Eq. (70), we can check that  $\mathbf{x}^{(t)} = a_t \mathbf{x}^{(0)} + b_t (\mathbf{x}^{(0)} - \mathbf{x}) + c_t \mathbf{x}$  for  $t \in [T_0, T]$  where  $a_{T_0} = 1$ ,  $b_{T_0} = c_{T_0} = 0$  and

$$\begin{cases} a_t = \gamma(t) a_{t-1} \\ b_t = \gamma(t) b_{t-1} + (1 - \gamma(t)) \frac{r_t + \delta_t - (r + \delta)}{2r_0 - (r + \delta)} \\ c_t = \gamma(t) c_{t-1} + (1 - \gamma(t)), \end{cases} \quad t \in [T_0 + 1, \dots, T_1 - 1].$$

The recursion for  $a_t$  and  $c_t$  is rather simple, using the convexity inequality  $\ln(1 - x) \leq -x$  for  $x \leq 1$ , we obtain

$$0 \leq a_T = 1 - c_T = \prod_{t=T_0+1}^T \gamma(t) \leq e^{-\sum_{t=T_0+1}^T \frac{1}{t}} \leq \frac{T_0 + 1}{T + 1}. \quad (74)$$

The recursion for  $b_t$  requires more work. First, recall that  $r + \delta \leq r_0$  and  $r_t + \delta_t \geq r + \delta$ . Therefore, if we define the inductive sequence  $\tilde{b}_{T_0} := 0$  and

$$\tilde{b}_t = \gamma(t) \tilde{b}_{t-1} + (1 - \gamma(t)) \frac{r_t + \delta_t - (r + \delta)}{r_0},$$

we have  $0 \leq b_t \leq \tilde{b}_t$  for all  $t \in [T_0, T]$ . Note that the induction can be rewritten as follows,

$$r_0 t \tilde{b}_t = r_0 (t - 1) \tilde{b}_{t-1} + r_t + \delta_t - (r + \delta).$$

As a result, letting  $T_2 = \min\{t \geq T_1, \delta_t = \delta\}$ , we obtain

$$\begin{aligned} r_0 T \tilde{b}_T &= \sum_{t=T_0+1}^T (r_t + \delta_t - r - \delta) \\ &= 2 \sum_{t=T_0+1}^{T_1-1} r_t - (T_1 - T_0 - 1)(r + \delta) + \sum_{t=T_1}^{T_2-1} \delta_t - (T_2 - T_1)\delta \\ &\stackrel{(i)}{\leq} 4\sqrt{\tilde{C}(T_1 - 2)} + \frac{\tilde{C}}{r} \left(1 + \ln \frac{T_2 - 2}{T_1 - 1} \mathbb{1}_{T_2 > T_1}\right), \end{aligned}$$

where in (i) we used  $r_t = \sqrt{\tilde{C}/(t - 1)}$  for  $t \in (T_0, T_1)$  and  $\delta_t = \tilde{C}/((t - 1)r)$  for  $t \in [T_1, T_2)$ , and used sum-integral inequalities. Now by definition of  $T_1$  and  $T_2$ , we have  $\sqrt{\frac{\tilde{C}}{T_1 - 2}} > r \geq \sqrt{\frac{\tilde{C}}{T_1 - 1}}$  and if  $T_2 > T_1$ , we have  $\delta < \tilde{C} \frac{1 - \gamma(T_2 - 1)}{\gamma(T_2 - 1)r} = \frac{\tilde{C}}{r(T_2 - 2)}$ . As a result, we obtained

$$0 \leq b_T \leq \frac{4\sqrt{\tilde{C}(T_1 - 2)}}{r_0 T} + \frac{\tilde{C}}{r_0 r T} \left(1 + \ln \frac{r}{\delta}\right) \leq \frac{\tilde{C}}{r_0 r T} \left(5 + \ln \frac{r}{\delta}\right). \quad (75)$$

**Step 2.** We now consider the original case in which  $B(\mathbf{x}, r + \delta) \subset \mathcal{U}^*$ ,  $\delta \leq r \leq r_0/2$ , and Eq. (69) holds with the constant  $c_0 = 8\tilde{c}_0 + 20\tilde{c}_0/r_0 + 4(T_0 + 1)r_0/C$ . We denote by  $C$  the term in the statement of Lemma 3 for parameters  $\mathbf{x}$  and  $r$ . Fix any  $\mathbf{z} \in B(\mathbf{x}, \delta/2)$ . Then, with  $\tilde{\delta} = \delta/2$ , we still have  $B(\mathbf{z}, r + \tilde{\delta}) \subset \mathcal{U}^*$ , and

$$r \geq \tilde{\delta} \geq \frac{c_0 C}{2} \frac{1 - \gamma(T)}{\gamma(T)r} \geq 4\tilde{c}_0 C \frac{1 - \gamma(T)}{\gamma(T)r}.$$

Note that the constant corresponding to the parameters  $\mathbf{z}, r$  in the statement of Lemma 3 is at most  $4C$  (note that  $\|\mathbf{z} - \mathbf{x}\| \leq \delta/2$  and since  $B(\mathbf{x}, r + \delta) \subset \mathcal{U}^*$ , this still leaves a margin at least  $\delta/2$  to the boundary of  $\mathcal{U}^*$  for  $B(\mathbf{z}, r)$ ). Therefore, the previous step shows that  $B(\mathbf{z}^{(T)}, r) \subset \mathcal{V}_T$  where

$$\mathbf{z}^{(T)} = a_T \mathbf{x}^{(0)} + b_T (\mathbf{x}^{(0)} - \mathbf{x}) + c_T \mathbf{z}.$$

Because this holds for all  $\mathbf{z} \in B(\mathbf{x}, \delta/2)$ , we obtained

$$B\left(a_T \mathbf{x}^{(0)} - b_T \mathbf{1} + c_T \mathbf{x}, r + \frac{\delta}{2}\right) \subset \mathcal{V}_T. \quad (76)$$

Now using Eqs. (74) and (75), we have

$$\begin{aligned} \|a_T \mathbf{x}^{(0)} - b_T \mathbf{1} + c_T \mathbf{x} - \mathbf{x}\| &\leq \frac{T_0 + 1}{T + 1} \|\mathbf{x} - \mathbf{x}^{(0)}\| + \frac{\tilde{C}(5 + \ln \frac{r}{\delta})}{r_0 r T} \|\mathbf{x} - \mathbf{x}^{(0)}\| \\ &\leq \frac{(T_0 + 1)r_0^2 + \tilde{C}(5 + \ln \frac{r}{\delta})}{r_0 r T} \sqrt{n} \max_{i \in [n]} \mathbb{E}[u_i] \end{aligned}$$

As a result, recalling the definition of  $c_0$ , we obtain

$$\|a_T \mathbf{x}^{(0)} - b_T \mathbf{1} + c_T \mathbf{x} - \mathbf{x}\| \leq \frac{c_0 C (1 + \ln \frac{r}{\delta})}{2r(T + 1)} \leq \frac{\delta}{2}.$$

Together with Eq. (76) this implies the desired result  $B(\mathbf{x}, r) \subset \mathcal{V}_T$ . This ends the proof of the first claim.

For the second claim, we simply note that by assumption, we have  $r \geq \delta \geq \frac{c_0 C}{r(T-1)}$ . In particular,  $\frac{r}{\delta} \leq \frac{r^2}{c_0 C} (T-1) \leq \frac{r_0^2}{c_0 C} T$ . Hence, up to modifying the constant  $c_0$ , the result holds if we replace the lower bound from Eq. (69) with  $c_0 C \frac{1 - \tilde{\gamma}(T)}{\tilde{\gamma}(T)}$  where  $\tilde{\gamma}(T) := 1 - (1 + \ln T)/T$ . ■

As a remark, note that when the margin  $\delta$  is of the same order as  $r$  within Lemma 31, then the logarithmic term  $\ln \frac{r}{\delta}$  is a constant, which essentially means that we can use the same techniques to prove lower bounds on  $\mathcal{V}_T$  as for  $\mathcal{U}_{\gamma(T)} = \mathcal{U}_{1-1/T}$ , up to worsening constants. This is significant since from Lemma 12, we also have  $\mathcal{V}_T \subset \mathcal{U}_{\gamma(T)}$ : in these cases, characterizing  $\mathcal{V}_T$  or  $\mathcal{U}_{\gamma(T)}$  with our tools is essentially equivalent.

On the other hand, as per the second claim of Lemma 31, the extra factor  $\ln \frac{r}{\delta}$  can be of order  $\ln T$ , which potentially induces additional logarithmic factors to translate results from the infinite-horizon setting to the finite-horizon setting.

Similarly as for Lemma 3, the optimized version Lemma 21 also holds in the finite-horizon setting up to worsening the constants.

**Lemma 32.** *Fix  $T \geq 2$  and  $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . There exist constants  $c_0 \geq 8$  and  $r_0 > 0$  such that the following holds. Suppose that all assumptions from Lemma 21 are satisfied with the exception that Eq. (13) is replaced with Eq. (69). Then, using the same notations,*

$$B(\mathbf{x}, r) \cap \{\mathbf{y} : \forall i \notin \tilde{I}, y_i = x_i\} \subseteq \mathbf{x} + \{0\}^{[n] \setminus \tilde{I}} \otimes \bigotimes_{s \in [q]} B_{I_s}(\mathbf{0}, r) \subseteq \mathcal{V}_T.$$

*Alternatively, suppose that all assumptions from Lemma 21 are satisfied with  $\gamma = \tilde{\gamma}(T) = 1 - \ln T/T$  and the constant  $C' = c_0 C$ , as well as  $r_0 \geq r \geq \delta$ . Then the previous conclusion also holds.*

**Proof** The same proof as in Lemma 21 carries to the finite-horizon setting: we can construct an allocation mechanism that treats separately agents by clusters  $I_1, \dots, I_q$  as described in the proof. The only difference is that instead of applying Lemma 3 we apply Lemma 31, which required exactly the assumptions made within Lemma 32. ■

As a conclusion, up to logarithmic factors, all lower bounds developed in the previous sections apply to the finite-horizon setting choosing  $\gamma = \gamma(T) = 1 - 1/T$ .

We now prove Corollaries 16 and 18 as examples of how the results from the  $\gamma$ -discounted setting can help for the finite-horizon setting.

**Proof of Corollary 16** It suffices to prove the result for  $T \geq 2$ . The lower bound is directly taken from Lemma 13 and the upper bound is a consequence of the second claim of Corollary 15 together with Lemma 32. ■

**Proof of Corollary 18** It suffices to prove the result for  $T \geq 2$ . The first claim is a consequence of the first result from Corollary 29 together with Lemma 12 for the lower bound and Lemma 31 for the upper bound. Note that there is no extra  $\ln T$  factor (we can take directly  $\gamma = \gamma(T) = 1 - 1/T$ ) in the upper bound because the proof of the convergence rate  $\mathcal{O}(\sqrt{1 - \gamma})$  comes from Theorem 27, which only uses Lemma 3 with parameters  $r = \delta$ . As a result, the extra term  $\ln \frac{r}{\delta}$  from Lemma 31 vanishes.

The second claim is a consequence of the second claim of Corollary 29 together with Lemma 32 for the upper bound (note that we do suffer a factor  $\ln T$  here). The lower bound is directly Lemma 13. ■

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## A Proofs of the facts from Section 2

First, we check that the utility regions are convex.

**Proof of Fact 1** Let  $\theta \in [0, 1]$ . For any resource allocation setting, given two allocation strategies  $\mathbf{S}_1$  and  $\mathbf{S}_2$  realizing utilities  $\mathbf{U}_1$  and  $\mathbf{U}_2$  respectively, the allocation which first samples a Bernoulli  $B \sim \mathcal{B}(\theta)$ , then implements  $\mathbf{S}_1$  if  $B = 1$  and  $\mathbf{S}_2$  otherwise is a valid allocation and realizes the utilities  $\theta\mathbf{U}_1 + (1 - \theta)\mathbf{U}_2$ . ■

We next prove Fact 2 which gives necessary and sufficient conditions for a utility region  $\mathcal{R}$  to be achievable in the  $\gamma$ -discounted setting.

**Proof of Fact 2** We start with showing that the condition is sufficient to show  $\mathcal{R}' \subset \mathcal{R} \subset \mathcal{U}_\gamma$ . To do so, we use the allocation function  $\mathbf{p}(\cdot | \mathbf{U})$  for  $\mathbf{U} \in \mathcal{R}$  as the strategy at all times. Precisely, fix an arbitrary vector  $\mathbf{U}_0 \in \mathcal{R}$ . Suppose we have constructed  $\mathbf{U}_{t-1}$  for  $t \geq 1$ , recalling the notation  $\mathbf{v}(t)$  for the agent reports at time  $t$ , we pose

$$\mathbf{p}(t) := \mathbf{p}(\mathbf{v}(t) | \mathbf{U}_{t-1}) \quad \text{and} \quad \mathbf{U}_t := \mathbf{W}(\mathbf{v}(t) | \mathbf{U}_{t-1}).$$

It is well defined since by induction  $\mathbf{U}_t \in \mathcal{R}$  for all  $t \geq 0$  from Eq (3). We denote by  $\mathbf{S}$  the constructed strategy. We now check that truthful reporting for all agents  $\boldsymbol{\sigma} = \mathbf{truth}$  forms a perfect Bayesian equilibrium.

To do so, we check that for all  $i \in [n]$ , any time  $t \geq 0$ , the expected remaining utility of agent  $i$  by reporting  $v$  at time  $t$  is exactly  $W_i(v | \mathbf{U}_{t-1})$ . Indeed, for any strategy  $\sigma_i$  for agent  $i$ , we have

$$\begin{aligned} & V_{i,t}(\mathbf{S}, \mathbf{truth}_{-i}, \sigma_i | u_i(t), \mathbf{h}(t)) \\ &= \sum_{t' \geq t} (1 - \gamma) \mathbb{E} \left[ \gamma^{t'-t} u_i(t') \mathbb{E}_{\mathbf{u}_{-i}(t')} [p_i(\mathbf{u}_{-i}(t'), v_i(t') | \mathbf{U}_{t'-1})] | u_i(t), \mathbf{h}(t) \right] \\ &= \sum_{t' \geq t} (1 - \gamma) \mathbb{E} \left[ \gamma^{t'-t} u_i(t') P_i(v_i(t') | \mathbf{U}_{t'-1}) | u_i(t), \mathbf{U}_{t-1} \right] \\ &\stackrel{(i)}{\leq} \sum_{t' \geq t} \gamma^{t'-t} \mathbb{E} [(1 - \gamma) u_i(t') P_i(u_i(t') | \mathbf{U}_{t'-1}) + \gamma W_i(u_i(t') | \mathbf{U}_{t'-1}) \\ &\quad - \gamma W_i(v_i(t') | \mathbf{U}_{t'-1}) | u_i(t), \mathbf{U}_{t-1}] \\ &\stackrel{(ii)}{=} (1 - \gamma) u_i(t) P_i(u_i(t) | \mathbf{U}_{t-1}) + \gamma (W_i(u_i(t) | \mathbf{U}_{t-1}) - W_i(v_i(t) | \mathbf{U}_{t-1})) \\ &\quad + \sum_{t' > t} \gamma^{t'-t} \mathbb{E} [U_{t'-1,i} - \gamma W_i(v_i(t') | \mathbf{U}_{t'-1}) | u_i(t), \mathbf{U}_{t-1}] \end{aligned}$$

In (i) we applied the incentive-compatibility equation Eq. (4), and in (ii) we took the expectation over  $u_i(t') \sim \mathcal{D}_i$  within each expectation and used Eq. (2). Now note that by construction, for  $t' \geq 1$  we have

$$W_i(v_i(t') | \mathbf{U}_{t'-1}) = \mathbb{E}_{\mathbf{u}_{-i}(t')} [W_i(\mathbf{u}_{-i}(t'), v_i(t') | \mathbf{U}_{t'-1})] = \mathbb{E}_{\mathbf{u}_{-i}(t')} [U_{t',i}].$$

As a result, with a telescoping argument, we obtain

$$V_{i,t}(\mathbf{S}, \mathbf{truth}_{-i}, \sigma_i | u_i(t), \mathbf{h}(t)) \leq (1 - \gamma) u_i(t) P_i(u_i(t) | \mathbf{U}_{t-1}) + \gamma (W_i(u_i(t) | \mathbf{U}_{t-1})),$$

where the only inequality used is that of (i), which is an equality for  $\sigma_i = \mathbf{truth}$ , that is when agent  $i$  reports truthfully. This exactly shows that Eq. (PBE) holds. We can now check that under truthful reporting,

$$\begin{aligned} V_i(\mathbf{S}, \mathbf{truth}) &= \mathbb{E}[V_i, 1(\mathbf{S}, \mathbf{truth} \mid u_i(1))] \\ &\stackrel{(i)}{=} \mathbb{E}[(1 - \gamma)u_i(1)P_i(u_i(1) \mid \mathbf{U}_0) + \gamma(W_i(u_i(1) \mid \mathbf{U}_0))] \stackrel{(ii)}{=} U_{0,i}, \end{aligned}$$

where in (i) we used the previous computations and in (ii) we used Eq. (2). Therefore we checked that  $\mathbf{U}_0 \in \mathcal{U}_\gamma$  for any arbitrary  $\mathbf{U}_0 \in \mathcal{R}$ .

We now prove that it is necessary. Let  $\mathcal{R}$  be an achievable region, that is  $\mathcal{R} \subset \mathcal{U}_\gamma$ . By definition of  $\mathcal{U}_\gamma$ , for all  $\mathbf{U} \in \mathcal{U}_\gamma$  there is a strategy  $\mathbf{S}(\mathbf{U})$  that realizes  $\mathbf{U}$ . By the revelation principle, without loss of generality, truthful reporting is a perfect Bayesian equilibrium. We define  $\mathbf{p}(\mathbf{v} \mid \mathbf{U}) := \mathbf{p}(1)$  where  $\mathbf{p}(1)$  is the first allocation of the strategy  $\mathbf{S}(\mathbf{U})$  having received reports  $\mathbf{v}$ . Let  $\tilde{u}_i(t) \stackrel{i.i.d.}{\sim} \mathcal{D}_i$  be independent sequences for all  $i \in [n]$ . We then let

$$\mathbf{W}(\mathbf{v} \mid \mathbf{U}) := (1 - \gamma)\mathbb{E} \left[ \sum_{t \geq 2} \gamma^{t-2} \tilde{u}_i(t) p_i(t) \mid \mathbf{v}(1) = \mathbf{v} \right], \quad \mathbf{v} \in [0, \bar{v}]^n.$$

That is,  $\mathbf{W}(\mathbf{v} \mid \mathbf{U})$  is the expected remaining utility knowing that the first reports were  $\mathbf{v}$ . By construction, since  $\mathbf{S}(\mathbf{U})$  realizes the utilities  $\mathbf{U}$ , Eq. (2) is satisfied. Next, note that for any first reports  $\mathbf{v}(1) = \mathbf{v}$ , the strategy  $\mathbf{S}(\mathbf{U})$  starting from time 2 exactly realizes the utility vector  $\mathbf{W}(\mathbf{v} \mid \mathbf{U})$  (the allocation problem is time-invariant): Eq. (3) holds. Last, we assumed that truthful reporting satisfies Eq. (PBE). Consider any strategy  $\sigma_i$  which reports an arbitrary value  $v_i(1) = v(u_i(1))$  but truthfully reports at all times  $t \geq 2$ . Taking the expectation of Eq. (PBE) for strategy  $\sigma_i$  and  $t = 1$ , over the realization of the first allocation  $\mathbf{p}(1)$  shows that Eq. (4) holds. ■

We now turn to the second characterization from Fact 3.

**Proof of Fact 3** Fix  $\gamma \in (0, 1)$ . Given Fact 2, it suffices to show that Eqs. (2) to (4) are equivalent to Eqs. (3) and (7).

We start by supposing that Eqs. (2) to (4) hold. We only need to show Eq. (7). Fix  $\mathbf{U} \in \mathcal{R}$ . For readability we omit  $\mid \mathbf{U}$  from all functions below. First, for any  $u, v \in [0, \bar{v}]$  with  $u \leq v$ , applying Eq. (4) to  $u$  and  $v$  implies that

$$\frac{1 - \gamma}{\gamma}(P_i(u) - P_i(v))v \leq W_i(v) - W_i(u) \leq \frac{1 - \gamma}{\gamma}(P_i(u) - P_i(v))u.$$

Note that the function  $P_i(\cdot)$  takes values in  $[0, 1]$  since  $\mathbf{p}(\cdot)$  takes values in  $\Delta_n$ . Also, the previous equation implies in particular that  $P_i(\cdot)$  is non-decreasing and that  $W_i(\cdot)$  is constant on any interval on which  $P_i(\cdot)$  is constant. We can therefore sum the previous equations for  $u = u_i$  and  $v = u_{i+1}$  for regular partitions  $0 \leq u_1, \dots, u_k = w$  for a fixed  $w \in [0, \bar{v}]$  interpreting the sums as Riemann integrals (possible because  $P_i$  is non-decreasing hence well-behaved) which implies

$$W_i(w) - W_i(0) = -\frac{1 - \gamma}{\gamma} \int_{P_i(0)}^{P_i(w)} P_i^{<-1>}(y) dy = -\frac{1 - \gamma}{\gamma} \left( w P_i(w) - \int_0^w P_i(x) dx \right) \quad (77)$$

This proves Eq. (5). We next use Eq. (2) to compute

$$\begin{aligned}
U_i &= \mathbb{E}_{u_i}[(1 - \gamma)u_i P_i(u_i) + \gamma W_i(u_i)] \\
&= \gamma W_i(0) + (1 - \gamma) \mathbb{E}_{u_i} \left[ \int_0^{u_i} P_i(x) dx \right] = \gamma W_i(0) + (1 - \gamma) \mathbb{E}_{u_i} \left[ \int_0^{\bar{v}} P_i(x) \mathbb{1}_{u_i \geq x} dx \right] \\
&= \gamma W_i(0) + (1 - \gamma) \int_0^{\bar{v}} F_i(x) P_i(x) dx.
\end{aligned}$$

Plugging this formula for  $W_i(0)$  into Eq. (77) gives the desired formula for the interim allocation Eq. (6). Eq. (7) holds by definition of the interim promise  $W_i(\cdot | \mathbf{U})$ .

Now suppose that Eqs. (3) and (7) hold. Using the previous computations, we note that Eq. (2) is satisfied. It only remains to check the incentive-compatibility constraint Eq. (4). Having fixed  $\mathbf{U} \in \mathcal{R}$ , for any  $i \in [n]$  and  $u, v \in [0, \bar{v}]$ , we have

$$\begin{aligned}
(1 - \gamma)u P_i(v) + \gamma W_i(v) &= U_i + (1 - \gamma) \left( P_i(v)(u - v) + \int_0^v P_i(x) dx - \int_0^{\bar{v}} F_i(x) P_i(x) dx \right) \\
&= U_i + (1 - \gamma) \left( \int_0^u P_i(x) dx - \int_0^{\bar{v}} F_i(x) P_i(x) dx \right) + (1 - \gamma) \int_u^v (P_i(x) - P_i(v)) dx.
\end{aligned}$$

Only the last term depends on  $v$ , hence the quantity is maximized for  $v = u$  because  $P_i(\cdot)$  is non-decreasing. This proves Eq. (4) and ends the proof.  $\blacksquare$

We next prove Fact 4 which is the finite-horizon counterpart of the previous facts.

**Proof of Fact 4** Suppose that Eqs. (2) and (4) for  $\gamma = \gamma(T)$ , and Eq. (19) are satisfied. Then, for any  $\mathbf{U} \in \mathcal{R}$ , we can use the allocation function  $p(\mathbf{v}(1) | \mathbf{U})$  for the first time  $t = 1$  and reports  $\mathbf{v}(1)$ , then realize the utility vector  $\mathbf{W}(\mathbf{v}(1) | \mathbf{U})$  since it belongs to the achievable region  $\mathcal{V}_{T-1}$  by Eq. (19). This strategy is incentive-compatible at least for time  $t = 1$  hence from Eq. (2), it realizes the vector  $\mathbf{U}$ , that is,  $\mathbf{U} \in \mathcal{V}_T$ . On the other hand, if  $\mathbf{U}$  can be realized, fix an incentive-compatible strategy to realize  $\mathbf{U}$ . Let  $p(\cdot | \mathbf{U})$  be the allocation function for the first time  $t = 1$  and define  $\mathbf{W}(\mathbf{v} | \mathbf{U}) = \frac{1}{T} \mathbb{E} \left[ \sum_{t \geq 2} \tilde{u}_i(t) p_i(t) | \mathbf{v}(1) = \mathbf{v} \right]$  the remaining utility for times in  $[2, T]$ . We can easily check that these satisfy all the desired equations.

The second claim can be proved with the same arguments as in the proof of Fact 3.  $\blacksquare$

Last, we prove Lemma 1 which shows that the couplings from Eqs. (10) and (11) are optimal.

**Proof of Lemma 1** From Eq. (9), the optimal value of Eq. (8) is upper bounded by  $\boldsymbol{\alpha}^\top \mathbb{E}[\tilde{\mathbf{Z}}]$ . For both couplings in Eqs. (10) and (11) we can easily check that for all realization  $\omega$ ,  $\mathbf{Z} \in \{\mathbf{y} : \boldsymbol{\alpha}^\top \mathbf{y} = \boldsymbol{\alpha}^\top \mathbb{E}[\tilde{\mathbf{Z}}]\}$ . Next, by independence of the variables  $\tilde{Z}_i$  for  $i \in [n]$ , we can also prove that in both cases for all  $i \in [n]$ ,  $\mathbb{E}[Z_i | \tilde{Z}_i] = \tilde{Z}_i$ . Hence these couplings satisfy the constraints and achieve the value  $\boldsymbol{\alpha}^\top \mathbb{E}[\tilde{\mathbf{Z}}]$  which ends the proof.  $\blacksquare$