Average-Case Performance of Rollout Algorithms for Knapsack Problems: Supplementary Material

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This document contains proofs that were omitted in the main paper in Sections S1 and S2, and evaluations of integrals in Section S3.

S1 Exhaustive Rollout

Exhaustive Rollout: Subset Sum Problem Analysis

Proof of Lemma 4.3. Again, fix $K = k$ for $k > 1$, $G = g$, $W_{k-1} = w_{k-1}$. Define the random variable V_k^u so that

$$V_k^u := \begin{cases} V_k, & \text{if } L_k = k-1, \\ 1, & \text{if } L_k \leq k-2 \vee L_k = k. \end{cases}$$

From Lemma 3.4 we are guaranteed that, given $G = g$, W_k follows distribution $\mathcal{U}[g, 1]$. Thus, to determine $\mathbb{P}(V_k^u > v | g, w_{k-1}, \overline{\mathcal{C}_1})$, we can use the same analysis for Lemma 4.2, but restricted to the interval $g \leq w_k \leq 1$. Taking the expression $\mathbb{P}(V^u > v | g, w_{K-1}, \overline{\mathcal{C}_1})$ in (29), removing the $(g-v)_+$ term, and normalizing by $(1-g)$, we have

$$\mathbb{P}(V_k^u > v | g, w_{k-1}, \overline{\mathcal{C}_1}) \quad = \quad \left(\frac{1}{1-g}\right) \left((w_{k-1} - v)_+ - (g + w_{k-1} - v - 1)_+ + (1 - g - w_{k-1})_+\right). \tag{S1}$$

This holds for all $k > 1$, so we replace k with K in the expression. □

Proof of Lemma 4.4. Fix $G = g$. Note that, for $K = 1$, the *jth* insertion gap can never be greater than g. Keeping the analysis for Lemma 4.2 in mind and using Lemma 3.4, we have that, for $v < g$,

$$\mathbb{P}(V_j > v | g, \mathcal{C}_1) = (g - v) + (1 - g), \tag{S2}$$

where $(g - v)$ corresponds to the case where $w_j \in [0, g]$ and $(1 - g)$ corresponds to $w_j \in]g, 1]$. □

Proof of Corollary 4.1 and Theorem 4.2. The sum terms may be bounded with integral approximations. For the upper bound, the argument of the sum is convex in m, so the midpoint rule provides an upper bound.

$$\sum_{m=0}^{n-2} \frac{9 + 2m}{3(3 + m)(4 + m)} \leq \int_{-\frac{1}{2}}^{n-\frac{3}{2}} \frac{9 + 2m}{3(3 + m)(4 + m)} \mathrm{d}m = \log\left[\left(\frac{3 + 2n}{5}\right)\left(\frac{7}{5 + 2n}\right)^{1/3}\right]. \tag{S3}$$

The asymptotic result then follows. □

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Exhaustive Rollout: 0-1 Knapsack Problem Analysis

We follow the same approach that was used for the subset sum problem and assume that the reader understands this analysis (and thus less detail is included here). We employ the results from Section 3 and we use the same definition for the jth insertion item that was used for the subset sum problem. Analogous to the jth insertion gap V_j, we define here the jth insertion gain Z_j for $j \geq 2$, where

$$Z_j := \max\left(0,\ \mathbb{1}(W_j \leq B)\left(P_j + \sum_{i=1}^{L_j-1} P_i \mathbb{1}(i \neq j)\right) - \sum_{i=1}^{K-1} P_i\right). \tag{S4}$$

The jth insertion gain is simply the positive part of the difference between the value of the solution obtained by using Blind-Greedy after moving item j to the front of the sequence, and the value of the solution from using Blind-Greedy on the original input sequence.

We will bound the expected insertion gains while conditioning on (G, W_{K-1}, P_{K-1}). Assuming $K > 1$, this is done in the following three lemmas for packed items, the critical item, and remaining items, respectively, just as we did for the subset sum problem. The lemma, after these three lemmas, handles the case where $K = 1$. We assume that $\sum_{i=1}^{n} W_i > B$ throughout the section.

Lemma S1.1. For $K > 1$ and $j = 2, \ldots, K-1$, the jth insertion gain satisfies

$$Z_j = 0 \tag{S5}$$

with probability one.

Proof. This follows by observation since changing the order of the packed items does not change the total profit. □

Lemma S1.2. For $K > 1$ and $j = K+1, \ldots, n$, the jth insertion gain satisfies $Z_j \geq Z_j^l$ with probability one, where Z_j^l is a deterministic function of $(G, W_{K-1}, W_j, P_{K-1}, P_j)$, and conditioning only on (G, W_{K-1}, P_{K-1}) gives

$$\begin{aligned}\mathbb{P}(Z_j^l \leq z | g, w_{K-1}, p_{K-1}, \overline{\mathcal{C}_1}) &= zg + \min(z + p_{K-1}, 1)\left(w_{K-1} - (g + w_{K-1} - 1)_+\right) \\ &\quad + (1 - g - w_{K-1})_+ \\ &=: \mathbb{P}(Z^l \leq z | g, w_{K-1}, p_{K-1}, \overline{\mathcal{C}_1}).\end{aligned} \tag{S6}$$

Proof. Fix $K = k$ for any $k > 1$, and let the event $\overline{\mathcal{C}_1}$ be implicit. We define the lower bounding random variable Z_j^l so that

$$Z_j^l := \begin{cases} Z_j, & \text{if } L_j = k \vee L_j = k-1, \\ 0, & \text{if } L_j \leq k-2 \vee L_j = j. \end{cases}$$

This means we have an exact characterization of the jth insertion gain when the insertion critical item is either k or $k-1$, and a worst-case gain of zero value in other cases. Thus it can be seen that $Z_j \geq Z_j^l$ with probability one, and Z_j^l uniquely depends on the random variables $(G, W_{K-1}, W_j, P_{K-1}, P_j)$. Let $\mathcal{D}_k$, $\mathcal{D}_{k-1}$, and $\mathcal{D}_{(k-2)-}$ indicate the events $L_j = k$, $L_j = k-1$, and $L_j \leq k-2 \vee L_j = j$, respectively. Using an illustration similar to Figure 3 under the assumption that $G = g$ and $\boldsymbol{W_S} = \boldsymbol{w_s}$, we have that, if we only allow W_j to be random, then by Lemma 3.4,

$$\mathbb{P}(\mathcal{D}_k | g, w_{k-1}, p_{k-1}, p_j) = g, \tag{S7}$$

$$\mathbb{P}(\mathcal{D}_{k-1} | g, w_{k-1}, p_{k-1}, p_j) = w_{k-1} - (g + w_{k-1} - 1)_+, \tag{S8}$$

$$\mathbb{P}(\mathcal{D}_{(k-2)-} | g, w_{k-1}, p_{k-1}, p_j) = (1 - g - w_{k-1})_+. \tag{S9}$$

Note that these expressions do not depend on any of the $\boldsymbol{P_S}$ values, since item weights and profits are independent. For each of the above cases, we can find the probability distribution for Z_j^l while allowing only P_j to be random, so that

$$\begin{aligned}
\mathbb{P}(Z_j^l \leq z|\mathcal{D}_k, g, w_{k-1}, w_j, p_{k-1}) &= \mathbb{P}(P_j \leq z) = z, && \text{(S10)}\\
\mathbb{P}(Z_j^l \leq z|\mathcal{D}_{k-1}, g, w_{k-1}, w_j, p_{k-1}) &= \mathbb{P}(P_j - P_{k-1} \leq z|p_{k-1}) = \min(z + p_{k-1}, 1), && \text{(S11)}\\
\mathbb{P}(Z_j^l \leq z|\mathcal{D}_{(k-2)-}, g, w_{k-1}, w_j, p_{k-1}) &= \mathbb{P}(0 \leq z) = 1. && \text{(S12)}
\end{aligned}$$

Again, these expressions do not depend on any of the $\boldsymbol{W_S}$ values by item profit and weight independence. Then, combining terms and noting that the above functions do not depend on all of the conditioned parameters, we have that, if we only condition on (G, W_{k-1}, P_{k-1}),

$$\begin{aligned}
\mathbb{P}(Z_j^l \leq z|g, w_{k-1}, p_{k-1}) &= \mathbb{P}(Z_j^l \leq z|\mathcal{D}_k, g, w_{k-1}, p_{k-1})\mathbb{P}(\mathcal{D}_k|g, w_{k-1}, p_{k-1})\\
&\quad +\mathbb{P}(Z_j^l \leq z|\mathcal{D}_{k-1}, g, w_{k-1}, p_{k-1})\mathbb{P}(\mathcal{D}_{k-1}|g, w_{k-1}, p_{k-1})\\
&\quad +\mathbb{P}(Z_j^l \leq z|\mathcal{D}_{(k-2)-}, g, w_{k-1}, p_{k-1})\mathbb{P}(\mathcal{D}_{(k-2)-}|g, w_{k-1}, p_{k-1})\\
&= zg + \min(z + p_{k-1}, 1)\left(w_{k-1} - (g + w_{k-1} - 1)_+\right)\\
&\quad +(1 - g - w_{k-1})_+. && \text{(S13)}
\end{aligned}$$

The analysis holds for all $k > 1$, so we replace k with K, which yields the expression in the lemma. ☐

Lemma S1.3. For $K > 1$, the Kth insertion gap satisfies $Z_K \geq Z_K^l$ with probability one, where Z_K^l is a deterministic function of $(G, W_{K-1}, W_K, P_{K-1}, P_K)$, and conditioning only on (G, W_{K-1}, P_{K-1}) gives

$$\begin{aligned}
\mathbb{P}(Z_K^l \leq z|g, w_{K-1}, p_{K-1}, \overline{\mathcal{C}_1}) &= \frac{1}{1-g}\left(\min(z + p_{K-1}, 1)\left(w_{k-1} - (g + w_{K-1} - 1)_+\right) + (1 - g - w_{K-1})_+\right)\\
&=: \mathbb{P}(\widetilde{Z}^l \leq z|g, w_{K-1}, p_{K-1}, \overline{\mathcal{C}_1}). && \text{(S14)}
\end{aligned}$$

Proof. Fix $K = k$ for $k > 1$ and make the event $\overline{\mathcal{C}_1}$ implicit. We define the lower bound random variable Z_k^l so that

$$Z_k^l := \begin{cases} Z_k, & \text{if } L_k = k-1,\\ 0, & \text{if } L_k \leq k-2 \vee L_k = k. \end{cases}$$

This random variable assumes a worst-case bound of zero gain if item $k-1$ becomes infeasible. By definition, we have $Z_k \geq Z_k^l$ with probability one and that Z_k^l is uniquely determined by $(G, W_{k-1}, W_j, P_{k-1}, P_j)$. Let $\mathcal{D}_{k-1}$ be the event that $L_k = k-1$ and let $\mathcal{D}_{(k-2)-}$ indicate the event $L_k \leq k-2 \vee L_k = k$. By Lemma 3.4, we have that, for $G = g$, item k has distribution $\mathcal{U}[g, 1]$. Using the analysis in the previous lemma, but restricted to the interval $[g, 1]$, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{D}_{k-1}|g, w_{k-1}, p_{k-1}, p_k) &= \frac{1}{1-g}\left(w_{k-1} - (g + w_{k-1} - 1)_+\right), && \text{(S15)}\\
\mathbb{P}(\mathcal{D}_{(k-2)-}|g, w_{k-1}, p_{k-1}, p_k) &= \frac{1}{1-g}(1 - g - w_{k-1})_+. && \text{(S16)}
\end{aligned}$$

By the independence of item weights and profits, the following results carry over from the proof of the previous lemma:

$$\begin{aligned}
\mathbb{P}(Z_k^l \leq z|\mathcal{D}_{k-1}, g, w_{k-1}, w_k, p_{k-1}) &= \mathbb{P}(P_k - P_{k-1} \leq z|p_{k-1}) = \min(z + p_{k-1}, 1), && \text{(S17)}\\
\mathbb{P}(Z_k^l \leq z|\mathcal{D}_{(k-2)-}, g, w_{k-1}, w_k, p_{k-1}) &= \mathbb{P}(0 \leq z) = 1. && \text{(S18)}
\end{aligned}$$

We then have that, if we only condition on (G, W_{k-1}, P_{k-1}),

$$\begin{aligned}\mathbb{P}(Z_k^l \leq z|g, w_{k-1}, p_{k-1}) &= \mathbb{P}(Z_k^l \leq z|\mathcal{D}_{k-1}, g, w_{k-1}, p_{k-1})\mathbb{P}(\mathcal{D}_{k-1}|g, w_{k-1}, p_{k-1}) \\ &\quad +\mathbb{P}(Z_k^l \leq z|\mathcal{D}_{(k-2)-}, g, w_{k-1}, p_{k-1})\mathbb{P}(\mathcal{D}_{(k-2)-}|g, w_{k-1}, p_{k-1}) \\ &= \frac{1}{1-g}\left(\min(z+p_{k-1}, 1)\left(w_{k-1} - (g+w_{k-1}-1)_+\right) + (1-g-w_{k-1})_+\right). \end{aligned} \tag{S19}$$

The analysis is valid for all $k > 1$, so we replace k with K to obtain the expression in the lemma. □

We now define $Z_*(n)$, which is the gain given by the first iteration of the rollout algorithm on an instance with n items,

$$Z_*(n) := \max(Z_2, \ldots, Z_n). \tag{S20}$$

For the rest of the section, we will usually refer to $Z_*(n)$ simply as Z_*.

Proof of Theorem 4.3. We proceed in a fashion nearly identical to the proof of Theorem 4.1. We have that for $K = k > 1$, $Z_* \geq Z_*^l$ with probability one, where

$$Z_*^l := \max(Z_k^l, Z_{k+1}^l, \ldots, Z_n^l). \tag{S21}$$

This makes use of Lemmas S1.1 - S1.3. By Lemmas S1.2 and S1.3, each Z_j^l for $j \geq k$ is a deterministic function of $(G, W_{k-1}, W_j, P_{k-1}, P_j)$. Lemma 3.4 gives that item weights W_j for $j > k$ independently follow the distribution $\mathcal{U}[0,1]$, and W_k independently follows the distribution $\mathcal{U}[g,1]$. As a result, conditioning on only (G, W_{k-1}, P_{k-1}) makes Z_j^l independent for $j \geq k$, and then, by the definition of the maximum,

$$\begin{aligned}\mathbb{P}(Z_*^l \leq z|g, w_{k-1}, p_{k-1}, k, \overline{\mathcal{C}_1}) &= \mathbb{P}(Z_k^l \leq z|g, w_{k-1}, p_{k-1}, \overline{\mathcal{C}_1}) \prod_{j=k+1}^{n} \mathbb{P}(Z_j^l \leq z|g, w_{k-1}, p_{k-1}, \overline{\mathcal{C}_1}) \\ &= \mathbb{P}(\widetilde{Z}^l \leq z|g, w_{k-1}, p_{k-1}, \overline{\mathcal{C}_1})\left(\mathbb{P}(Z^l \leq z|g, w_{k-1}, p_{k-1}, \overline{\mathcal{C}_1})\right)^{(n-k)}. \end{aligned} \tag{S22}$$

In the remainder of the proof, we first integrate over the conditioned variables and then consider the case $\mathcal{C}_1$. For the integrals, we adopt some simplified notation to make expressions more manageable. As with the subset sum problem, let $M := n - K$. Moreover, define

$$\pi_+ := g, \tag{S23}$$

$$\pi_0 := w_{K-1} - (g + w_{k-1} - 1)_+, \tag{S24}$$

$$\pi_- := (1 - g - w_{k-1}), \tag{S25}$$

$$\widetilde{\pi}_0 := \frac{1}{1-g}\left(w_{k-1} - (g + w_{k-1} - 1)_+\right), \tag{S26}$$

$$\widetilde{\pi}_- := \frac{1}{1-g}(1 - g - w_{k-1})_+. \tag{S27}$$

This allows us to write (S22) as

$$\mathbb{P}(Z_*^l \leq z|g, w_{k-1}, p_{k-1}, m, \overline{\mathcal{C}_1}) = \left(\min(z+p_{k-1}, 1)\widetilde{\pi}_0 + \widetilde{\pi}_-\right)\left(z\pi_+ + \min(z+p_{k-1}, 1)\pi_0 + \pi_-\right)^m. \tag{S28}$$

Integrating over p_{k-1}, which follows density $\mathcal{U}[0,1]$,

$$\begin{aligned}
\mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1}) &= \int \mathbb{P}(Z^l_* \leq z|g, w_{k-1}, p_{k-1}, m, \overline{\mathcal{C}_1}) f_{P_{k-1}}(p_{k-1}) \mathrm{d}p_{k-1} \\
&= \int_0^{1-z} \left((z+p_{k-1})\widetilde{\pi}_0 + \widetilde{\pi}_-\right)\left(z\pi_+ + (z+p_{k-1})\pi_0 + \pi_-\right)^m \mathrm{d}p_{k-1} \\
&\quad + \int_{1-z}^1 \left(\widetilde{\pi}_0 + \widetilde{\pi}_-\right)\left(z\pi_+ + \pi_0 + \pi_-\right)^m \mathrm{d}p_{k-1} \\
&= (\widetilde{\pi}_0 + \widetilde{\pi}_-)(\pi_0 + \pi_- + \pi_+ z)^m z + \frac{1}{(m+1)(m+2)\pi_0^2} \cdot \\
&\quad \big((\pi_0 + P_n + \pi_+ z)^{m+1}(\pi_0\widetilde{\pi}_0(m+1) + \pi_0\widetilde{\pi}_-(m+2) - \widetilde{\pi}_0\pi_- - \widetilde{\pi}_0\pi_+ z) \\
&\quad -(\pi_- + (\pi_0+\pi_+)z)^{m+1}((2+m)\pi_0\widetilde{\pi}_- - \pi_-\widetilde{\pi}_0 + \widetilde{\pi}_0(\pi_0 + m\pi_0 - \pi_+)z)\big).
\end{aligned} \tag{S29}$$

At this point it is useful to evaluate separately the cases where $g + w_{k-1} < 1$ and $g + w_{k-1} \geq 1$. Let $\mathcal{E}$ indicate the event that $g + w_{k-1} < 1$ holds, and let $\overline{\mathcal{E}}$ be the complement of this event. Furthermore, define $A := W_{K-1}$. This allows us to define

$$\mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1})_{\mathcal{E}} := \mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1})\mathbb{1}(g + w_{k-1} < 1), \tag{S30}$$

$$\mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} := \mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1})\mathbb{1}(g + w_{k-1} \geq 1), \tag{S31}$$

so that

$$\mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1}) = \mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1})_{\mathcal{E}} + \mathbb{P}(Z^l_* \leq z|g, w_{k-1}, m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}}. \tag{S32}$$

Starting with the case where $\mathcal{E}$ holds and substituting A for W_{k-1},

$$\begin{aligned}
\mathbb{P}(Z^l_* \leq z|g, a, m, \overline{\mathcal{C}_1})_{\mathcal{E}} &= z(1-g+gz)^m + \frac{(1-g+gz)^{m+1}(1-g+m-gm-gz)}{(1-g)a(m+1)(m+2)} \\
&\quad - \frac{(1-g+gz+a(1-z))^{m+1}(1-g+m-gm-gz+a(-1-m+z+mz))}{(1-g)a(m+1)(m+2)}.
\end{aligned} \tag{S33}$$

We now wish to compute

$$\mathbb{P}(Z^l_* \leq z|m, \overline{\mathcal{C}_1})_{\mathcal{E}} := \int_0^1 \int_0^{1-g} \mathbb{P}(Z^l_* \leq z|g, a, m, \overline{\mathcal{C}_1})_{\mathcal{E}} f_A(a) f_G(g) \mathrm{d}a\mathrm{d}g. \tag{S34}$$

The evaluation of this integral is given in Section S3.2, which shows

$$\mathbb{P}(Z^l_* \leq z|m, \overline{\mathcal{C}_1})_{\mathcal{E}} = \rho_1(m,z) + \sum_{j=1}^{m+1} \rho_{2j}(m,z) + \rho_3(m,z) + \rho_4(m,z), \tag{S35}$$

where

$$\rho_1(m,z) = -\frac{2z\left(2+m^2(-1+z)^2+m(-1+z)(-3+5z)-2z\left(3-3z+z^{2+m}\right)\right)}{(1+m)(2+m)(3+m)(-1+z)^3}, \tag{S36}$$

$$\rho_{2j}(m,z) = \frac{2z^{3+m}(j+(2+m)(-2+z)-jz)}{j(-3+j-m)(-2+j-m)(1+m)(2+m)(-1+z)^2}, \tag{S37}$$

$$+\frac{2z^j(-j(1+m)(-1+z)+(2+m)(-1+m(-1+z)+2z))}{j(-3+j-m)(-2+j-m)(1+m)(2+m)(-1+z)^2}, \tag{S38}$$

$$\rho_3(m,z) = -\frac{2H(m+1)\left(-1+m(-1+z)+2z+(-2+z)z^{3+m}\right)}{(1+m)(2+m)(3+m)(-1+z)^2}, \tag{S39}$$

$$\rho_4(m,z) = -\frac{2}{(2+m)^2(3+m)(-1+z)} - \frac{2z^{2+m}}{(2+m)^2} + \frac{2z^{3+m}}{(2+m)^2(3+m)(-1+z)}. \tag{S40}$$

Since we are ultimately interested in the expected value of Z_*^l, we wish to evaluate

$$\overline{\mathbb{E}}[Z_*^l|m,\overline{\mathcal{C}_1}]_{\mathcal{E}} := \int_0^1 \mathbb{P}(Z_*^l \le z|m,\overline{\mathcal{C}_1})_{\mathcal{E}}\mathrm{d}z. \tag{S41}$$

Recall that $\overline{\mathbb{E}}[\cdot] := 1-\mathbb{E}[\cdot]$. Using the definition

$$\xi_j(m) := \int_0^1 \rho_j(m,z)\mathrm{d}z, \tag{S42}$$

we have

$$\overline{\mathbb{E}}[Z_*^l|m,\overline{\mathcal{C}_1}]_{\mathcal{E}} = \xi_1(m) + \sum_{j=1}^{m+1}\xi_{2j}(m) + \xi_3(m) + \xi_4(m), \tag{S43}$$

where

$$\xi_1(m) = -\frac{2H(m+1)(3+m-H(m+3)(2+m))}{(m+1)(m+2)(m+3)}, \tag{S44}$$

$$\xi_{2j}(m) = \frac{2\left(-(-3+j-m)(2+m)+\left(j+(2+m)^2\right)H(j)-\left(j+(2+m)^2\right)H(m+3)\right)}{j(-3+j-m)(-2+j-m)(1+m)(2+m)}, \tag{S45}$$

$$\xi_3(m) = \frac{2(H(m+3)-1)}{(2+m)^2(3+m)}, \tag{S46}$$

$$\xi_4(m) = -\frac{(2+m)(17+5m)-2(3+m)(4+m)H(m+2)}{(m+1)(m+2)(m+3)}. \tag{S47}$$

This completes the case for the event $\mathcal{E}$ (i.e. $g+w_{k-1}<1$). Now, when the event $\overline{\mathcal{E}}$ holds,

$$\begin{aligned}\mathbb{P}(Z_*^l \le z|g,a,m,\overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} &= z(1-g+gz)^m - \frac{(1-2g+(1-g)m)z^{2+m}}{(1-g)^2(1+m)(2+m)} \\ &\quad + \frac{((1-g)(1+m)-gz)(1-g+gz)^{1+m}}{(1-g)^2(1+m)(2+m)}.\end{aligned} \tag{S48}$$

Continuing as we did with the case $\mathcal{E}$,

$$\begin{aligned}\mathbb{P}(Z_*^l \le z|m,\overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} &:= \int_0^1\int_{1-g}^1 \mathbb{P}(Z_*^l \le z|g,a,m,\overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} f_A(a)f_G(g)\mathrm{d}a\mathrm{d}g \\ &= \int_0^1 g\mathbb{P}(Z_*^l \le z|g,a,m,\overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} f_G(g)\mathrm{d}g,\end{aligned} \tag{S49}$$

where we have used the fact that the expression $\mathbb{P}(Z_*^l \leq z|g,a,m,\overline{\mathcal{C}_1})_{\overline{\mathcal{E}}}$ is not a function of a. Evaluation of this integral is given in Section S3.3; the expression is

$$\begin{aligned}
\mathbb{P}(Z_*^l \leq z|m,\overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} \quad = \quad & -\frac{2z\left(1+m-3z-mz+z^{2+m}(3+m-(1+m)z)\right)}{(1+m)(2+m)(3+m)(-1+z)^3} + \frac{-2z}{(m+1)(m+2)}\sum_{j=1}^{m+1}\frac{z^{m+1-j}}{j} \\
& +\frac{2z}{(m+1)(m+2)^2(1-z)} + \frac{2(1+m+z)}{(m+1)(m+2)^2(m+3)(1-z)^2} - \frac{(6+2m)z^{m+2}}{(m+1)(m+2)} \\
& +\frac{z^{m+2}}{m+1} + \frac{2H(m+1)z^{m+2}}{(m+1)(m+2)} - \frac{2(1+m+2z)z^{m+2}}{(m+1)(m+2)^2(1-z)} \\
& -\frac{2(1+m+z)z^{m+3}}{(m+1)(m+2)^2(m+3)(1-z)^2}.
\end{aligned} \tag{S50}$$

We again calculate the following term for the expected value

$$\begin{aligned}
\overline{\mathbb{E}}[Z_*^l|m,\overline{\mathcal{C}_1}]_{\overline{\mathcal{E}}} \quad := \quad & \int_0^1 \mathbb{P}(Z_*^l \leq z|m)\mathrm{d}z \\
= \quad & \frac{20+10m+m^2-2(3+m)H(1+m)}{(2+m)(3+m)^2} + \sum_{j=1}^{m+1}\frac{2}{j(-3+j-m)(1+m)(2+m)}.
\end{aligned} \tag{S51}$$

Bringing together both cases $\mathcal{E}$ and $\overline{\mathcal{E}}$, we have

$$\begin{aligned}
\mathbb{E}[Z_*^l|m,\overline{C}_1] \quad = \quad & \int_0^1 (1-\mathbb{P}(Z_*^l \leq z|m,\overline{\mathcal{C}_1}))\mathrm{d}z \\
= \quad & 1 - \int_0^1 \mathbb{P}(Z_*^l \leq z|m,\overline{\mathcal{C}_1})\mathrm{d}z \\
= \quad & 1 - \overline{\mathbb{E}}[Z_*^l|m,\overline{\mathcal{C}_1}]_{\mathcal{E}} - \overline{\mathbb{E}}[Z_*^l|m,\overline{\mathcal{C}_1}]_{\overline{\mathcal{E}}} \\
= \quad & 1 + \frac{1}{(m+1)(m+2)^3(m+3)^2}\left((186+472m+448m^2+203m^3+45m^4+4m^5)\right. \\
& +(-244-454m-334m^2-124m^3-24m^4-2m^5)H(m+1) \\
& \left.+(-48-88m-60m^2-18m^3-2m^4)(H(m+1))^2\right) \\
& +\sum_{j=1}^{m+1}\frac{2\left(-4+j-4m+jm-m^2-\left(j+(2+m)^2\right)H(j)+\left(j+(2+m)^2\right)H(3+m)\right)}{j(-3+j-m)(-2+j-m)(1+m)(2+m)}.
\end{aligned} \tag{S52}$$

If the first item is critical, then

$$\mathbb{P}(Z_* \leq z|g,m,\mathcal{C}_1) = (1-g+gz)^m. \tag{S53}$$

Marginalizing over G and taking the expectation gives

$$\begin{aligned}
\mathbb{P}(Z_* \leq z|m,\mathcal{C}_1) \quad = \quad & \int_0^1 \mathbb{P}(Z_*^l \leq z|g,m,\mathcal{C}_1)f_G(g)\mathrm{d}g \\
= \quad & \int_0^1 (1-g+gz)^m(2-2g)\mathrm{d}g \\
= \quad & \frac{2\left(1+m-2z-mz+z^{2+m}\right)}{(1+m)(2+m)(-1+z)^2}.
\end{aligned} \tag{S54}$$

$$\begin{aligned}\mathbb{E}(Z_*|m,\mathcal{C}_1) &= 1-\int_0^1 \mathbb{P}(Z_*\leq z|m,\mathcal{C}_1)\mathrm{d}z \\ &= 1+\frac{2}{2+m}-\frac{2H(m+1)}{m+1}. \qquad \text{(S55)}\end{aligned}$$

Since the event $\mathcal{C}_1$ indicates $M=n-1$,

$$\mathbb{E}(Z_*|\mathcal{C}_1) = 1+\frac{2}{n+1}-\frac{2H(n)}{n}. \qquad \text{(S56)}$$

Finally, accounting for the distribution of M with Lemma 3.1 gives the expression in the theorem:

$$\begin{aligned}&\mathbb{E}\left[Z_*(n)\left|\sum_{i=1}^{n}W_i>B\right.\right]\geq 1+\frac{2}{n(n+1)}-\frac{2H(n)}{n^2}\\ &+\frac{1}{n}\sum_{m=0}^{n-2}\left(\sum_{j=1}^{m+1}T(j,m)+\Big((186+472m+448m^2+203m^3+45m^4+4m^5)\right.\\ &-(244+454m+334m^2+124m^3+24m^4+2m^5)H(m+1)\\ &\left.-(48+88m+60m^2+18m^3+2m^4)(H(m+1))^2\Big)\frac{1}{(m+1)(m+2)^3(m+3)^2}\right),\end{aligned} \qquad \text{(S57)}$$

where

$$\begin{aligned}T(j,m) &:= \frac{2\left(-4+j-4m+jm-m^2-\left(j+(2+m)^2\right)H(j)\right)}{j(-3+j-m)(-2+j-m)(1+m)(2+m)}\\ &\quad+\frac{2(j+(2+m)^2)H(3+m)}{j(-3+j-m)(-2+j-m)(1+m)(2+m)}. \qquad \text{(S58)}\end{aligned}$$

□

We observe that the nested summation term may be omitted without significant loss in the performance bound. This is accomplished by showing that the argument of the sum is always positive.

Lemma S1.4. For all $m>0$ and $1\leq j\leq m+1$,

$$\frac{\left(-4+j-4m+jm-m^2-\left(j+(2+m)^2\right)H(j)+\left(j+(2+m)^2\right)H(3+m)\right)}{j(-3+j-m)(-2+j-m)(1+m)(2+m)}>0. \qquad \text{(S59)}$$

Proof. The denominator is always positive, so we focus on the numerator. There numerator consists of two parts,

$$N_1(j,m) := (4-j)(m+1)+m^2, \qquad \text{(S60)}$$

$$N_2(j,m) := (j+(2+m)^2)\sum_{i=j+1}^{m+3}\frac{1}{i}. \qquad \text{(S61)}$$

Our goal is to show that $N_2(j,m)>N_1(j,m)$ always holds. The difference equation for $N_2(j,m)$ with respect to j satisfies

$$
\begin{aligned}
\Delta(N_2(j,m)) &:= N_2(j+1,m) - N_2(j,m) \\
&= \sum_{i=j+2}^{m+3} \frac{1}{i} + (j+(2+m)^2) \sum_{i=j+2}^{m+3} \frac{1}{i} - (j+(2+m)^2) \sum_{i=j+1}^{m+3} \frac{1}{i} \\
&= \sum_{i=j+2}^{m+3} \frac{1}{i} - \frac{j+(2+m)^2}{j+1} \\
&\leq \frac{m-j+2}{j+2} - \frac{j+(2+m)^2}{j+1} \\
&< \frac{m-j+2}{j+1} - \frac{j+(2+m)^2}{j+1} \\
&= \frac{-2-3m-m^2-2j}{j+1} \\
&\leq \frac{-4-3m-m^2}{m+2}.
\end{aligned} \tag{S62}
$$

For the other term, we have

$$\Delta(N_1(j,m)) = -(m+1). \tag{S63}$$

Both $N_1(j,m)$ and $N_2(j,m)$ are decreasing in j and $N_2(j,m)$ decreases at a greater rate. We approximate $N_2(j,m)$ with the following:

$$H(m+3) - H(j) = \sum_{i=j+1}^{m+3} \frac{1}{i} \geq \int_{j+1}^{m+4} \frac{1}{x} \mathrm{d}x = \log\left(\frac{m+4}{j+1}\right). \tag{S64}$$

Looking at $j=1$,

$$N_1(1,m) = 3+3m+m^2, \tag{S65}$$

$$N_2(1,m) = (5+4m+m^2)\log\left(\frac{m+4}{2}\right), \tag{S66}$$

guaranteeing $N_2(1,m) > N_1(1,m)$. With consideration of starting points and slopes for the two numerator terms, ensuring that $N_2(m+1,m) > N_1(m+1,m)$ is sufficient for the lemma. We have

$$N_1(m+1,m) = 3+2m, \tag{S67}$$

$$
\begin{aligned}
N_2(m+1,m) &= (m+1+(2+m)^2)\left(\frac{1}{m+2} + \frac{1}{m+3}\right) \\
&> (5+5m+m^2)\left(\frac{2}{m+3}\right) \\
&= \frac{10+10m+2m^2}{m+3} \\
&> 3+2m.
\end{aligned} \tag{S68}
$$

□

Proof of Theorem 4.4. We will show that $\lim_{n\to\infty} \mathbb{E}[Z_*|\cdot] = 1$, so we are interested in bounding the rate at which $1 - \mathbb{E}[Z_*|\cdot]$ approaches 0. Accordingly, we are only concerned with the negative terms in (S57). The magnitudes of these terms are

$$T_1(n) = \frac{2H(n)}{n^2}, \tag{S69}$$

$$T_2(n) = \frac{1}{n}\sum_{m=0}^{n-2} \frac{(244+454m+334m^2+24m^4+2m^5)H(m+1)}{(m+1)(m+2)^3(m+3)^2}, \tag{S70}$$

$$T_3(n) = \frac{1}{n}\sum_{m=0}^{n-2} \frac{(48+88m+60m^2+18m^3+2m^4)(H(m+1))^2}{(m+1)(m+2)^3(m+3)^2}. \tag{S71}$$

The second and third terms are decreasing in m, so they are bounded by their respective integrals. Using a logarithmic bound on the harmonic numbers, we have

$$T_1(n) = O\left(\frac{\log n}{n^2}\right), \tag{S72}$$

$$\begin{aligned} T_2(n) &= \frac{1}{n}\sum_{m=0}^{n-2} O\left(\frac{\log m}{m}\right) \\ &= \frac{1}{n}\int_0^{n-1} O\left(\frac{\log m}{m}\right) \mathrm{d}m \\ &= O\left(\frac{\log^2 n}{n}\right), \end{aligned} \tag{S73}$$

$$\begin{aligned} T_3(n) &= \frac{1}{n}\sum_{m=0}^{n-2} O\left(\frac{\log^2 m}{m^2}\right) \\ &= \frac{1}{n}\int_0^{n-1} O\left(\frac{\log^2 m}{m^2}\right) \mathrm{d}m \\ &= O\left(\frac{\log^2 n}{n^2}\right). \end{aligned} \tag{S74}$$

The largest growth rate is $O(\frac{\log^2 n}{n})$. Furthermore, we have that $\lim_{n\to\infty} \mathbb{E}[Z_*(n)|\cdot] = 1$ since the gain has a natural upper bound of unit value. □

S2 Consecutive Rollout

Consecutive Rollout: Subset Sum Problem Analysis

The proof method for Theorem 5.1 is similar to the approach used for Theorem 4.1. Keeping Figure 1 in mind, we look at modifications to the Blind-Greedy solution caused by removing the first item. Removing the first item causes the other items to slide to the left and may make some remaining items feasible to pack. We determine bounds on the gap produced by this procedure while conditioning on the greedy gap G, critical item K, and the item weights (W_K, W_{K+1}). We then take the minimum of this gap and the greedy gap, and integrate over conditioned variables to obtain the final bound. Our analysis is divided into lemmas based on the critical item K, where a separate lemma is given for the cases $K = 1$, $2 \le K \le n-1$, and $K = n$.

To formalize the behavior of Consecutive-Rollout, we introduce the following two definitions. The drop critical item L_1 is the index of the item that becomes critical when the first item is removed and thus satisfies

$$\begin{cases} \sum_{i=2}^{L_1-1} W_i \le B < \sum_{i=2}^{L_1} W_i, & \text{if } \sum_{i=2}^{n} W_i > B, \\ L_1 = n+1, & \text{if } \sum_{i=2}^{n} W_i \le B, \end{cases}$$

where the latter case signifies that all remaining items can be packed. The drop gap V_1 then has definition

$$V_1 := B - \sum_{i=2}^{L_1-1} W_i. \tag{S75}$$

We are ultimately interested in the minimum of the drop gap and the greedy gap, which we refer to as the minimum gap, and is the value obtained by the first iteration of the rollout algorithm:

$$V_*(n) := \min(G, V_1). \tag{S76}$$

We will often write $V_*(n)$ simply as V_*. We will also use $\mathcal{C}_i$ to denote the event that item i is critical and $\overline{\mathcal{C}_{1n}}$ for the event that $2 \le K \le n-1$. Furthermore, recall that we have $\boldsymbol{P_I} = \boldsymbol{W_I}$ for the subset sum problem.

Lemma S2.1. For $2 \le K \le n-1$, the expected minimum gap satisfies

$$\mathbb{E}[V_*(n) | 2 \le K \le n-1] \le \frac{13}{60}. \tag{S77}$$

Proof. Fix $K = k$ for $2 \le k \le n-1$. The drop gap, in general, may be a function of the weights of all remaining items. To make things more tractable, we define the random variable V_1^u that satisfies $V_1 \le V_1^u$ with probability one, and as we will show, is a deterministic function of only (G, W_1, W_k, W_{k+1}). The variable V_1^u is specifically defined as

$$V_1^u := \begin{cases} V_1, & \text{if } L_1 = k \vee L_1 = k+1, \\ B - \sum_{i=2}^{k+1} W_i, & \text{if } L_1 \ge k+2. \end{cases} \tag{S78}$$

In effect, V_1^u does not account for the additional reduction in the gap given if any of the items $i \ge k+2$ become feasible, so clearly, $V_1^u \ge V_1$.

To determine the distribution of V_1^u, we start by considering scenarios where $L_1 \ge k+2$ is not possible, and thus $V_1^u = V_1$. For $G = g$ and $\boldsymbol{W_I} = \boldsymbol{w_I}$, an illustration of the drop gap, as determined by (g, w_1, w_k, w_{k+1}), is shown in Figure S1. The knapsack is shown at the top of the figure with items packed from left to right, and at the bottom the drop gap v_1 is shown as a function of w_1. The shape of the function is justified by considering different sizes of w_1. As long as w_1 is smaller than $w_k - g$, the gap given by removing the first item increases at unit rate. As soon as $w_1 = w_k - g$, item k becomes feasible and the gap jumps to zero. The gap then increases at unit rate, and another jump occurs when w_1 reaches $w_k - g + w_{k-1}$. The case shown in the figure satisfies $w_k - g + w_{k+1} + w_{k+2} > 1$. It can be seen that this is a sufficient condition for the event $L_1 \ge k+2$ to be impossible, since even if $w_1 = 1$, item $k+2$ cannot become feasible. It is for this reason that v_1 is uniquely determined by (g, w_1, w_k, w_{k+1}) here.

Continuing with the case shown in the figure, if we only condition on (g, w_k, w_{k+1}), we have by Lemma 3.4 that W_1 follows distribution $\mathcal{U}[0,1]$, meaning that the event $V_1 > v$ is given by the length of the bold regions on the w_1 axis. We explicitly describe the size of these regions. Assuming that $L_1 \le k+1$, we derive the following expression:

$$\begin{aligned} \mathbb{P}(V_1 > v | g, w_k, w_{k+1}, \overline{\mathcal{C}_{1n}}, L_1 \le k+1) \quad &= \quad (w_k - g) + (w_{k+1} - v)_+ + (1 - w_k + g - w_{k+1} - v)_+ \\ &\quad -(w_k - g + w_{k+1} - 1)_+, \quad v < g. \end{aligned} \tag{S79}$$

The first three terms in the expression come from the three bold regions shown in Figure S1. We have specified that $v < g$, so the length of the first segment is always $w_k - g$. For the second term, it is possible that $v > w_{k+1}$, so we only take the positive portion of $w_{k+1} - v$. In the third term, we take the positive portion to account for the cases where item $k+1$ does not become feasible, meaning $w_k - g + w_{k+1} > 1$, and if it is feasible, where v is greater than the height of the third peak, where $v > 1 - w_k + g - w_{k+1}$.

The last term is required for the case where item $k+1$ does not become feasible, as we must subtract the length of the bold region that potentially extends beyond $w_1 = 1$. Note that we always subtract one in this expression since it is not possible for the w_1 value, where $v_1 = v$ on the second peak, to be greater than one.

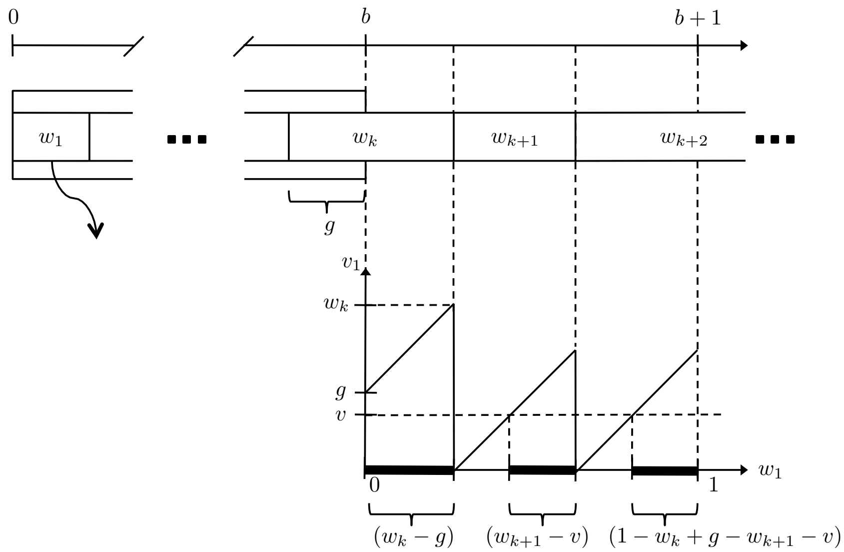

Figure S1: Gap v_1 as a function of w_1, parameterized by (g, w_k, w_{k+1}), resulting from the removal of the first item and assuming that $K = k$ with $2 \leq k \leq n-1$. The function starts at g and increases at unit rate, except at $w_1 = w_k - g$ and $w_1 = w_k - g + w_{k+1}$, where the function drops to zero. If we only condition on (g, w_k, w_{k+1}), the probability of the event $V_1 > v$ is given by the total length of the bold regions for $v < g$. Note that in the figure, $w_k - g + w_{k+1} < 1$ and the second two bold segments have positive length; these properties do not hold in general.

To see this, assume the contrary, so that $v + w_k - g > 1$. This inequality is obtained since, on the second peak, we have $v_1 = g - w_k + w_1$ and the w_1 value that satisfies $v_1 = v$ is equal to $v + w_k - g$. The statement $v + w_k - g > 1$, however, violates our previously stated assumption that $g < v$.

We now argue that we, in fact, have $V_1 \leq V_1^u$ with probability one, where

$$\begin{aligned}\mathbb{P}(V_1^u > v | g, w_k, w_{k+1}, \overline{\mathcal{C}_{1n}}) &= (w_k - g) + (w_{k+1} - v)_+ + (1 - w_k + g - w_{k+1} - v)_+ \\ &\quad -(w_k - g + w_{k+1} - 1)_+, \quad v < g. \end{aligned} \tag{S80}$$

We have simply replaced V_1 with V_1^u in (S79) and removed the condition $L_1 \leq k+1$. We already know that the expression is true for $L_1 \leq k+1$. For $L_1 \geq k+2$, we refer to Figure S1 and visualize the effect of a much smaller w_{k+2}, so that $w_k - g + w_{k+1} + w_{k+2} < 1$. This would yield four (or more) peaks in the v_1 function. To determine the probability of the event $V_1 > v$ while W_1 is random, we would have to evaluate the sizes of these extra peaks, which would be a function of w_{k+2}, w_{k+3}, etc. However, our definition of V_1^u does not account for the additional reductions in the gap given by items beyond $k+1$. We have already shown that $V_1 \leq V_1^u$, and now, clearly, V_1^u is a deterministic function of (G, W_1, W_k, W_{k+1}), and (S80) is justified.

We now evaluate the minimum of V_1^u and G and integrate over the conditioned variables. To begin, note that conditioning on the gap G makes V_1^u and G independent, so,

$$\mathbb{P}(V_1^u > v, G > v | \overline{\mathcal{C}_{1n}}, g, w_k, w_{k+1}) = \mathbb{P}(V_1^u > v | \overline{\mathcal{C}_{1n}}, g, w_k, w_{k+1}) \mathbb{1}(v < g). \tag{S81}$$

Marginalizing over W_{k+1}, which has uniform density according to Lemma 3.4, gives

$$\begin{aligned}\mathbb{P}(V_1^u > v, G > v|\overline{\mathcal{C}_{1n}}, g, w_k) &= \int_0^1 \mathbb{P}(V_1^u > v, g > v|\overline{\mathcal{C}_{1n}}, g, w_k, w_{k+1}) f_{W_{k+1}}(w_{k+1})\mathrm{d}w_{k+1} \\ &= \Big((w_k - g) + \frac{1}{2}(1-v)^2 - \frac{1}{2}(w_k - g)^2 \\ &\quad + \frac{1}{2}(1 - w_k + g - v)_+^2\Big)\mathbb{1}(v < g). \end{aligned} \tag{S82}$$

Using Lemma 3.3, we have

$$\begin{aligned}\mathbb{P}(V_1^u > v, G > v|\overline{\mathcal{C}_{1n}}, w_k) &= \int_0^{w_k} \mathbb{P}(V_1^u > v, G > v|\overline{\mathcal{C}_{1n}}, g, w_k) f_{G|\overline{\mathcal{C}_{1n}}, W_k}(g|\overline{\mathcal{C}_{1n}}, w_k)\mathrm{d}g \\ &= 1 - 2v - \frac{v}{w_k} + \frac{2v^2}{w_k} - \frac{v^3}{2w_k} + \frac{vw_k}{2}. \end{aligned} \tag{S83}$$

Finally, we integrate over W_k according to Lemma 3.2

$$\begin{aligned}\mathbb{P}(V_1^u > v, G > v|\overline{\mathcal{C}_{1n}}) &\leq \int_v^1 \mathbb{P}(V_1^u > v, G > v|\overline{\mathcal{C}_{1n}}, w_k) f_{W_k}(w_k)\mathrm{d}w_k \\ &= 1 - \frac{11v}{3} + 5v^2 - 3v^3 + \frac{2v^4}{3}. \end{aligned} \tag{S84}$$

This term is sufficient for calculating the expected value bound. ☐

Lemma S2.2. For $K = n$, the expected minimum gap satisfies

$$\mathbb{E}[V^*(n)|K = n] = \frac{1}{4}. \tag{S85}$$

Proof. We follow the same approach that we used for Lemma S2.1. Figure S2 shows the drop gap V_1 as a function of w_1, given w_n and g. The figure is justified using the same arguments that are in the proof of Lemma S2.1, but since no other items can become feasible, we can derive an exact expression for the probability of the event $V_1 > v$ when only conditioning on (g, w_n). Since W_1 has distribution $\mathcal{U}[0, 1]$ via Lemma 3.4, we can simply take the total length of the bold regions to find $\mathbb{P}(V_1 > v|\mathcal{C}_n, w_n, g)$. Thus,

$$\mathbb{P}(V_1 > v|\mathcal{C}_n, w_n, g) = (w_n - g) + (1 - w_n + g - v) = (1 - v), \qquad v < g, \tag{S86}$$

where we have that $1 - w_n + g - v$ is non-negative since $v < g$ and $w_n \leq 1$. To find the probability of the event $V_* > v$, we note that the events $V > v$ and $G > v$ are conditionally independent given $G = g$, so

$$\mathbb{P}(V > v, G > v|\mathcal{C}_n, w_n, g) = (1 - v)\mathbb{1}(v < g), \tag{S87}$$

Marginalizing over G using Lemma 3.3 gives

$$\begin{aligned}\mathbb{P}(V > v, G > v|\mathcal{C}_n, w_n) &= \int_0^1 \mathbb{P}(V > v, G > v|\mathcal{C}_n, w_n, g) f_{G|\mathcal{C}_n, W_n}(g|\mathcal{C}_n, w_n)\mathrm{d}g \\ &= \frac{(w_n - v)(1 - v)}{w_n}. \end{aligned} \tag{S88}$$

Noting the distribution of the critical item from Lemma 3.2,

$$\begin{aligned}\mathbb{P}(V > v, G > v|\mathcal{C}_n) &= \int_0^1 \mathbb{P}(V > v, G > v|\mathcal{C}_n, w_n) f_{W_n|\mathcal{C}_n}(w_n|\mathcal{C}_n)\mathrm{d}w_n \\ &= 1 - 3v + 3v^2 - v^3 = \mathbb{P}(V^* > v|\mathcal{C}_n). \end{aligned} \tag{S89}$$

This is sufficient for calculating the expected value. ☐

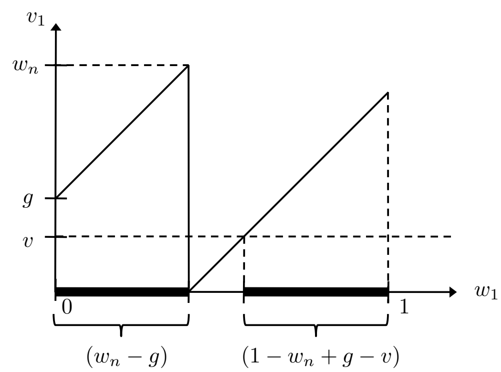

Figure S2: Drop gap v_1 as a function of w_1, parameterized by (w_n, g), resulting from the removal of the first item and assuming that the last item is critical ($K = n$). The function starts at g and increases at unit rate until $w_1 = w_n - g$, where it drops to zero, and then continues to increase at unit rate. If we only condition on $(w_n,\, g)$, the probability of the event $V_1 > v$ is given by the total length of the bold regions for $v < g$.

Lemma S2.3. For $K = 1$, the expected minimum gap satisfies

$$\mathbb{E}[V^*(n)|K = 1] \leq \frac{7}{30}. \tag{S90}$$

Proof. We use a more direct approach when the first item is critical, since W_1 no longer has a uniform distribution (from Lemma 3.2). However, the analysis here is similar to the proof of Lemma S2.1 in how we bound the drop gap. Note that we have $B = G$ for this case. Additionally, the gap given by the minimum gap will always be equal to the drop gap since Blind-Greedy does not pack any items. We define a variable V_1^u that satisfies $V_1 \leq V_1^u$ with probability one, where

$$V_1^u := \begin{cases} V_1, & \text{if } L_1 = 2 \vee L_1 = 3, \\ G - W_2 - W_3, & \text{if } L_1 \geq 4. \end{cases} \tag{S91}$$

We let the event $L_1 \geq 4$ also account for the case where $n = 3$, and the two remaining items are feasible. If, in fact, $n \geq 4$ and $L_1 \geq 4$, then V_1^u does not account for the additional reductions in the gap caused by more items becoming feasible. Thus we see that V_1^u is a deterministic function of (G, W_2, W_3).

To further simplify our expressions, we define $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_{4+}$ to be the events $L_1 = 2$, $L_1 = 3$, and $L_1 \geq 4$, respectively. Based on these cases, the drop gap bound V_1^u is given by the values shown in Table S1.

Table S1: Drop gap bound values when the first item is critical ($\mathcal{C}_1$).

Case	Defining inequalities	Minimum gap bound
$\mathcal{D}_2$	$W_2 > G$	$V_1^u = G$
$\mathcal{D}_3$	$W_2 \leq G, W_2 + W_3 > G$	$V_1^u = G - W_2$
$\mathcal{D}_{4+}$	$W_2 + W_3 \leq G$	$V_1^u = G - W_2 - W_3$

We begin by finding some necessary distributions for the cases. For case $\mathcal{D}_3$, the posterior distribution of W_2 is needed. We have

$$f_{W_2|\mathcal{C}_1,\mathcal{D}_3,G}(w_2|\mathcal{C}_1, \mathcal{D}_3, g) \;=\; \frac{\mathbb{P}(W_2 \leq G, W_2 + W_3 > G|\mathcal{C}_1, g, w_2) f_{W_2}(w_2)}{\mathbb{P}(W_2 \leq G, W_2 + W_3 > G|\mathcal{C}_1, g)}, \tag{S92}$$

where we have used Bayes' Theorem and that $f_{W_2|\mathcal{C}_1,G}(w_2, \mathcal{C}_1, g) = f_{W_2}(w_2) = \mathcal{U}[0, 1]$ by Lemma 3.4. For the numerator, we have

$$\mathbb{P}(W_2 \leq G, W_2 + W_3 > G|\mathcal{C}_1, g, w_2) = (1 - g + w_2)\mathbb{1}(w_2 \leq g), \tag{S93}$$

which follows using Lemma 3.4 for the distribution of W_3. Integrating over W_2 gives

$$\mathbb{P}(\mathcal{D}_3|\mathcal{C}_1,g) = \int_0^1 \mathbb{P}(W_2 \leq g, W_2+W_3 > G|\mathcal{C}_1,w_2) f_{W_2}(w_2)\mathrm{d}w_2 = g - \frac{g^2}{2}. \tag{S94}$$

Returning to the posterior distribution of W_2,

$$\begin{aligned} f_{W_2|\mathcal{C}_1,\mathcal{D}_3,G}(w_2|\mathcal{C}_1,\mathcal{D}_3,g) &= \frac{2(1-g+w_2)}{(2-g)g}, \quad 0 \leq w_2 \leq g, & \text{(S95)} \\ \mathbb{P}(W_2 \leq w_2|\mathcal{C}_1,\mathcal{D}_3,g) &= \frac{(2-2g+w_2)w_2}{(2-g)g}, \quad 0 \leq w_2 \leq g. & \text{(S96)} \end{aligned}$$

Moving to the case $\mathcal{D}_{4+}$, define $W' := W_2 + W_3$;

$$\mathbb{P}(\mathcal{D}_{4+}|\mathcal{C}_1,g) = \mathbb{P}(W' \leq g|\mathcal{C}_1,g) = \frac{g^2}{2}, \tag{S97}$$

where we have used that the distribution of W', conditioned on the first item being critical, is the distribution for the sum of two independent uniform random variables (via Lemma 3.4). Finally, for the posterior distribution of $W_2 + W_3$, we have

$$\mathbb{P}(W' \leq w'|\mathcal{C}_1,\mathcal{D}_{4+},g) = \frac{w'^2}{g^2}, \quad 0 \leq w' \leq g. \tag{S98}$$

We can now find distributions for V_1^u conditioned on all cases for the drop critical item. For case $\mathcal{D}_2$, clearly, $V_1^u = G$, and

$$\mathbb{P}(\mathcal{D}_2|\mathcal{C}_1,g) = \mathbb{P}(W_2 > g) = 1-g. \tag{S99}$$

For $\mathcal{D}_3$, we have

$$\begin{aligned} \mathbb{P}(V_1^u > v|\mathcal{C}_1,\mathcal{D}_3,g) &= \mathbb{P}(W_2 < G-v|\mathcal{C}_1,\mathcal{D}_3,g) \\ &= \frac{(2-g-v)(g-v)}{(2-g)g}, \quad 0 \leq v < g. & \text{(S100)} \end{aligned}$$

Then, for $\mathcal{D}_{4+}$,

$$\begin{aligned} \mathbb{P}(V_1^u > v|\mathcal{C}_1,\mathcal{D}_{4+},g) &= \mathbb{P}(W' < G-v|\mathcal{C}_1,\mathcal{D}_{4+},g) \\ &= \frac{(g-v)^2}{g^2}, \quad 0 \leq v < g. & \text{(S101)} \end{aligned}$$

Considering all three cases, we have

$$\begin{aligned} \mathbb{P}(V_1^u > v|\mathcal{C}_1,g) &= \mathbb{P}(V_1^u > v|\mathcal{C}_1,\mathcal{D}_2,g)\mathbb{P}(\mathcal{D}_2|\mathcal{C}_1,g) + \mathbb{P}(V_1^u > v|\mathcal{C}_1,\mathcal{D}_3,g)\mathbb{P}(\mathcal{D}_3|\mathcal{C}_1,g) \\ &\quad + \mathbb{P}(V_1^u > v|\mathcal{C}_1,\mathcal{D}_{4+},g)\mathbb{P}(\mathcal{D}_{4+}|\mathcal{C}_1,g) \\ &= (1-v-gv+v^2)\mathbb{1}(v<g). & \text{(S102)} \end{aligned}$$

This gives the expected value bound

$$\mathbb{E}[V_*|\mathcal{C}_1,g] \leq \mathbb{E}[V_1^u|\mathcal{C}_1,g] = g - \frac{g^2}{2} - \frac{g^3}{6}. \tag{S103}$$

Finally, integrating over G using Theorem 3.1,

$$\mathbb{E}[V_*|\mathcal{C}_1] \leq \int_0^1 \mathbb{E}[V_*|\mathcal{C}_1,g] f_G(g)\mathrm{d}g = \frac{7}{30}. \tag{S104}$$

□

The final result for the subset sum problem follows easily from the stated lemmas.

Proof of Theorem 5.1 Using the above lemmas and noting that the events $\mathcal{C}_1$, $\overline{\mathcal{C}_{1n}}$, and C_n form a partition of the event $\sum_{i \in I} W_i > B$, the result follows using the total expectation theorem and Lemma 3.1. $\square$

Consecutive Rollout: 0-1 Knapsack Problem Analysis

The analysis of the consecutive rollout algorithm for the 0-1 knapsack problem follows the same structure as the analysis for the subset sum problem, and makes use of the properties described in Section 3. The development here assumes that the reader is familiar with the subset sum analysis, so less detail is presented.

We use the same definition of the drop critical item L_1 that was used on the subset sum problem. From the algorithm description of Consecutive-Rollout and the gain definition in (7), we have that the gain $Z_*(n)$ satisfies

$$Z_*(n) = \max\left(0, \sum_{i=2}^{L_1-1} P_i - \sum_{i=1}^{K-1} P_i\right). \tag{S105}$$

We will sometimes write $Z_*(n)$ simply as Z_*. The following three lemmas bound the gain $Z_*(n)$ for different cases of the critical item, assuming $n \geq 3$. Theorem 5.2 then follows easily. We implicitly assume that $\sum_{i=1}^{n} W_i > B$ holds for the section.

Lemma S2.4. For $K = n$, the expected gain satisfies

$$\mathbb{E}[Z_*(n)|K = n] = \frac{1}{9}. \tag{S106}$$

Proof. A positive gain can only obtained in the case where the last item becomes feasible when removing the first. Consistent with our subset sum notation, let $\mathcal{D}_{n+1}$ denote the event that item n becomes feasible when the first item is removed. Using Lemma 3.4 and the perspective of Figure 1, this probability is given by

$$\mathbb{P}(\mathcal{D}_{n+1}|g, w_n, \mathcal{C}_n) = \mathbb{P}(W_1 \geq W_n - G|w_n, g, \mathcal{C}_n) = (1 - w_n + g). \tag{S107}$$

Integrating over G using Lemma 3.3 and W_n using Lemma 3.4 gives

$$\mathbb{P}(\mathcal{D}_{n+1}|\mathcal{C}_n, w_n) = \int_0^{w_n} (1 - w_n + g)\frac{1}{w_n}\mathrm{d}g = 1 - \frac{w_n}{2}, \tag{S108}$$

$$\mathbb{P}(\mathcal{D}_{n+1}|\mathcal{C}_n) = \int_0^1 \left(1 - \frac{w_n}{2}\right) 2w_n \mathrm{d}w_n = \frac{2}{3}. \tag{S109}$$

Now, assuming that item n becomes feasible, we are interested in the case where it provides a larger value. This is simply given by the probability

$$\mathbb{P}(P_n \geq P_1) = \frac{1}{2}, \tag{S110}$$

following from the symmetry of the distributions of P_1 and P_n. Conditioned on the event $P_n \geq P_1$, we are interested in the distribution of the gain, which is equal to $P_n - P_1$. We have

$$\mathbb{P}(P_n - P_1 \leq q|P_n \geq P_1) = \frac{\mathbb{P}(0 \leq P_n - P_1 \leq q)}{\mathbb{P}(P_n \geq P_1)}. \tag{S111}$$

For the numerator,

$$\mathbb{P}(0 \leq P_n - P_1 \leq q) \quad = \quad \int_0^{1-q}\int_{p_1}^{p_1+q} \mathrm{d}p_n \mathrm{d}p_1 + \int_{1-q}^{1}\int_{p_1}^{1} \mathrm{d}p_n \mathrm{d}p_1 = q - \frac{q^2}{2}, \tag{S112}$$

which gives

$$\mathbb{P}(P_n - P_1 \le q | P_n \ge P_1) = 2q - q^2, \tag{S113}$$

$$\mathbb{E}[P_n - P_1 | P_n \ge P_1] = \frac{1}{3}. \tag{S114}$$

Finally, by the independence of item weight and profit, we have

$$\mathbb{E}[Z_*(n)|\mathcal{C}_n] = \mathbb{E}[P_n - P_1 | P_n \ge P_1]\mathbb{P}(P_n \ge P_1)\mathbb{P}(\mathcal{D}_{n+1}|\mathcal{C}_n) = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{9}. \tag{S115}$$

□

Lemma S2.5. For $2 \le K \le n-1$, the expected gain satisfies

$$\mathbb{E}[Z_*(n)|2 \le K \le n] \ge \frac{59}{288} \approx 0.205. \tag{S116}$$

Proof. We again let $\overline{\mathcal{C}_{1n}}$ be the event that $2 \le K \le n-1$. We fix $K = k$, and the proof holds for all valid values of k. In the case of event $\overline{\mathcal{C}_{1n}}$, it is possible that removing the first item allows for the critical item to become feasible, as well as additional items (i.e. $L_1 \ge k+2$). However, we are only guaranteed the existence of one item beyond the critical item, since it is possible that $k = n-1$. Let $\mathcal{D}_{k+1}$ indicate the event $L_1 = k+1$ and let $\mathcal{D}_{(k+2)+}$ indicate the event $L_1 \ge k+2$. If item $k+2$ does not exist (i.e. $k = n-1$), then this event means that all remaining items are packed.

For the probability of the event $\mathcal{D}_{k+1}$, we have from Lemma 3.4 that W_1 has distribution $\mathcal{U}[0,1]$. Then,

$$\mathbb{P}(\mathcal{D}_{k+1}|g, w_k, w_{k+1}, \overline{\mathcal{C}_{1n}}) = w_{k+1} - (w_k - g + w_{k+1} - 1)_+. \tag{S117}$$

This can be argued using an illustration similar to Figure S1, where the second term mitigates that case where w_{k+1} extends beyond $b+1$ for $B = b$. Likewise, for the event $\mathcal{D}_{(k+2)+}$, we have

$$\mathbb{P}(\mathcal{D}_{(k+2)+}|g, w_k, w_{k+1}, \overline{\mathcal{C}_{1n}}) = (1 - w_k + g - w_{k+1})_+. \tag{S118}$$

Starting with event $\mathcal{D}_{k+1}$, we integrate over W_{k+1}, which has uniform density by Lemma 3.4.

$$\begin{aligned}
\mathbb{P}(\mathcal{D}_{k+1}|g, w_k, \overline{\mathcal{C}_{1n}}) &= \int_0^1 \mathbb{P}(\mathcal{D}_{k+1}|g, w_k, w_{k+1}, \overline{\mathcal{C}_{1n}}) f_{W_{k+1}}(w_{k+1})\mathrm{d}w_{k+1} \\
&= \int_0^1 w_{k+1}\mathrm{d}w_{k+1} - \int_{1+g-w_k}^1 (w_k - g + w_{k+1} - 1)\mathrm{d}w_{k+1} \\
&= \frac{1}{2} - \frac{g^2}{2} + g w_k - \frac{w_k^2}{2}.
\end{aligned} \tag{S119}$$

Marginalizing over G with Lemma 3.3 gives

$$\begin{aligned}
\mathbb{P}(\mathcal{D}_{k+1}|w_k, \overline{\mathcal{C}_{1n}}) &= \int_0^1 \mathbb{P}(\mathcal{D}_{k+1}|g, w_k, \overline{\mathcal{C}_{1n}}) f_{G|W_k}(g|w_k)\mathrm{d}g \\
&= \int_0^{w_k} \left(\frac{1}{2} - \frac{g^2}{2} + g w_k - \frac{w_k^2}{2}\right)\frac{1}{w_k}\mathrm{d}g \\
&= \frac{1}{2} - \frac{w_k^2}{6}.
\end{aligned} \tag{S120}$$

Finally by Lemma 3.2,

$$\mathbb{P}(\mathcal{D}_{k+1}|\overline{\mathcal{C}_{1n}}) = \int_0^1 \mathbb{P}(\mathcal{D}_{k+1}|w_k, \overline{\mathcal{C}_{1n}}) f_{W_k}(w_k)\mathrm{d}w_k = \int_0^1 \left(\frac{1}{2} - \frac{w_k^2}{6}\right) 2w_k\mathrm{d}w_k = \frac{5}{12}. \tag{S121}$$

Now, for the event $\mathcal{D}_{(k+2)+}$, we integrate in the same order, using the same lemmas.

$$\begin{aligned}\mathbb{P}(\mathcal{D}_{(k+2)+}|g,w_k,\overline{\mathcal{C}_{1n}}) &= \int_0^1 \mathbb{P}(\mathcal{D}_{(k+2)+}|g,w_k,w_{k+1},\overline{\mathcal{C}_{1n}})f_{W_{k+1}}(w_{k+1})\mathrm{d}w_{k+1} \\ &= \int_0^{1-w_k+g}(1-w_k+g-w_{k+1})\mathrm{d}w_{k+1} \\ &= \frac{1}{2}+g+\frac{g^2}{2}-w_k-gw_k+\frac{w_k^2}{2}. \end{aligned} \tag{S122}$$

$$\begin{aligned}\mathbb{P}(\mathcal{D}_{(k+2)+}|w_k,\overline{\mathcal{C}_{1n}}) &= \int_0^1 \mathbb{P}(\mathcal{D}_{(k+2)+}|g,w_k,\overline{\mathcal{C}_{1n}})f_{G|W_k}(g|w_k)\mathrm{d}g \\ &= \int_0^{w_k}\left(\frac{1}{2}+g+\frac{g^2}{2}-w_k-gw_k+\frac{w_k^2}{2}\right)\frac{1}{w_k}\mathrm{d}g \\ &= \frac{1}{2}-\frac{w_k}{2}+\frac{w_k^2}{6}. \end{aligned} \tag{S123}$$

$$\mathbb{P}(\mathcal{D}_{(k+2)+}|\overline{\mathcal{C}_{1n}}) = \int_0^1 \mathbb{P}(\mathcal{D}_{(k+2)+}|\overline{\mathcal{C}_{1n}},w_k)f_{W_k}(w_k)\mathrm{d}w_k = \int_0^1\left(\frac{1}{2}-\frac{w_k}{2}+\frac{w_k^2}{6}\right)2w_k\mathrm{d}w_k = \frac{1}{4}. \tag{S124}$$

Equipped with these probabilities, we now consider the gain from the rollout for the different drop critical item cases. For the case where only one item becomes feasible ($\mathcal{D}_{k+1}$), the analysis in the previous lemma holds, so we have

$$\mathbb{E}[P_n-P_1|\overline{\mathcal{C}_{1n}},\mathcal{D}_{k+1}] = \mathbb{E}[P_n-P_1|P_n>P_1]\mathbb{P}(P_n>P_1)\mathbb{P}(\mathcal{D}_{k+1}|\overline{\mathcal{C}_{1n}}) = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{5}{12}=\frac{5}{72}. \tag{S125}$$

If two or more items become feasible ($\mathcal{D}_{(k+1)+}$), we only consider the gain resulting from adding two items, and this serves as a lower bound for the case of more items becoming feasible. Accordingly, define

$$P' := P_k + P_{k+1}. \tag{S126}$$

The probability that the profits of the two items are greater than P_1 is given by

$$\mathbb{P}(P'\geq P_1) = 1-\mathbb{P}(P'<P_1) = 1-\int_0^1\int_0^{p_1}p'\mathrm{d}p'\mathrm{d}p_1 = 1-\int_0^1\frac{p_1^2}{2}\mathrm{d}p_1 = \frac{5}{6}. \tag{S127}$$

The gain conditioned on the event $P'>P_1$ is given by

$$\mathbb{P}(P'-P_1\leq q|P'\geq P_1) = \frac{\mathbb{P}(0\leq P'-P_1\leq q)}{\mathbb{P}(P'\geq P_1)}. \tag{S128}$$

Proceeding with the numerator and assuming $0\leq q\leq 1$,

$$\begin{aligned}\mathbb{P}(0\leq P'-P_1\leq q) &= \int_0^{1-q}\int_{p_1}^{p_1+q}p'\mathrm{d}p'\mathrm{d}p_1+\int_{1-q}^1\int_{p_1}^1 p'\mathrm{d}p'\mathrm{d}p_1+\int_{1-q}^1\int_1^{p_1+q}(2-p')\mathrm{d}p'\mathrm{d}p_1 \\ &= \int_0^{1-q}\left(p_1q+\frac{q^2}{2}\right)\mathrm{d}p_1+\int_{1-q}^1\left(\frac{1}{2}-\frac{p_1^2}{2}\right)\mathrm{d}p_1 \\ &\quad+\int_{1-q}^1\left(-\frac{3}{2}+2p_1-\frac{p_1^2}{2}+2q-p_1q-\frac{q^2}{2}\right)\mathrm{d}p_1 \\ &= \frac{q}{2}+\frac{q^2}{2}-\frac{q^3}{3},\ 0\leq q\leq 1. \end{aligned} \tag{S129}$$

Now, for $1 < q \leq 2$,

$$\begin{aligned}
\mathbb{P}(0 \leq P' - P_1 \leq q) &= \int_0^1 \int_{p_1}^1 p' \mathrm{d}p' \mathrm{d}p_1 + \int_0^{2-q} \int_1^{p_1+q} (2-p')\mathrm{d}p'\mathrm{d}p_1 + \int_{2-q}^1 \int_1^2 (2-p')\mathrm{d}p'\mathrm{d}p_1 \\
&= \int_0^1 \left(\frac{1}{2} - \frac{p_1^2}{2}\right) \mathrm{d}p_1 + \int_0^{2-q} \left(-\frac{3}{2} + 2p_1 - \frac{p_1^2}{2} + 2q - p_1 q - \frac{q^2}{2}\right) \mathrm{d}p_1 \\
&\quad + \frac{1}{2}\int_{2-q}^1 \mathrm{d}p_1 \\
&= -\frac{1}{2} + 2q - q^2 + \frac{q^3}{6},\ 1 < q \leq 2. \qquad \text{(S130)}
\end{aligned}$$

The distribution for the gain is thus given by

$$\mathbb{P}(P' - P_1 \leq q | P' - P_1 \geq 0) = \begin{cases} \frac{3}{5}q + \frac{3}{5}q^2 - \frac{2}{5}q^3, & 0 \leq q \leq 1, \\ -\frac{3}{5} + \frac{12}{5}q - \frac{6}{5}q^2 + \frac{1}{5}q^3, & 1 < q \leq 2. \end{cases} \qquad \text{(S131)}$$

The expected value is

$$\mathbb{E}[P' - P_1 | P' - P_1 \geq 0] = \int_0^1 q\left(\frac{3}{5} + \frac{6}{5}q - \frac{6}{5}q^2\right) \mathrm{d}q + \int_1^2 q\left(\frac{12}{5} - \frac{12}{5}q + \frac{3}{5}q^2\right) \mathrm{d}q = \frac{13}{20}. \qquad \text{(S132)}$$

Recalling that it is possible for more than two items to be added in the case $\mathcal{D}_{(k+2)+}$, let P'' be the total value of items added for the case. We may bound the expected gain as follows, where the term $\mathbb{P}(\mathcal{D}_k|\overline{\mathcal{C}_{1n}})$ is omitted since it provides zero gain. We are implicitly using the fact that item weights and profits are independent.

$$\begin{aligned}
\mathbb{E}[Z_*(n)|\overline{\mathcal{C}_{1n}}] &= \mathbb{E}[P'' - P_1 | P'' > P_1]\mathbb{P}(P'' \geq P_1)\mathbb{P}(\mathcal{D}_{(k+2)+}|\overline{\mathcal{C}_{1n}}) \\
&\quad + \mathbb{E}[P_n - P_1 | P_n \geq P_1]\mathbb{P}(P_n \geq P_1)\mathbb{P}(\mathcal{D}_{k+1}|\overline{\mathcal{C}_{1n}}) \\
&\geq \mathbb{E}[P' - P_1 | P' > P_1]\mathbb{P}(P' \geq P_1)\mathbb{P}(\mathcal{D}_{(k+2)+}|\overline{\mathcal{C}_{1n}}) \\
&\quad + \mathbb{E}[P_n - P_1 | P_n \geq P_1]\mathbb{P}(P_n \geq P_1)\mathbb{P}(\mathcal{D}_{k+1}|\overline{\mathcal{C}_{1n}}) \\
&= \frac{13}{20} \cdot \frac{5}{6} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{5}{12} = \frac{59}{288}. \qquad \text{(S133)}
\end{aligned}$$

□

Lemma S2.6. For $K = 1$, the expected gain satisfies

$$\mathbb{E}[Z_*(n)|K = 1] \geq \frac{5}{24} \approx 0.208. \qquad \text{(S134)}$$

Proof. We use the drop events $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_{4+}$ just as we did for the subset sum problem. The event probabilities given $G = g$ are the same as those for the subset sum problem. Accordingly,

$$\mathbb{P}(\mathcal{D}_2|\mathcal{C}_1) = \int_0^1 \mathbb{P}(\mathcal{D}_2|\mathcal{C}_1, g) f_G(g)\mathrm{d}g = \int_0^1 (1-g)(2-2g)\mathrm{d}g = \frac{2}{3}, \qquad \text{(S135)}$$

$$\mathbb{P}(\mathcal{D}_3|\mathcal{C}_1) = \int_0^1 \mathbb{P}(\mathcal{D}_3|\mathcal{C}_1, g) f_G(g)\mathrm{d}g = \int_0^1 \left(g - \frac{g^2}{2}\right)(2-2g)\mathrm{d}g = \frac{1}{4}, \qquad \text{(S136)}$$

$$\mathbb{P}(\mathcal{D}_{4+}|\mathcal{C}_1) = \frac{1}{12}. \qquad \text{(S137)}$$

The greedy solution gives zero value, so the expected gain is easily determined using independence of item weights and profits,

$$\mathbb{E}[Z_*|\mathcal{C}_1, \mathcal{D}_2] = 0, \qquad \text{(S138)}$$

$$\mathbb{E}[Z_*|\mathcal{C}_1, \mathcal{D}_3] = \mathbb{E}[P_2] = \frac{1}{2}, \tag{S139}$$

$$\mathbb{E}[Z_*|\mathcal{C}_1, \mathcal{D}_{4+}] \geq \mathbb{E}[P_2 + P_3] = 1. \tag{S140}$$

Combining all cases for the drop critical item,

$$\begin{aligned}
\mathbb{E}[Z_*|\mathcal{C}_1] &= \mathbb{E}[Z_*|\mathcal{C}_1, \mathcal{D}_3]\mathbb{P}(\mathcal{D}_3|\mathcal{C}_1) + \mathbb{E}[Z_*|\mathcal{C}_1, \mathcal{D}_{4+}]\mathbb{P}(\mathcal{D}_{4+}|\mathcal{C}_1) \\
&\geq \frac{1}{2} \cdot \frac{1}{4} + 1 \cdot \frac{1}{12} = \frac{5}{24}.
\end{aligned} \tag{S141}$$

□

The result for the 0-1 knapsack problem then follows.

Proof of Theorem 5.2. The events $\mathcal{C}_1$, $\overline{\mathcal{C}_{1n}}$, and $\mathcal{C}_n$ form a partition of the event $\sum_{i=1}^n W_i > B$, so using Lemma 3.1 gives

$$\begin{aligned}
\mathbb{E}\left[Z_*(n) \middle| \sum_{i=1}^n W_i > B\right] &= \mathbb{E}[Z_*(n)|\mathcal{C}_1]\mathbb{P}(\mathcal{C}_1) + \mathbb{E}[Z_*(n)|\overline{\mathcal{C}_{1n}}]\mathbb{P}(\overline{\mathcal{C}_{1n}}) + \mathbb{E}[Z_*(n)|\mathcal{C}_n]\mathbb{P}(\mathcal{C}_n) \\
&\geq \frac{5}{24}\left(\frac{1}{n}\right) + \frac{59}{288}\left(\frac{n-2}{n}\right) + \frac{1}{9}\left(\frac{1}{n}\right) = \frac{-26 + 59n}{288n}.
\end{aligned} \tag{S142}$$

□

S3 Evaluation of Integrals

The following lemma is used in the evaluation of integrals described in this section.

Lemma S3.1. For constant values κ_1, κ_2 and non-negative integer θ,

$$\int \frac{(\kappa_1 + \kappa_2 x)^\theta}{x} \mathrm{d}x = \kappa_1^\theta \log(x) + \sum_{j=1}^{\theta} \frac{\kappa_1^{\theta-j}(\kappa_1 + \kappa_2 x)^j}{j}. \tag{S143}$$

Proof. We begin by noting that

$$\begin{aligned}
\int \frac{(\kappa_1 + \kappa_2 x)^\theta}{x} \mathrm{d}x &= \int \kappa_2(\kappa_1 + \kappa_2 x)^{\theta-1} \mathrm{d}x + \int \frac{\kappa_1(\kappa_1 + \kappa_2 x)^{\theta-1}}{x} \mathrm{d}x \\
&= \frac{(\kappa_1 + \kappa_2 x)^\theta}{\theta} + \kappa_1 \int \frac{(\kappa_1 + \kappa_2 x)^{\theta-1}}{x} \mathrm{d}x.
\end{aligned} \tag{S144}$$

The statement of the lemma clearly holds for $\theta = 0$. Assuming that it holds for $\theta = t$, we have for $\theta = t+1$,

$$\begin{aligned}
\int \frac{(\kappa_1 + \kappa_2 x)^{t+1}}{x} \mathrm{d}x &= \frac{(\kappa_1 + \kappa_2 x)^{t+1}}{t+1} + \kappa_1 \left(\kappa_1^t \log(x) + \sum_{j=1}^{t} \frac{\kappa_1^{t-j}(\kappa_1 + \kappa_2 x)^j}{j}\right) \\
&= \kappa_1^{t+1} \log(x) + \sum_{j=1}^{t+1} \frac{\kappa_1^{t+1-j}(\kappa_1 + \kappa_2 x)^j}{j}.
\end{aligned} \tag{S145}$$

The property then holds for all θ by induction.

□

S3.1 Evaluation of Integral (35)

To simplify expressions, we use $A := W_{K-1}$. Moreover, recall that $M := n - K$. The integral is

$$\mathbb{P}(V_*^u > v|m, \overline{\mathcal{C}_1}) = \int_0^1 \int_0^1 \mathbb{P}(G > v|a, g, \overline{\mathcal{C}_1}) \mathbb{P}(\widetilde{V}^u > v|a, g, \overline{\mathcal{C}_1}) \left(\mathbb{P}(V^u > v|a, g, \overline{\mathcal{C}_1})\right)^m f_A(a) f_G(g) \mathrm{d}a\mathrm{d}g. \tag{S146}$$

This may be evaluated by considering regions where the arguments have simple analytical descriptions as a function of a and g. We begin by noting that $\mathbb{P}(G > v|a, g, \overline{\mathcal{C}_1}) = \mathbb{1}(v < g)$, so we may restrict our analysis to regions where $v < g$. For the integral evaluation of $(a, g) \in R_j$, we use the notation

$$\rho_j(v, m) = \iint_{R_j} \mathbb{P}(G > v|a, g, \overline{\mathcal{C}_1}) \mathbb{P}(\widetilde{V}^u > v|a, g, \overline{\mathcal{C}_1}) \left(\mathbb{P}(V^u > v|a, g, \overline{\mathcal{C}_1})\right)^m f_A(a) f_G(g) \mathrm{d}a\mathrm{d}g. \tag{S147}$$

The relevant regions are shown in Figure S3, where different enumerations are necessary for $v \leq \frac{1}{2}$ and $v > \frac{1}{2}$. The values of $(\mathbb{P}(V^u > v|a, g, \overline{\mathcal{C}_1}))^m$ and $\mathbb{P}(\widetilde{V}^u > v|a, g, \overline{\mathcal{C}_1})$ are shown in Table S2. Note that, in many cases, the $\frac{1}{1-g}$ factor from $\mathbb{P}(G > v|a, g, \overline{\mathcal{C}_1})$ cancels with the $(1-g)$ factor from $f_G(g)$, which simplifies the expression.

Table S2: Arguments of (35) for regions shown in Figure S3.

Region	$(\mathbb{P}(V^u > v\|a, g, \overline{\mathcal{C}_1}))^m$	$\mathbb{P}(\widetilde{V}^u > v\|a, g, \overline{\mathcal{C}_1})$
R_1	$(1 - v - a)^m$	$(1 - g - a)/(1 - g)$
R_2	$(g - v)^m$	0
R_3	$(1 - 2v)^m$	$(1 - g - v)/(1 - g)$
R_4	$(a + g - 2v)^m$	$(a - v)/(1 - g)$
R_5	$(a + g - 2v)^m$	$(a - v)/(1 - g)$
R_6	$(1 - v)^m$	1
R_7	$(1 - v - a)^m$	$(1 - g - a)/(1 - g)$
R_8	$(g - v)^m$	0
R_9	$(a + g - 2v)^m$	$(a - v)/(1 - g)$
R_{10}	$(1 - v)^m$	1

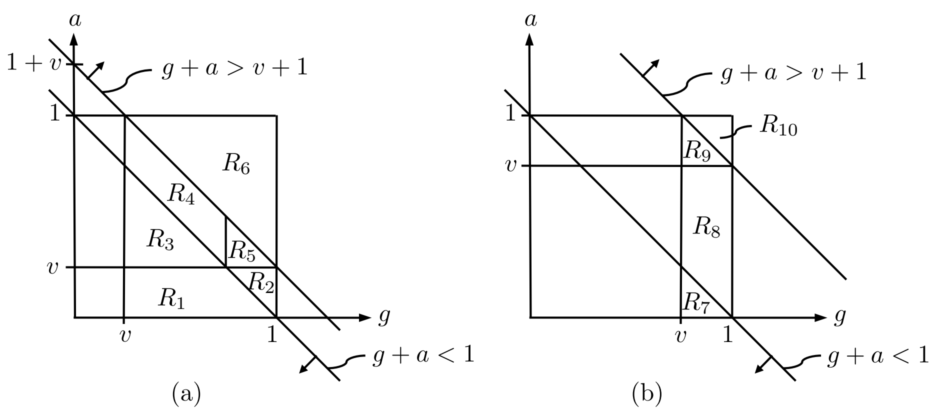

Figure S3: Integration regions for (a) $v \leq \frac{1}{2}$ and (b) $v > \frac{1}{2}$.

Regions 1-6 correspond to the case where $v \leq \frac{1}{2}$.

$$\begin{aligned}\rho_1(v,m) &= \int_0^v \int_v^{1-a} 2(1-v-a)^m(1-g-a)\mathrm{d}g\mathrm{d}a = \int_0^v (1-a-v)^{2+m}\mathrm{d}a \\ &= \frac{-(1-2v)^{3+m}+(1-v)^{3+m}}{3+m}. \end{aligned} \tag{S148}$$

$$\rho_2(v,m) = 0. \tag{S149}$$

$$\begin{aligned}\rho_3(v,m) &= \int_v^{1-v} \int_v^{1-a} 2(1-2v)^m(1-g-v)\mathrm{d}g\mathrm{d}a = \int_v^{1-v} (3v-a-1)(1-2v)^m(v+a-1)\mathrm{d}a \\ &= \frac{2}{3}(1-2v)^{3+m}. \end{aligned} \tag{S150}$$

$$\begin{aligned}\rho_4(v,m) &= \int_v^{1-v} \int_{1-g}^{v+1-g} 2(a+g-2v)^m(a-v)\mathrm{d}a\mathrm{d}g \\ &= \frac{1}{(1+m)(2+m)} \int_v^{1-v} \big(-2(1-2v)^{1+m} + 4g(1-2v)^{1+m} - 2m(1-2v)^{1+m} + 2gm(1-2v)^{1+m} \\ &\quad +2(1-v)^{1+m} - 4g(1-v)^{1+m} + 2m(1-v)^{1+m} - 2gm(1-v)^{1+m} + 2m(1-2v)^{1+m}v \\ &\quad + 2(1-v)^{1+m}v\big)\,\mathrm{d}g \\ &= \frac{1}{(1+m)(2+m)} \big(m(1-2v)^{3+m} + m(1-v)^m + 2(1-v)^m v - 3m(1-v)^m v \\ &\quad -6(1-v)^m v^2 + 2m(1-v)^m v^2 + 4(1-v)^m v^3\big)\,. \end{aligned} \tag{S151}$$

$$\begin{aligned}\rho_5(v,m) &= \int_{1-v}^{1} \int_v^{1+v-g} 2(a+g-2v)^m(a-v)\mathrm{d}a\mathrm{d}g \\ &= \frac{1}{(1+m)(2+m)} \int_{1-v}^{1} \big(2(1-v)^{1+m} - 4g(1-v)^{1+m} + 2m(1-v)^{1+m} - 2gm(1-v)^{1+m} \\ &\quad +2(g-v)^{2+m} + 2(1-v)^{1+m}v\big)\,\mathrm{d}g \\ &= \frac{1}{(1+m)(2+m)(3+m)} \big(-2(1-2v)^{3+m} + 2(1-v)^{1+m} - 10(1-v)^{1+m}v - 2m(1-v)^{1+m}v \\ &\quad +14(1-v)^{1+m}v^2 + 7m(1-v)^{1+m}v^2 + m^2(1-v)^{1+m}v^2\big)\,. \end{aligned} \tag{S152}$$

$$\rho_6(v,m) = \int_v^1 \int_{1+v-g}^{1} (1-v)^m(2-2g)\mathrm{d}a\mathrm{d}g = \int_v^1 (2-2g)(1-v)^m(g-v)\mathrm{d}g = \frac{1}{3}(1-v)^{3+m}. \tag{S153}$$

Summing all terms of $\mathbb{P}(V_*^u > v|m, \overline{\mathcal{C}_1})$ for $v \leq \frac{1}{2}$ gives

$$\begin{aligned}\mathbb{P}(V_*^u > v|m, \overline{\mathcal{C}_1})_{\leq \frac{1}{2}} &:= \rho_1(v,m)+\rho_2(v,m)+\rho_3(v,m)+\rho_4(v,m)+\rho_5(v,m)+\rho_6(v,m) \\ &= \frac{1}{3(3+m)} \big(2m(1-2v)^m + m(1-v)^m + 9(1-v)^{3+m} - 12m(1-2v)^m v \\ &\quad -3m(1-v)^m v + 24m(1-2v)^m v^2 + 3m(1-v)^m v^2 - 16m(1-2v)^m v^3 \\ &\quad -m(1-v)^m v^3\big)\,. \end{aligned} \tag{S154}$$

Regions 7-10 are for the case $v > \frac{1}{2}$.

$$\begin{aligned}\rho_7(v,m) &= \int_v^1 \int_0^{1-g} 2(1-v-a)^m(1-g-a)\mathrm{d}a\mathrm{d}g \\ &= \frac{1}{(1+m)(2+m)} \int_v^1 \big(2(1-v)^{1+m} - 4g(1-v)^{1+m} + 2m(1-v)^{1+m} - 2gm(1-v)^{1+m} \\ &\quad +2(g-v)^{2+m} + 2(1-v)^{1+m}v\big)\,\mathrm{d}g = \frac{(1-v)^{3+m}}{3+m}. \end{aligned} \tag{S155}$$

$$\rho_8(v,m) = 0. \tag{S156}$$

$$\begin{aligned}\rho_9(v,m) &= \int_v^1 \int_v^{1+v-g} 2(a+g-2v)^m(a-v)\mathrm{d}a\mathrm{d}g \\ &= \frac{1}{(1+m)(2+m)} \int_v^1 \big(2(1-v)^{1+m} - 4g(1-v)^{1+m} + 2m(1-v)^{1+m} - 2gm(1-v)^{1+m} \\ &\quad +2(g-v)^{2+m} + 2(1-v)^{1+m}v\big)\,\mathrm{d}g = \frac{(1-v)^{3+m}}{3+m}. \end{aligned} \tag{S157}$$

$$\rho_{10}(v,m) = \int_v^1 \int_{1+v-g}^1 (1-v)^m(2-2g)\mathrm{d}a\mathrm{d}g = \int_v^1 (2-2g)(1-v)^m(g-v)\mathrm{d}g = \frac{1}{3}(1-v)^{3+m}. \tag{S158}$$

Summing these terms yields, for $v > \frac{1}{2}$,

$$\begin{aligned}\mathbb{P}(V_*^u > v|m,\overline{\mathcal{C}_1})_{>\frac{1}{2}} &:= \rho_7(v,m) + \rho_8(v,m) + \rho_9(v,m) + \rho_{10}(v,m) \\ &= \frac{1}{3}(1-v)^{3+m} + \frac{2(1-v)^{3+m}}{3+m}. \end{aligned} \tag{S159}$$

In summary, we have

$$\mathbb{P}(V_*^u > v|m,\overline{\mathcal{C}_1}) = \begin{cases} \mathbb{P}(V_*^u > v|m,\overline{\mathcal{C}_1})_{\leq\frac{1}{2}}, & v \leq \frac{1}{2}, \\ \mathbb{P}(V_*^u > v|m,\overline{\mathcal{C}_1})_{>\frac{1}{2}}, & v > \frac{1}{2}. \end{cases} \tag{S160}$$

S3.2 Evaluation of Integral (S34)

We wish to evaluate

$$\mathbb{P}(Z_*^l \leq z|m,\overline{\mathcal{C}_1})_{\mathcal{E}} := \int_0^1 \int_0^{1-g} \mathbb{P}(Z_*^l \leq z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}} f_A(a) f_G(g)\mathrm{d}a\mathrm{d}g, \tag{S161}$$

where

$$\begin{aligned}\mathbb{P}(Z_*^l \leq z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}} &= z(1-g+gz)^m + \frac{(1-g+gz)^{m+1}(1-g+m-gm-gz)}{(1-g)a(m+1)(m+2)} \\ &\quad - \frac{(1-g+gz+a(1-z))^{m+1}\,(1-g+m-gm-gz+a(-1-m+z+mz))}{(1-g)a(m+1)(m+2)}. \end{aligned} \tag{S162}$$

We first determine the following using the fact that A follows distribution $\mathcal{U}[0,1]$

$$\int \mathbb{P}(Z_*^l \leq z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}} f_A(a)\mathrm{d}a = \int \mathbb{P}(Z_*^l \leq z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}}\mathrm{d}a. \tag{S163}$$

The following constants simplify the expression:

$$\begin{aligned}
\lambda_1 &:= 1-g+gz, && \text{(S164)}\\
\lambda_2 &:= z-1, && \text{(S165)}\\
\lambda_3 &:= 1-g+m-gm-gz, && \text{(S166)}\\
\lambda_4 &:= -1-m+z+mz, && \text{(S167)}\\
\lambda_5 &:= \frac{-1}{(1-g)(m+1)(m+2)}. && \text{(S168)}
\end{aligned}$$

This gives

$$\begin{aligned}
\int \mathbb{P}(Z_*^l \le z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}}\mathrm{d}a &= \int \left(z\lambda_1^m - \frac{\lambda_5\lambda_3\lambda_1^{m+1}}{a} + \lambda_5\lambda_3\frac{(\lambda_1+a\lambda_2)^{m+1}}{a} + \lambda_5\lambda_4(\lambda_1+a\lambda_2)^{m+1}\right)\mathrm{d}a\\
&= az\lambda_1^m - \lambda_5\lambda_3\lambda_1^{m+1}\log(a) + \lambda_5\lambda_3\left(\lambda_1^{m+1}\log(a) + \sum_{j=1}^{m+1}\frac{\lambda_1^{m+1-j}(\lambda_1+\lambda_2 a)^j}{j}\right)\\
&\quad + \frac{\lambda_5\lambda_4}{\lambda_2(m+2)}(\lambda_1+\lambda_2 a)^{m+2}\\
&= az\lambda_1^m + \lambda_5\lambda_3\sum_{j=1}^{m+1}\frac{\lambda_1^{m+1-j}(\lambda_1+\lambda_2 a)^j}{j} + \frac{\lambda_5\lambda_4}{\lambda_2(m+2)}(\lambda_1+\lambda_2 a)^{m+2}, \qquad \text{(S169)}
\end{aligned}$$

where we have made use of the integral identity from Lemma S3.1. Evaluating over the domain of integration gives

$$\begin{aligned}
\int_0^{1-g} \mathbb{P}(Z_*^l \le z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}}\mathrm{d}a &= (1-g)z\lambda_1^m + \lambda_5\lambda_3\left(\sum_{j=1}^{m+1}\frac{\lambda_1^{m+1-j}z^j}{j} - \lambda_1^{m+1}H(m+1)\right)\\
&\quad + \frac{\lambda_5\lambda_4(z^{m+2}-\lambda_1^{m+2})}{\lambda_2(m+2)}. \qquad \text{(S170)}
\end{aligned}$$

Next, we calculate

$$\int_0^1\int_0^{1-g} \mathbb{P}(Z_*^l \le z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}} f_A(a)f_G(g)\mathrm{d}a\mathrm{d}g = \int_0^1\left(\int_0^{1-g}\mathbb{P}(Z_*^l \le z|g,a,m,\overline{\mathcal{C}_1})_{\mathcal{E}}\mathrm{d}a\right)(2-2g)\mathrm{d}g. \qquad \text{(S171)}$$

We integrate each additive term separately:

$$\begin{aligned}
\rho_1(m,z) &= \int_0^1 (1-g)z\lambda_1^m(2-2g)\mathrm{d}g\\
&= \int_0^1 2(1-g)^2 z(1-g+gz)^m\mathrm{d}g\\
&= -\frac{2z\left(2+m^2(-1+z)^2+m(-1+z)(-3+5z)-2z\left(3-3z+z^{2+m}\right)\right)}{(1+m)(2+m)(3+m)(-1+z)^3}. \qquad \text{(S172)}
\end{aligned}$$

$$
\begin{aligned}
\rho_{2j}(m,z) &= \int_0^1 \lambda_5\lambda_3 \frac{\lambda_1^{m+1-j} z^j}{j}(2-2g)\mathrm{d}g \\
&= \int_0^1 -\frac{2(1-g+m-gm-gz)(1-g+gz)^{m+1-j} z^j}{(m+1)(m+2)j}\mathrm{d}g \\
&= \frac{2z^{3+m}(j+(2+m)(-2+z)-jz)}{j(-3+j-m)(-2+j-m)(1+m)(2+m)(-1+z)^2} \\
&\quad + \frac{2z^j(-j(1+m)(-1+z)+(2+m)(-1+m(-1+z)+2z))}{j(-3+j-m)(-2+j-m)(1+m)(2+m)(-1+z)^2}. \qquad \text{(S173)}
\end{aligned}
$$

$$
\begin{aligned}
\rho_3(m,z) &= \int_0^1 -\lambda_5\lambda_3\lambda_1^{m+1} H(m+1)(2-2g)\mathrm{d}g \\
&= \int_0^1 \frac{2H(m+1)(1-g+m-gm-gz)(1-g+gz)^{m+1}}{(m+1)(m+2)}\mathrm{d}g \\
&= -\frac{2H(m+1)\left(-1+m(-1+z)+2z+(-2+z)z^{3+m}\right)}{(1+m)(2+m)(3+m)(-1+z)^2}. \qquad \text{(S174)}
\end{aligned}
$$

$$
\begin{aligned}
\rho_4(m,z) &= \int_0^1 \frac{\lambda_5\lambda_4(z^{m+2}-\lambda_1^{m+2})}{\lambda_2(m+2)}(2-2g)\mathrm{d}g \\
&= -\frac{2(-1-m+z+mz)(z^{m+2}-(1-g+gz)^{m+2})}{(m+1)(m+2)^2(-1+z)}\mathrm{d}g \\
&= -\frac{2}{(2+m)^2(3+m)(-1+z)} - \frac{2z^{2+m}}{(2+m)^2} + \frac{2z^{3+m}}{(2+m)^2(3+m)(-1+z)}. \qquad \text{(S175)}
\end{aligned}
$$

With these terms, we have

$$
\mathbb{P}(Z_*^l \le z | m, \mathcal{E}, \overline{\mathcal{C}_1}) = \rho_1(m,z) + \sum_{j=1}^{m+1} \rho_{2j}(m,z) + \rho_3(m,z) + \rho_4(m,z). \qquad \text{(S176)}
$$

S3.3 Evaluation of Integral (S49)

The integral is

$$
\mathbb{P}(Z_*^l \le z | m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} = \int_0^1 g\mathbb{P}(Z_*^l \le z | g, a, m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} f_G(g)\mathrm{d}g, \qquad \text{(S177)}
$$

where

$$
\begin{aligned}
\mathbb{P}(Z_*^l \le z | g, a, m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}} &= z(1-g+gz)^m - \frac{(1-2g+(1-g)m)z^{2+m}}{(1-g)^2(1+m)(2+m)} \\
&\quad + \frac{((1-g)(1+m)-gz)(1-g+gz)^{1+m}}{(1-g)^2(1+m)(2+m)}. \qquad \text{(S178)}
\end{aligned}
$$

For the first term in $\mathbb{P}(Z_*^l \le z | m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}}$,

$$
\int_0^1 gz(1-g+gz)^m(2-2g)\mathrm{d}g = -\frac{2z\left(1+m-3z-mz+z^{2+m}(3+m-(1+m)z)\right)}{(1+m)(2+m)(3+m)(-1+z)^3}. \qquad \text{(S179)}
$$

To find the indefinite integral of the second term in $\mathbb{P}(Z_*^l \leq z|m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}}$, we use the substitution $g = 1 - e$.

$$\begin{aligned}
& \int -\frac{2g(1-2g+(1-g)m)z^{2+m}}{(1-g)(1+m)(2+m)}\mathrm{d}g \\
= & \int \frac{2(1-e)(1-2(1-e)+em)z^{m+2}}{e(m+1)(m+2)}\mathrm{d}e \\
= & \int \frac{(-2+2e(m+3)-2e^2(m+2))z^{m+2}}{e(m+1)(m+2)}\mathrm{d}e \\
= & \int \frac{-2z^{m+2}}{e(m+1)(m+2)}\mathrm{d}e + \int \frac{2(m+3)z^{m+2}}{(m+1)(m+2)}\mathrm{d}e + \int \frac{-2ez^{m+2}}{(m+1)}\mathrm{d}e \\
= & \frac{-2z^{m+2}}{(m+1)(m+2)}\log(e) + \frac{2e(3+m)z^{m+2}}{(m+1)(m+2)} - \frac{e^2 z^{m+2}}{(m+1)}.
\end{aligned} \tag{S180}$$

For the indefinite integral of the final term in $\mathbb{P}(Z_*^l \leq z|m, \overline{\mathcal{C}_1})_{\overline{\mathcal{E}}}$, we again use the substitution $g = 1 - e$.

$$\begin{aligned}
& \int \frac{2g((1-g)(1+m)-gz)(1-g+gz)^{1+m}}{(1-g)(1+m)(2+m)}\mathrm{d}g \\
= & \int \frac{-2(1-e)(e(m+1)-(1-e)z)(1+(1-e)(z-1))^{m+1}}{e(m+1)(m+2)}\mathrm{d}e \\
= & \int \frac{(2z-2e(1+m+2z)-2e^2(-1-m-z))(z+e(1-z))^{m+1}}{e(m+1)(m+2)}\mathrm{d}e \\
= & \int \frac{2z(z+e(1-z))^{m+1}}{e(m+1)(m+2)}\mathrm{d}e + \int \frac{-2(1+m+2z)(z+e(1-z))^{m+1}}{(m+1)(m+2)}\mathrm{d}e \\
& + \int \frac{-2e(-1-m-z)(z+e(1-z))^{m+1}}{(m+1)(m+2)}\mathrm{d}e \\
= & \frac{2z}{(m+1)(m+2)}\left(z^{m+1}\log(e) + \sum_{j=1}^{m+1} \frac{z^{m+1-j}(z+e(1-z))^j}{j}\right) \\
& - \frac{2(1+m+2z)(z+e(1-z))^{m+2}}{(m+1)(m+2)^2(1-z)} - \frac{2e(-1-m-z)(z+e(1-z))^{m+2}}{(m+1)(m+2)^2(1-z)} \\
& + \frac{2(-1-m-z)(z+e(1-z))^{m+3}}{(m+1)(m+2)^2(m+3)(1-z)^2}.
\end{aligned} \tag{S181}$$

Note that we have used the integral identity from Lemma S3.1. For the second and third terms, we have

$$
\begin{aligned}
&\int_0^1 \left(\frac{((1-g)(1+m)-gz)(1-g+gz)^{1+m}}{(1-g)^2(1+m)(2+m)} - \frac{(1-2g+(1-g)m)z^{2+m}}{(1-g)^2(1+m)(2+m)} \right) \mathrm{d}g \\
= \quad & \frac{2z}{(m+1)(m+2)} \sum_{j=1}^{m+1} \frac{z^{m+1-j}(z+e(1-z))^j}{j} - \frac{2(1+m+2z)(z+e(1-z))^{m+2}}{(m+1)(m+2)^2(1-z)} \\
& - \frac{2e(-1-m-z)(z+e(1-z))^{m+2}}{(m+1)(m+2)^2(1-z)} + \frac{2(-1-m-z)(z+e(1-z))^{m+3}}{(m+1)(m+2)^2(m+3)(1-z)^2} \\
& + \frac{2e(3+m)z^{m+2}}{(m+1)(m+2)} - \frac{e^2 z^{m+2}}{(m+1)} \Bigg|_{e=1}^{e=0} \\
= \quad & \frac{-2z}{(m+1)(m+2)} \sum_{j=1}^{m+1} \frac{z^{m+1-j}}{j} + \frac{2z}{(m+1)(m+2)^2(1-z)} + \frac{2(1+m+z)}{(m+1)(m+2)^2(m+3)(1-z)^2} \\
& - \frac{(6+2m)z^{m+2}}{(m+1)(m+2)} + \frac{z^{m+2}}{m+1} + \frac{2H(m+1)z^{m+2}}{(m+1)(m+2)} - \frac{2(1+m+2z)z^{m+2}}{(m+1)(m+2)^2(1-z)} \\
& - \frac{2(1+m+z)z^{m+3}}{(m+1)(m+2)^2(m+3)(1-z)^2}.
\end{aligned} \tag{S182}
$$

Altogether,

$$
\begin{aligned}
\mathbb{P}(Z_*^l \leq z | m, \overline{\mathcal{C}}_1)_{\overline{\mathcal{E}}} \quad = \quad & -\frac{2z\left(1+m-3z-mz+z^{2+m}(3+m-(1+m)z)\right)}{(1+m)(2+m)(3+m)(-1+z)^3} + \frac{-2z}{(m+1)(m+2)} \sum_{j=1}^{m+1} \frac{z^{m+1-j}}{j} \\
& + \frac{2z}{(m+1)(m+2)^2(1-z)} + \frac{2(1+m+z)}{(m+1)(m+2)^2(m+3)(1-z)^2} - \frac{(6+2m)z^{m+2}}{(m+1)(m+2)} \\
& + \frac{z^{m+2}}{m+1} + \frac{2H(m+1)z^{m+2}}{(m+1)(m+2)} - \frac{2(1+m+2z)z^{m+2}}{(m+1)(m+2)^2(1-z)} \\
& - \frac{2(1+m+z)z^{m+3}}{(m+1)(m+2)^2(m+3)(1-z)^2}.
\end{aligned} \tag{S183}
$$