An average-case asymptotic analysis of the Container Relocation Problem

V. Galle [a,*], S. Borjian Boroujeni [a], V.H. Manshadi [b], C. Barnhart [a,c], P. Jaillet [a,d]

[a] *Operations Research Center, MIT, 77 Massachusetts Ave, Cambridge, MA 02139, USA*
[b] *School of Management, Yale, 165 Whitney Avenue, New Haven, CT 06511, USA*
[c] *Civil & Environmental Engineering, MIT, 77 Massachusetts Ave, Cambridge, MA 02139, USA*
[d] *Electrical Engineering & Computer Science, MIT, 77 Massachusetts Ave, Cambridge, MA 02139, USA*

ARTICLE INFO

ABSTRACT

The Container Relocation Problem (CRP) involves finding a sequence of moves of containers that minimizes the number of relocations needed to retrieve all containers in a given order. In this paper, we focus on average case analysis of the CRP when the number of columns grows asymptotically. We show that the expected minimum number of relocations converges to a simple and intuitive lower-bound for which we give an analytical formula.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

Due to limited space in the storage area in container terminals, containers are stacked in columns on top of each other. As shown in Fig. 1, several columns of containers create bays of containers. If a container that needs to be retrieved (*target* container) is not on the topmost tier of a column and is covered by other containers, the *blocking* containers must be relocated to other slots. As a result, during the retrieval process, one or more relocation moves are performed by the yard cranes. Finding the sequence of moves that minimizes the total number of relocations while retrieving containers from a bay in a pre-defined order is referred to as the Container Relocation Problem (CRP) or the Block Relocation Problem (BRP). For reviews and classification surveys of the existing literature on the CRP, we refer the reader to [3,6].

A common assumption of the CRP is that only the containers that are blocking the target container can be relocated. We refer to the CRP with this setting as the *restricted CRP*. In this paper, unless stated otherwise, CRP refers to the restricted CRP. On the other hand, if we relax this assumption, we will refer to the problem as *unrestricted CRP*.

In this paper, we study the CRP for large randomly distributed bays and we show that the ratio between the expected minimum number of relocations and a simple lower bound (developed by [2] and denoted below by S_0) approaches 1. While the problem is known to be NP-hard, this gives strong evidence that the CRP is "easier" to solve for large instances, and that heuristics can find near-optimal solutions.

Let us define the problem more formally: we are given a bay with C columns, P tiers; initially N containers are stored in the bay with exactly h containers in each column, where $h \leqslant P - 1$, so $N = h \times C$. We denote such a bay $B_{h,C}$. We label the containers based on their required departure order, i.e., container 1 is the first one to be retrieved. The CRP corresponds to finding a sequence of moves to retrieve containers $1, 2, \ldots, N$ (respecting the order) with a minimum number of relocations. For bay $B_{h,C}$, we denote the minimum number of relocations by $z_{opt}(B_{h,C})$. We focus on an average case analysis when the number of columns grows asymptotically. In our model, since $N = h \times C$, when C grows to infinity, N also grows to infinity.

Average case analysis of CRP is fairly new. The only other paper found in the literature is by [5]. They also provide a probabilistic analysis of the asymptotic CRP when both the number of columns and tiers grow to infinity. They show that there exists a polynomial time algorithm that solves this problem close to optimality with high probability. Our model departs from theirs in two aspects: (i) We keep the maximum height (number of tiers) a constant whereas in [5] the height also grows. Our assumption is motivated by the fact that the maximum height is limited by the crane

* Corresponding author.
E-mail addresses: vgalle@mit.edu (V. Galle), setareh@alum.mit.edu (S.B. Boroujeni), vahideh.manshadi@yale.edu (V.H. Manshadi), cbarnhar@mit.edu (C. Barnhart), jaillet@mit.edu (P. Jaillet).
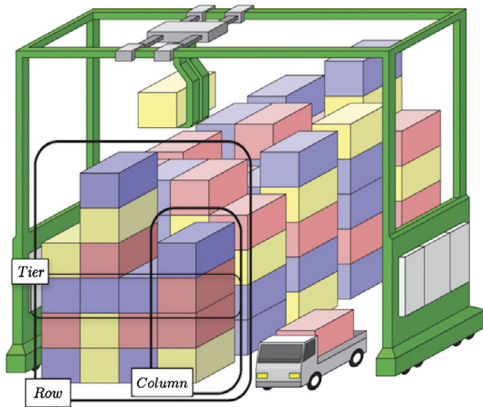
Fig. 1. Bays of containers in storage area.

Fig. 2. Bay with 4 tiers and 4 columns.

height, and it cannot grow arbitrarily; and (ii) We assume the ratio of the number of containers initially in the bay to the bay size (i.e., number of columns) stays constant (i.e., the bay is almost full at the beginning) and is equal to h. On the other hand, in [5], the ratio of the number of containers initially in the bay to the bay size decreases (and it approaches zero) as the number of columns grows. In other words, in the model of [5], in large bays, the bay is underutilized.

Before stating the main result in Section 3, we first provide four main ingredients in the next section: the notion of a uniformly random bay, the simple lower bound S_0 on the minimum number of relocations introduced by [2], a heuristic developed by [1] that performs well in large bays, and the notion of "special" columns.

2. Background

2.1. Uniformly random bay

We view a bay as an array of $P \times C$ slots (see Fig. 2). The slots are numbered from bottom to top, and left to right from 1 to $P \times C$. The goal is to generate a bay $B_{h,C}$ with uniform probability, meaning each container is equally likely to be anywhere in the configuration, with the restriction that there are h containers per column. We first generate a uniformly random permutation of $\{1, \ldots, h \times C\}$ called π. Then we assign a slot for each container with the following relation: $B_{h,C}(i,j) = \pi(h \times (j-1) + i)$ for $i \leqslant h$ and $B_{h,C}(i,j) = 0$ for $i \geqslant h+1$. One can see that each bay is generated with probability $1/N!$. There is a one to one mapping between configurations with C columns and permutations of $\{1, \ldots, h \times C\}$, denoted by $\mathcal{S}_{h \times C}$. Finally, we denote the expectation of random variable X over this uniform distribution by $\mathbb{E}_{h,C}[X]$.

2.2. The counting lower bound S_0

This bound was introduced in [2] and it is based on the following simple observation. In the initial configuration, if a container is blocking, then it must be relocated at least once. Thus we count the number of blocking containers in $B_{h,C}$, we denote it as $S_0(B_{h,C})$, and we have $z_{opt}(B_{h,C}) \geqslant S_0(B_{h,C})$. Note that if a container blocks

more than one container, it is counted only once. In Lemma 1 we give an explicit formula for the expectation of S_0 under the uniform distribution.

Lemma 1. *Let C, $h \in \mathbb{N}$ and S_0 be the counting lower bound defined above, we have*

$$\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right] = \alpha_h \times C, \tag{1}$$

where $\alpha_h = h - \sum_{i=1}^{h} 1/i$ is the expected number of blocking containers in one column.

Fact 2. *Note that α_h only depends on h.*

Proof of Lemma 1. Let $S_0^i(B_{h,C})$ be the number of blocking containers in column i. By the linearity of expectation, we have $\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right] = \mathbb{E}_{h,C}\left[\sum_{i=1}^{C} S_0^i(B_{h,C})\right] = \sum_{i=1}^{C} \mathbb{E}_{h,C}\left[S_0^i(B_{h,C})\right] = \alpha_h \times C$, where $\alpha_h = \mathbb{E}_{h,C}\left[S_0^1(B_{h,C})\right] = \mathbb{E}_{h,1}\left[S_0(B_{h,1})\right]$. This relies on the fact that each column is identically distributed.

Now let us compute α_h. It is clear that $\alpha_1 = 0$. For $h \geqslant 2$, by conditioning on the event that the topmost container is the smallest number in the column or not, we obtain the recursive equation $\alpha_h = \alpha_{h-1} + (h-1)/h$. Finally by induction we have $\alpha_h = h - \sum_{i=1}^{h} 1/i$ which completes the proof.

2.3. The heuristic H ([1])

Suppose n is the target container located in column c, and r is the topmost blocking container in c. For convenience, we denote by $\min(c_i)$ the minimum of column c_i (note that $\min(c_i) = N+1$ if c_i is empty). H uses the following rule to determine $c^* \neq c$, the column where r should be relocated to. If there is a column c_i with $|c_i| < P$, where $\min(c_i)$ is greater than r, then H chooses such a column where $\min(c_i)$ is minimized, since columns with larger minimums can be useful for larger blocking containers (as r will never be relocated again, we say this relocation of r is a "good" move). If there is no column satisfying $\min(c_i) > r$ (any relocation of r can only result in a "bad" move), then H chooses the column where $\min(c_i)$ is maximized in order to delay the next unavoidable relocation of r as much as possible. We will refer to this heuristic as heuristic H and denote its number of relocations by $z_H(B_{h,C})$. Notice that $z_{opt}(B_{h,C}) \leqslant z_H(B_{h,C})$. Finally we state the following simple fact, which does not require a formal proof:

Fact 3. *For any configuration B with C columns and at most C containers, we have*

$$S_0(B) = z_{opt}(B) = z_H(B). \tag{2}$$

2.4. Definition of "special" columns

For $h, C \in \mathbb{N}$, a column in $B_{h,C}$ is called "special" if all of its containers belong to the C highest. Given this definition, a column in $B_{h,C+1}$ is "special" if all containers belong to the $C+1$ highest, or equivalently, if each of its containers has index at least $\omega_{h,C} = (h-1)(C+1) + 1$. We will also consider the following event:

$$\Omega_{h,C} = \left\{B_{h,C+1} \text{ has at least 1 "special" column}\right\}. \tag{3}$$

Lemma 4 states that the event $\Omega_{h,C}$ has a probability that increases exponentially fast to 1 as a function of C. The proof of Lemma 4 can be found in the Appendix.

Lemma 4. *Let $h, C \in \mathbb{N}$ such that $C \geqslant h+1$ and $\Omega_{h,C}$ be the event defined by Eq. (3), then we have*

$$\mathbb{P}(\overline{\Omega_{h,C}}) \leqslant e^{-\theta_h(C+1)}, \tag{4}$$

where

$$\theta_h = \frac{1}{8h}\left(\frac{2}{h(h+1)}\right)^{2h} > 0. \tag{5}$$

3. An average-case asymptotic analysis of CRP

The major result of this paper states that when the number of columns increases to infinity, the expected optimal number of relocations is asymptotically proportional to the expected number of blocking containers.

Theorem 5. *Let S_0 be the counting lower bound and z_{opt} be the optimal number of relocations defined in Section* 1. *Then for $C, h \in \mathbb{N}$ such that $C \geqslant h + 1$, we have*

$$1 \leqslant \frac{\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]} \leqslant f_h(C), \tag{6}$$

where

$$f_h(C) = 1 + \frac{K_h}{C} \xrightarrow[C \to \infty]{} 1, \tag{7}$$

where K_h is a constant defined by Eq. (13).

Proof of Theorem 5. The basic intuition is that, as the number of columns grows, for any blocking container, we can find a "good" column with high probability. This implies that with high probability, each container is only relocated once. More formally, as C grows, with high probability $\mathbb{E}_{h,C+1}[z_{opt}(B_{h,C+1})] - \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]$ is exactly α_h. Therefore, for large enough C, $\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]$ essentially behaves like $\alpha_h \times C$, which is equal to $\mathbb{E}_{h,C}[S_0(B_{h,C})]$ (according to Lemma 1).

Since for all configurations $B_{h,C}$, $z_{opt}(B_{h,C}) \geqslant S_0(B_{h,C})$ then $\frac{\mathbb{E}_{h,C}[z_{opt}(B_{h,C})]}{\mathbb{E}_{h,C}[S_0(B_{h,C})]} \geqslant 1$.

Moreover, we have:

$$\frac{\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]} = 1 + \frac{\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right] - \mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]}$$

$$= 1 + \frac{1}{\alpha_h C}\left(\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right] - \alpha_h C\right)$$

$$= 1 + \frac{g_h(C)}{\alpha_h C} \tag{8}$$

where

$$g_h(C) = \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right] - \alpha_h C. \tag{9}$$

Now we claim that there exists a constant $\theta_h > 0$ (defined in Eq. (5)) such that:

$$\mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1})\right] \leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right] + \alpha_h$$
$$+ h(P-1)(C+1)e^{-\theta_h(C+1)}, \quad \forall C \geqslant h+1. \tag{10}$$

Eq. (10) studies how $\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]$ evolves and states that it increases almost linearly in α_h which shows that the function $g_h(.)$ is essentially bounded. Before proving Eq. (10) we conclude the proof of the theorem. Using Eq. (10), we have for all $C \geqslant h+1$:

$$\mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1})\right] \leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right] + \alpha_h$$
$$+ h(P-1)(C+1)e^{-\theta_h(C+1)}$$

$$\Longrightarrow \mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1})\right] - \alpha_h(C+1) \leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]$$
$$- \alpha_h C + h(P-1)(C+1)e^{-\theta_h(C+1)}$$

$$\Longrightarrow g_h(C+1) \leqslant g_h(C) + h(P-1)(C+1)e^{-\theta_h(C+1)}$$

$$\Longrightarrow g_h(C) \leqslant g_h(h+1) + h(P-1)\sum_{i=h+2}^{C}\left(ie^{-\theta_h i}\right)$$

$$\Longrightarrow g_h(C) \leqslant g_h(h+1) + h(P-1)\sum_{i=1}^{\infty}\left(ie^{-\theta_h i}\right)$$
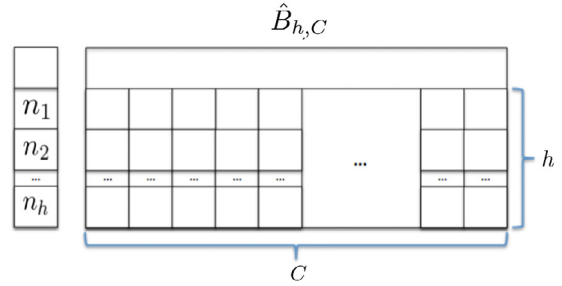

Fig. 3. Bay decomposition of $B_{h,C+1}$. (The right part has C columns.)

$$\Longrightarrow g_h(C) \leqslant g_h(h+1) + \frac{e^{\theta_h}h(P-1)}{(e^{\theta_h}-1)^2} = K_h'. \tag{11}$$

Therefore using Eqs. (8) and (11), we have

$$\frac{\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]} \leqslant 1 + \frac{K_h}{C} = f_h(C), \tag{12}$$

where

$$K_h = \frac{K_h'}{\alpha_h} = \frac{g_h(h+1) + \frac{e^{\theta_h}h(P-1)}{(e^{\theta_h}-1)^2}}{\alpha_h}. \tag{13}$$

Now let us prove Eq. (10). Recall that a column is defined to be "special" if none of its containers are smaller than $\omega_{h,C} = (h-1)(C+1) + 1$ and that $\Omega_{h,C} = \left\{B_{h,C+1}\text{ has at least one "special" column}\right\}$.

The intuition is the following: the probability of having a "special" column grows quickly to 1 as a function of C, implying that the event $\Omega_{h,C}$ happens with high probability. Now, conditioned on $\Omega_{h,C}$, we more easily express the difference between bays of size $C + 1$ and C in the following way. We claim that

$$\mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1})|\Omega_{h,C}\right] \leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right] + \alpha_h. \tag{14}$$

Let $B_{h,C+1}$ be a given bay with $C + 1$ columns that verifies $\Omega_{h,C}$. Since columns in bays can be interchanged, we suppose that a "special" column is the first (leftmost) column of the bay. We also denote $n_1, n_2, \ldots, n_h$ the containers of the first column. We know that $n_1, n_2, \ldots, n_h \geqslant \omega_{h,C}$ and $n_1 \neq n_2 \neq \ldots \neq n_h$. Finally let $\hat{B}_{h,C}$ be the bay $B_{h,C+1}$ without its first column (see Fig. 3).

First we prove that

$$z_{opt}(B_{h,C+1}) \leqslant z_{opt}(\hat{B}_{h,C}) + S_0\left(\begin{bmatrix}n_1 \\ \ldots \\ n_h\end{bmatrix}\right). \tag{15}$$

To prove Eq. (15), we construct a feasible sequence σ for the bay of size $C + 1$ for which the number of relocations is equal to the right side of Eq. (15). Let $\sigma_{opt}(\hat{B}_{h,C})$ the optimal sequence for $\hat{B}_{h,C}$, $t' = \min\{n_1, \ldots, n_h\}$ be the first time step when the target container in $\sigma_{opt}(\hat{B}_{h,C})$ is larger than $\min\{n_1, n_2, \ldots, n_h\}$ and $B_{h,C}'$ be the bay obtained at t' using $\sigma_{opt}(\hat{B}_{h,C})$. Let the first $t' - 1$ moves of σ be the first $t' - 1$ moves of $\sigma_{opt}(\hat{B}_{h,C})$. Note that $B_{h,C}'$ has at most $C + 1 - h$ (which is at most C) containers due to the choice of $\omega_{h,C}$. By Fact 3, the number of relocations performed by $\sigma_{opt}(\hat{B}_{h,C})$ from t' until the end is $S_0(B_{h,C}')$. Therefore

$$z_{opt}(\hat{B}_{h,C}) = \left\{\begin{array}{c}\text{\# relocations up to } t' \\ \text{done by } \sigma_{opt}(\hat{B}_{h,C})\end{array}\right\} + S_0(B_{h,C}'). \tag{16}$$

After t', we run heuristic H on

$$B'_{h,C+1} = \left(\begin{bmatrix} n_1 & \cdots & n_h \end{bmatrix}^T \cup B'_{h,C} \right).$$

We claim that z_σ (number of relocations performed by the feasible sequence σ constructed above) is exactly the right side of Eq. (15). There are at most $C + 1$ containers in $B'_{h,C+1}$, therefore using Fact 3, we know that if we apply the heuristic H to this configuration, then the number of relocations done by H is $S_0(B'_{h,C+1}) = S_0(B'_{h,C}) + S_0\left(\begin{bmatrix} n_1 & \cdots & n_h \end{bmatrix}^T \right)$. Therefore $z_{opt}(B_{h,C+1}) \leqslant z_\sigma(B_{h,C+1})$ and

$$z_\sigma(B_{h,C+1}) = \left\{ \begin{array}{c} \# \text{ relocations up to } t' \\ \text{done by } \sigma_{opt}(\hat{B}_{h,C}) \end{array} \right\} + S_0(B'_{h,C}) + S_0\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right) \tag{17}$$

which gives us

$$z_{opt}(B_{h,C+1}) \leqslant z_{opt}(\hat{B}_{h,C}) + S_0\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right),$$

and proves Eq. (15).

Now we can take the expectation from both sides of Eq. (15) over a uniform distribution of the rest of the $h \times C$ containers that are not in the first column. We claim that the first term on the right hand-side of Eq. (15) is exactly $\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C}) \right]$. For any configuration that appears in $\hat{B}_{h,C}$ we can map it to a unique configuration $B_{h,C}$ where all containers are between 1 and hC, and vice versa. Thus,

$$\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C+1}) \,\middle|\, \begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right] \leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C}) \right] + S_0\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right). \tag{18}$$

Next, we take the expectation of both sides of Eq. (18) over possible first columns, which is a "special" column. Now notice that if $B_{h,C+1}$ is generated uniformly in the sets of bays of size $C+1$, then conditioned on $\Omega_{h,C}$, the probability of having a certain column $[n_1, \ldots, n_h]^T$ is identical for any $n_1 \neq \ldots \neq n_h \geqslant \omega_{h,C}$ and it is given by

$$\mathbb{P}\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \,\middle|\, \Omega_{h,C} \right) = \frac{(C+1-h)!}{(C+1)!} = \frac{1}{\binom{C+1}{h} h!}.$$

Therefore we can write:

$$\mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \mid \Omega_{h,C} \right]$$

$$= \sum_{\substack{(n_1,\ldots,n_h) \\ n_i \neq n_j \\ n_i \geqslant \omega_{h,C}}} \left(\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C+1}) \,\middle|\, \begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix}, \Omega_{h,C} \right] \right.$$

$$\left. \times \mathbb{P}\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \,\middle|\, \Omega_{h,C} \right) \right) \tag{19}$$

$$= \sum_{\substack{(n_1,\ldots,n_h) \\ n_i \neq n_j \\ n_i \geqslant \omega_{h,C}}} \left(\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C+1}) \,\middle|\, \begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right] \right.$$

$$\left. \times \mathbb{P}\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \,\middle|\, \Omega_{h,C} \right) \right) \tag{20}$$

$$\leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C}) \right] \sum_{\substack{(n_1,\ldots,n_h) \\ n_i \neq n_j \\ n_i \geqslant \omega_{h,C}}} \mathbb{P}\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \,\middle|\, \Omega_{h,C} \right)$$

$$+ \sum_{\substack{(n_1,\ldots,n_h) \\ n_i \neq n_j \\ n_i \geqslant \omega_{h,C}}} S_0\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right) \times \mathbb{P}\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \,\middle|\, \Omega_{h,C} \right)$$

$$\leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C}) \right]$$

$$+ \sum_{\substack{(n_1,\ldots,n_h) \\ n_i \neq n_j \\ n_i \geqslant \omega_{h,C}}} S_0\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right) \times \mathbb{P}\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \,\middle|\, \Omega_{h,C} \right). \tag{21}$$

The equality between (19) and (20) comes from the fact that if we know that $B_{h,C+1}$ has a "special" column, then we do not need to condition on $\Omega_{h,C}$. Eq. (21) uses the fact that

$$\sum_{\substack{n_1 \neq \ldots \neq n_h \\ n_i \geqslant \omega_{h,C}}} \mathbb{P}\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \,\middle|\, \Omega_{h,C} \right) = 1.$$

Note that, given any $(n_1, \ldots, n_h)$ such that $n_i \neq n_j$, we have

$$\mathbb{E}_{h,1}\left[S_0\left(\begin{bmatrix} n_1 \\ \cdots \\ n_h \end{bmatrix} \right) \right] = \alpha_h,$$

where the expectation is over a random order of $(n_1, \ldots, n_h)$. This is true regardless of the set $(n_1, \ldots, n_h)$ that is drawn from (see Fact 2). This implies that the second term on the right hand side of Eq. (21) is equal to α_h; Therefore, we get Eq. (14).

Now we want to focus on the event $\overline{\Omega_{h,C}}$. We give an upper bound on $\mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \mid \overline{\Omega_{h,C}} \right]$. For any configuration, in order to retrieve one container, we need at most $P - 1$ relocations (since at most $P - 1$ containers are blocking it), thus for any configuration, the optimal number of relocations is at most $P - 1$ times the number of containers $(h(C + 1))$ which gives us $h(P - 1)(C + 1)$ as an upper bound on the optimal number of relocations. We use this universal bound to get

$$\mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \mid \overline{\Omega_{h,C}} \right] \leqslant h(P - 1)(C + 1). \tag{22}$$

Finally using Lemma 4, we have

$$\mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \right] = \mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \mid \Omega_{h,C} \right] \mathbb{P}(\Omega_{h,C})$$

$$+ \mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \mid \overline{\Omega_{h,C}} \right] \mathbb{P}(\overline{\Omega_{h,C}})$$

$$\leqslant \mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \mid \Omega_{h,C} \right]$$

$$+ \mathbb{E}_{h,C+1}\left[z_{opt}(B_{h,C+1}) \mid \overline{\Omega_{h,C}} \right] e^{-\theta_h(C+1)}$$

$$\leqslant \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C}) \right] + \alpha_h + h(P - 1)(C + 1)e^{-\theta_h(C+1)}$$

which proves Eq. (10) and hence completes the proof of the theorem.

In the next corollary, we show that the optimal solution of the unrestricted CRP has a similar asymptotic behavior. We remind that the unrestricted CRP refers to the problem where we can also relocate non-blocking containers. The proof is trivial since by definition $S_0(B_{h,C}) \leqslant z_{unr}(B_{h,C}) \leqslant z_{opt}(B_{h,C})$.

Corollary 6. Let $z_{unr}(B_{h,C})$ be the optimal number of relocations for the unrestricted CRP. For $C \geqslant h + 1$, we have

$$1 \leqslant \frac{\mathbb{E}_{h,C}\left[z_{unr}(B_{h,C}) \right]}{\alpha_h C} \leqslant f_h(C),$$

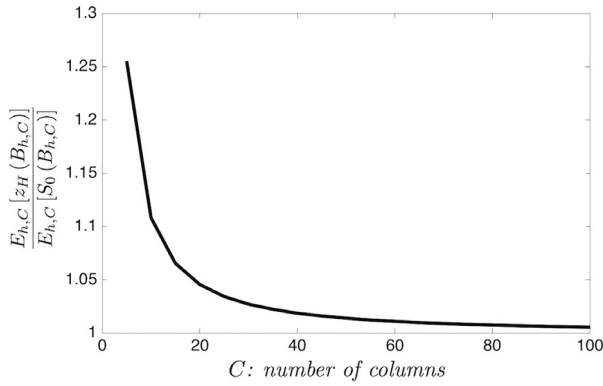where f_h is the function defined in Theorem 5.

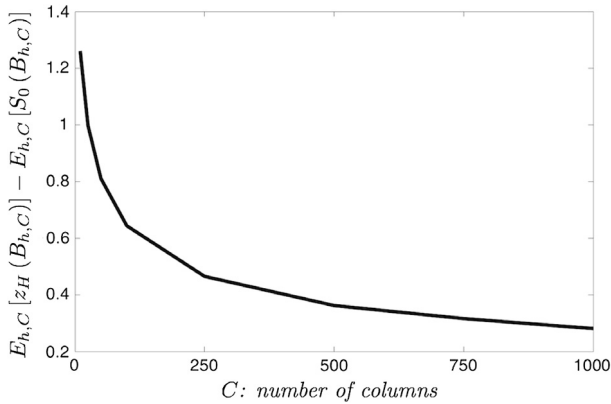**Fig. 4.** Simulation of the convergence of the ratio.

Fig. 5. Simulation of the convergence of the difference.

3.1. Experimental results on the efficiency of heuristic H

Theorem 5 gives insights on how the expected optimal solution of the CRP behaves asymptotically on random bays. To give more insights on CRP, we show experimentally that the same result holds for heuristic H, i.e., the ratio of $\mathbb{E}_{h,C}\left[z_H(B_{h,C})\right]$ and $\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]$ converges to 1 as C goes to infinity. We take $h = P - 1 = 4$ and for each size C, we compute both expectations over a million instances generated uniformly, take their ratio and plot the result in Fig. 4. Notice that we have

$$1 \leqslant \frac{\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]} \leqslant \frac{\mathbb{E}_{h,C}\left[z_H(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]},$$

so Fig. 4 also shows experimentally that Theorem 5 holds.

First, note that Fig. 4 implies that the relative gap between heuristic H and S_0 shrinks to 0 as C increases. Moreover we have

$$\frac{\mathbb{E}_{h,C}\left[z_H(B_{h,C})\right] - \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right]} \leqslant \frac{\mathbb{E}_{h,C}\left[z_H(B_{h,C})\right] - \mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]}{\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]},$$

and thus the relative gap of H with optimality also converges to 0 as C grows to infinity.

In the proof of Theorem 5, we also study the function $g_h(C) = \mathbb{E}_{h,C}\left[z_{opt}(B_{h,C})\right] - \mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]$. Note that $g_h(C) \leqslant \mathbb{E}_{h,C}\left[z_H(B_{h,C})\right] - \mathbb{E}_{h,C}\left[S_0(B_{h,C})\right]$ where the right-hand side of the inequality is the function plotted in Fig. 5. The plot shows that $g_h(C) \leqslant 1.25$ for all C, meaning that $g_h(C)$ is bounded as we proved in Theorem 5. Moreover, the plot implies that heuristic H is on average at most 1.25 away from the optimal solution, so heuristic H is relatively more efficient in the case of large bays. Intuitively, the probability of having a good column converges to 1, as we increase the number of columns; hence the problem tends to become easier as C grows.

Finally, in the proof, we note that the rate of convergence of the minimum z_{opt} to S_0 is at least $1/C$. Interestingly, we can infer from Fig. 4. that the rate of convergence of the ratio for heuristic H is also proportional to $1/C$.

4. Conclusion

The Container Relocation Problem (CRP) is known for its computational intractability, so most research studies have designed heuristics to solve the problem, particularly for large bays. The main purpose of this paper is to show a new theoretical result stating that the ratio between the expected minimum number of relocations and a simple lower bound (given by Lemma 1) approaches 1. The main insight of this result is that in large bays each blocking container is relocated at most once with high probability. This leads us to believe that the same theoretical result should hold for heuristic H and we confirm this intuition by simulation.

Furthermore direct extensions of this paper include the formal proof of a similar result for the heuristic H, the proof of convergence of the difference between the optimal solution and this lower bound. The study of the CRP with distributions other than the uniform one could also be very interesting theoretically as well as experimentally.

Acknowledgment

Appendix. Proof of Lemma 4

Recall that

$$\Omega_{h,C} = \left\{B_{h,C+1} \text{ has at least one "special" column}\right\}.$$

We know that each bay of size $C + 1$ can be mapped to a permutation π of $\mathcal{S}_{h(C+1)}$ taken uniformly at random. Let $q(.)$ be the function from $\mathcal{S}_{h(C+1)}$ to $\mathbb{R}^+$ defined by

$q : \pi \longmapsto$ number of "special" columns in the resulting bay of π.

Note that

$$\mathbb{P}\left(\overline{\Omega_{h,C}}\right) = \mathbb{P}\left(q(\pi) = 0\right).$$

First we compute the expected value of $q(.)$

$$\mathbb{E}_{h,C+1}[q] = \mathbb{E}_{h,C+1}\left[\sum_{i=1}^{C+1} \chi\left(c_i \text{ is a "special" column}\right)\right]$$
$$= (C + 1) \times \mathbb{P}\left(\{c_1 \text{ is a "special" column}\}\right),$$

where we use linearity of expectation and the fact that columns are identically distributed.

A simple counting implies that:

$$\mathbb{P}\left(\{c_1 \text{ is a "special" column}\}\right)$$
$$= \frac{(C+1)[(C+1) - 1]\dots[(C+1) - h + 1]}{h(C+1)[h(C+1) - 1]\dots[h(C+1) - h + 1]}$$
$$\geqslant \left(\frac{(C+1) - h + 1}{h(C+1)}\right)^h \geqslant \left(\frac{2}{h(h+1)}\right)^h,$$

where we use $C + 1 \geqslant h + 1$ to show the last inequality (notice that when $C \to \infty$, the probability is equivalent to $(1/h)^h$ which would guarantee a faster convergence rate).

Therefore we know that

$$\mathbb{E}_{h,C+1}[q] \geqslant (C + 1) \times \left(\frac{2}{h(h + 1)}\right)^h. \tag{A.1}$$

We claim that $q(.)$ is well concentrated around its mean. In order to do so, we prove that $q(.)$ is 1-Lipschitz.

Define ρ the distance between two permutations $\pi_1, \pi_2 \in \mathcal{S}_{h(C+1)}$ as $\rho(\pi_1, \pi_2) = |\{i \in [h(C + 1)] : \pi_1(i) \neq \pi_2(i)\}|$. We want to prove that

$$|q(\pi_1) - q(\pi_2)| \leqslant \rho(\pi_1, \pi_2), \quad \forall (\pi_1, \pi_2) \in \mathcal{S}_{h(C+1)}.$$

Let $\pi_1, \pi_2 \in \mathcal{S}_{h(C+1)}$. Let us first consider the case where $\rho(\pi_1, \pi_2) = 2$. (Notice that if $\rho(\pi_1, \pi_2) \neq 0$ then $\rho(\pi_1, \pi_2) \geqslant 2$). In that case, we have $i, j \in \{1, \ldots, n\}$ such that $\pi_1(i) = \pi_2(j)$ and $\pi_1(j) = \pi_2(i)$. Let $B^{(1)}$ and $B^{(2)}$ be the configurations generated by π_1 and π_2. Having $\rho(\pi_1, \pi_2) = 2$ corresponds to the fact that if we swap 2 containers in $B^{(1)}$, we get $B^{(2)}$ and we denote those containers $a = \pi_1(i)$ and $b = \pi_1(j)$. We have three cases:

- a and b are both in "special" columns in $B^{(1)}$. In this case, swapping them will not change anything since both their new columns in $B^{(2)}$ will also be "special" and hence $|q(\pi_1) - q(\pi_2)| = 0$.
- a and b are both in columns that are not "special" columns in $B^{(1)}$. If $a, b \geqslant \omega_{h,C}$ or $a, b < \omega_{h,C}$ then we do not create any new special column in $B^{(2)}$. Now suppose that $a \geqslant \omega_{h,C}$ and $b < \omega_{h,C}$, then the column of a in $B^{(2)}$ might be a "special" column, but the column of b in $B^{(2)}$ cannot be "special". Therefore in that case, $|q(\pi_1) - q(\pi_2)| \leqslant 1$.
- a is in a "special" column in $B^{(1)}$ but b is not. Now we know that $a \geqslant \omega_{h,C}$. If $b < \omega_{h,C}$ then the column of b in $B^{(2)}$ cannot be "special" but the column of a might be and in that case $|q(\pi_1) - q(\pi_2)| \leqslant 1$. If $b \geqslant \omega_{h,C}$, then the column of b in $B^{(2)}$ is "special" and the column of a in $B^{(2)}$ is not "special" which gives us $|q(\pi_1) - q(\pi_2)| = 0$. Note that the proof is identical if b is in a "special" column in $B^{(1)}$ but a is not.

So far we have shown that

If $\rho(\pi_1, \pi_2) = 2$, then $|q(\pi_1) - q(\pi_2)| \leqslant 1$. $\tag{A.2}$

Now we suppose that $\rho(\pi_1, \pi_2) = k$ where $2 \leqslant k \leqslant h(C + 1)$. Note that we can construct a sequence of permutations $(\pi_1', \pi_2', \ldots, \pi_k')$ such that $\pi_1' = \pi_1, \pi_k' = \pi_2$ and $\rho(\pi_i', \pi_{i+1}') = 2$.

Now using this fact and Eq. (A.2),

$$
\begin{aligned}
|q(\pi_1) - q(\pi_2)| &= \left|\sum_{i=1}^{k-1} q(\pi_i') - q(\pi_{i+1}')\right| \\
&\leqslant \sum_{i=1}^{k-1} |q(\pi_i') - q(\pi_{i+1}')| \leqslant \sum_{i=1}^{k-1} 1 = k - 1 \\
&\leqslant k = \rho(\pi_1, \pi_2),
\end{aligned}
$$

which proves that $q(.)$ is 1-Lipschitz.

Now we use Theorem 8.3.3 of [4] which states that $\mathbb{P}\left(q \leqslant \mathbb{E}_{h,C+1}[q] - t\right) \leqslant e^{-\frac{t^2}{8h(C+1)}}$ and apply it with $t = \mathbb{E}_{h,C+1}[q]$ and Eq. (A.1) to get

$$\mathbb{P}(q = 0) = \mathbb{P}\left(q \leqslant \mathbb{E}_{h,C+1}[q] - \mathbb{E}_{h,C+1}[q]\right) \leqslant e^{-\frac{(\mathbb{E}_{h,C+1}[q])^2}{8h(C+1)}} \leqslant e^{-\theta_h(C+1)},$$

where

$$\theta_h = \frac{1}{8h}\left(\frac{2}{h(h+1)}\right)^{2h} > 0,$$

which concludes the proof.

References

[1] M. Caserta, S. Schwarze, S. Voß, A mathematical formulation and complexity considerations for the blocks relocation problem, European J. Oper. Res. 219 (1) (2012) 96–104. http://dx.doi.org/10.1016/j.ejor.2011.12.039.
[2] K.H. Kim, G.-P. Hong, A heuristic rule for relocating blocks, Comput. Oper. Res. 33 (4) (2006) 940–954. http://dx.doi.org/10.1016/j.cor.2004.08.005.
[3] J. Lehnfeld, S. Knust, Loading, unloading and premarshalling of stacks in storage areas: Survey and classification, European J. Oper. Res. 239 (2) (2014) 297–312. http://dx.doi.org/10.1016/j.ejor.2014.03.011.
[4] J. Matoušek, J. Vondrák, The probabilistic method: Lecture notes, in: LectureNotes, 2008. https://www.cs.cmu.edu/afs/cs.cmu.edu/academic/class/15859-f09/www/handouts/matousek-vondrak-prob-ln.pdf (accessed 29.08.16).
[5] M. Olsen, A. Gross, Average case analysis of blocks relocation heuristics, in: 5th International Conference ICCL Proceedings, Vol. 8760, 2014, pp. 81–92. http://dx.doi.org/10.1007/978-3-319-11421-7_6.
[6] R. Stahlbock, S. Voß, Operations research at container terminals: a literature update, OR Spectrum 30 (1) (2008) 1–52. http://dx.doi.org/10.1007/s00291-007-0100-9.