Scattered Data Interpolation with Wavelet Trees

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Abstract. This paper describes a new result on a wavelet scheme for scattered data interpolation extensively described in [2]. The interpolation scheme of [2] already has several advantages as compared to more classical RBF: it is highly stable and highly adaptive. In this paper, we describe a new property of this scheme: that the approximation error locally only depends on a local sample density or fill distance. We give a definition for this local fill distance, state the error estimate and outline the proof.

§1. Introduction

The interpolation scheme devised and studied in [2] aims, given a set of samples $\mathcal{X} \subset \mathbb{R}^d$ and the corresponding sample values $f(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{X}$, at fitting a function we call $f_{\mathcal{X}}$ to these samples.

The scheme consists in finding the interpolant as a finite linear combination of interpolating wavelets in a predefined basis $(\phi_{j\boldsymbol{k}})_{(j,\boldsymbol{k})\in\mathcal{J}}$. A number of features of this scheme have been already thoroughly described in [2]. Mainly, the process has an error decay rate that is optimal w.r.t the smoothness of the unknown function f. The local decay also depends on the local function smoothness.

The result described in this paper extends this set of features and can be stated as follows: the local error decay rate depends on the local density of samples (or fill distance). Usually, error estimates are given in terms of a uniform (or worst) fill distance across all the considered interpolation domain.

In §2, we give an overview of the scheme and of its properties. In §3, we define a local fill distance, state this local error estimate and give an outline of the proof. §4 concludes the paper.

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§2. Scattered Data Wavelet Interpolation

The interpolation scheme is built in three steps. A first step, called allocation, consists in picking a subset $(\phi_{j\boldsymbol{k}})_{(j,\boldsymbol{k})\in\mathcal{J}'}$ of an interpolating wavelet basis $(\phi_{j\boldsymbol{k}})_{(j,\boldsymbol{k})\in\mathcal{J}}$ [4]. The wavelets $\phi_{j\boldsymbol{k}}$ are defined by

$$\phi_{j\boldsymbol{k}}(\boldsymbol{x}) = \phi(2^j\boldsymbol{x} - \boldsymbol{k}),$$

where j is a resolution index, and $\boldsymbol{k}$ is the translation index. (In a bounded domain, the boundary wavelets may have slightly different shapes due to boundary effects, so the above formula must be refined, but to keep our statement simple, we will ignore this in this paper). The fundamental function ϕ is a compactly supported [3] interpolating function, i.e.

$$\begin{aligned} \phi(\boldsymbol{k}) &= 0 \qquad \text{if } \boldsymbol{k} \in \mathbb{Z}^d - \{0\}, \\ \phi(\boldsymbol{k}) &= 1 \qquad \text{if } \boldsymbol{k} = 0. \end{aligned}$$

The wavelet basis is then $(\phi_{j\boldsymbol{k}})_{(j,\boldsymbol{k})\in\mathcal{J}}$, where the index set is

$$\mathcal{J} = \{(0,\boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^d\} \cup \bigcup_{j=1}^{+\infty} \{(j,\boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^d,\ \boldsymbol{k} \notin 2\mathbb{Z}^d\}.$$

2.1. Allocation

Each wavelet $\phi_{j\boldsymbol{k}}$ has a center node $\nu_{j\boldsymbol{k}} = 2^{-j}\boldsymbol{k}$. Each wavelet in the basis also has a basin $B_{j\boldsymbol{k}}$ which is Voronoï cell of $\nu_{j\boldsymbol{k}}$ in the mesh of all the nodes $\nu_{j\boldsymbol{k}}$ of basis functions *of the same resolution*. We can assume that for each j, the set of basins $B_{j\boldsymbol{k}}$ is a partition of the domain.

Definition 1. *An* allocation *is a 1-1 mapping from $\mathcal{X}$ to the wavelet basis which is constructed iteratively as follows: $\mathcal{X}_f$ is initialized as $\mathcal{X}_f = \mathcal{X}$, and for each j, starting from $j = 0$, for each $\boldsymbol{k}$ such that $B_{j\boldsymbol{k}} \cap \mathcal{X}_f \neq \emptyset$:*

- *let $\boldsymbol{x}$ be the closest point to $\nu_{j\boldsymbol{k}}$ in $B_{j\boldsymbol{k}} \cap \mathcal{X}_f \neq \emptyset$;*
- *set $A(\boldsymbol{x}) := (j,\boldsymbol{k})$*
- *subtract $\boldsymbol{x}$ from $\mathcal{X}_f$, i.e. set $\mathcal{X}_f := \mathcal{X}_f - \{\boldsymbol{x}\}$.*

If the fill distance *$h(\mathcal{X}) = \min\{|\boldsymbol{x}-\boldsymbol{x}'| : \boldsymbol{x} \in \mathcal{X},\ \boldsymbol{x}' \in \mathcal{X},\ \boldsymbol{x} \neq \boldsymbol{x}'\}$ is nonzero, this process finishes for a finite j, i.e., $A(\boldsymbol{x})$ is defined for all $\boldsymbol{x} \in \mathcal{X}$ and $\mathcal{X}_f = \emptyset$. The selected subfamily is then indexed by $\mathcal{J}' = \{A(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{X}\}$.*

This allocation procedure and its properties are detailed in [2,1]. In univariate interpolation, we end up selecting a subset of the wavelet family that has a tree structure. An example is given in Fig. 1.

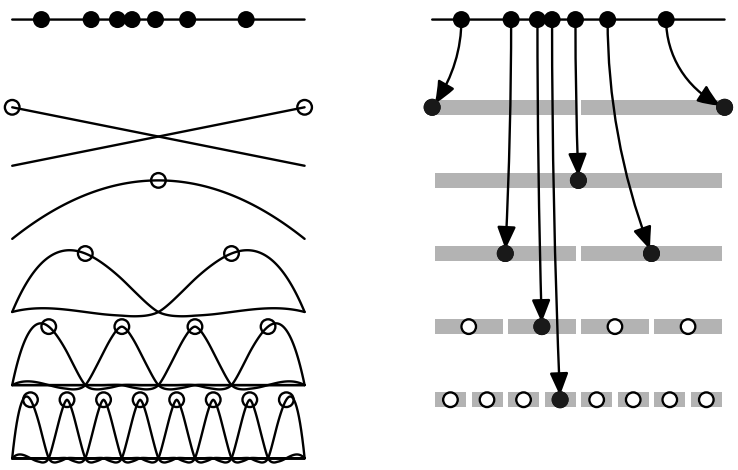

Fig. 1. Left: a wavelet family, with the location of the centers marked with black circles on the wavelets. The black dots on the top line are the samples. Right: the 1-1 mapping between samples and wavelets.

So far, if we expand the interpolant $f_{\mathcal{X}}$ as

$$f_{\mathcal{X}} = \sum_{(j,\boldsymbol{k})\in\mathcal{J}'} c_{j\boldsymbol{k}}\phi_{j\boldsymbol{k}},$$

the interpolation constraints on the expansion coefficients $(c_{j\boldsymbol{k}})$ make up a square linear system, since the allocation mapping A is a 1-1 mapping between the unknowns $c_{j\boldsymbol{k}}$ and the constraints. A second step, called subsystem selection, is however necessary to guarantee that we end up with a *stable* system.

2.2. Subsystem selection

Given an integer parameter P and a real parameter ρ, we define a (P,ρ)-placement condition. This is a geometric criterion relating the positions of the samples and the nodes of the wavelets that is simple to verify, and we use it to guarantee that we end up with a stable system.

Definition 2. *A sample point $\boldsymbol{x}$ and its allocated wavelet $\phi_{j\boldsymbol{k}}$ (where $(j,\boldsymbol{k}) = A(\boldsymbol{x})$) fulfill a (P,ρ)-*placement condition *if and only if the following conditions are fulfilled:*

1) *for all $j' < j$, all $\boldsymbol{k}'$ such that $\|\nu_{j\boldsymbol{k}} - \nu_{j'\boldsymbol{k}'}\| \leq P2^{-j}$, there exists $\boldsymbol{x}' \in \mathcal{X}$ such that $A(\boldsymbol{x}') = (j',\boldsymbol{k}')$, and $\|\nu_{j'\boldsymbol{k}'} - \boldsymbol{x}'\| \leq \rho 2^{-j}$;*

2) *$\|\nu_{j\boldsymbol{k}} - \boldsymbol{x}\| \leq \rho 2^{-j}$.*

Definition 3. *The (P,ρ)-*subsystem selection *step consists in running through all the pairs $(\boldsymbol{x},(j\boldsymbol{k}))$ where $(j,\boldsymbol{k}) = A(\boldsymbol{x})$, from large to fine scales, and removing all such pairs that do not fulfill the (P,ρ)-placement condition.*

With the remaining set of wavelets $(\phi_{j\boldsymbol{k}})_{(j,\boldsymbol{k})\in\mathcal{J}_S}$*, and the remaining set of samples* $\boldsymbol{x}\in\mathcal{X}_S$ *we obtain a new square linear system:*

$$\sum_{(j,\boldsymbol{k})\in\mathcal{J}_S} c_{j\boldsymbol{k}}\phi_{j\boldsymbol{k}}(\boldsymbol{x}) = f(\boldsymbol{x}) \qquad \forall \boldsymbol{x}\in\mathcal{X}_S, \tag{1}$$

and $f_{\mathcal{X}}$ *is defined to be* $f_{\mathcal{X}_S} = \sum_{(j,\boldsymbol{k})\in\mathcal{J}_S} c_{j\boldsymbol{k}}\phi_{j\boldsymbol{k}}$.

2.3. System solving

Once this has been done, we obtain a smaller square linear system (1). For well chosen values of P and ρ (which depend on the shape of the interpolating wavelet ϕ), we can guarantee that the system is stable. The stability is obtained in the L_∞ norm.

Theorem 4. *If the function* ϕ *has a finite support (and if the boundary wavelets have a finite number of different shapes and are all of finite support), then there exist an integer* $P>0$ *and a real* $\rho>0$ *such that for any set of samples* $\mathcal{X}$*, the subsystem obtained by* (P,ρ) *subsystem selection is stable. Moreover:*

1) *There exists a constant bound* M *such that the subsystem solution* $f_{\mathcal{X}}$ *fulfills* $\|f_{\mathcal{X}}\|_\infty \le M\times\|f\|_\infty$ *for any* $\mathcal{X}$*,*

2) *The norm of the system matrix inverse* $\|M^{-1}\|_{\infty,\infty}$ *is bounded by some* $M'\times|\log q(\mathcal{X})|$*, where* $q(\mathcal{X})$ *is the* separation distance *of the sample set* $\mathcal{X}$*, i.e.*

$$q(\mathcal{X}) = \min_{\boldsymbol{x}\in\mathcal{X}}\min_{\boldsymbol{x}'\in\mathcal{X}-\{\boldsymbol{x}\}}\|\boldsymbol{x}-\boldsymbol{x}'\|.$$

We give a short overview of the proof. We write the above subsystem in a new function basis of *relocated wavelets*.

Definition 5. *The* relocation *of a wavelet subfamily* $(\phi_{j\boldsymbol{k}})_{(j,\boldsymbol{k})\in\mathcal{J}'}$ *is a new family* $(\varphi_{j\boldsymbol{k}})_{(j,\boldsymbol{k})\in\mathcal{J}'}$ *spanning the same subspace, and fulfilling for all* $(j,\boldsymbol{k})\in\mathcal{J}'$

$$\varphi_{j\boldsymbol{k}}(\nu_{j'\boldsymbol{k}'}) = 1_{(j,\boldsymbol{k})=(j',\boldsymbol{k}')}.$$

The φ's are uniquely defined by this property. Since we can prove that

$$\varphi_{j\boldsymbol{k}} - \phi_{j\boldsymbol{k}} \in \operatorname{span}\left\{\phi_{j'\boldsymbol{k}'} : (j',\boldsymbol{k}')\in\mathcal{J}',\ j'>j\right\},$$

the matrix C of the basis change is triangular. We can show that it is invertible, and its norm (as well as that of its inverse) is bounded by some $M'\times|\log q(\mathcal{X})|$. In the new relocated basis, the subsystem is diagonally

dominant and of bounded inverse. These ingredients are used to prove item 1) of Th. 4. To prove item 2), we show that

$$\left\| \sum_{j\boldsymbol{k}} c_{j\boldsymbol{k}} \varphi_{j\boldsymbol{k}} \right\|_\infty \leq M'' \times \max_{j\boldsymbol{k}} |c_{j\boldsymbol{k}}|,$$

where M'' does not depend on the sample set $\mathcal{X}$.

From this, standard arguments can be used to prove that the interpolation scheme inherits the wavelet approximation properties, i.e., that for instance

$$\|f - f_{\mathcal{X}}\|_\infty \leq M''' \times h(\mathcal{X})^\alpha \|f\|_\alpha$$

if f is α-Lipschitz and $\|f\|_\alpha$ is its α-Lipschitz norm, provided that the wavelet system has enough vanishing moments (i.e. the dual wavelet of ϕ, which is a Radon measure, has at least $\lceil \alpha \rceil$ vanishing moments).

This error estimate can be refined: we can prove that a local error decay rate depends on the local Lipschitz smoothness of the unknown function f: for example, if f is Lipschitz-α in a vicinity of a point $\boldsymbol{x}$, then

$$|f(\boldsymbol{x}) - f_{\mathcal{X}}(\boldsymbol{x})| \leq M \times h(\mathcal{X})^\alpha.$$

The recent result which has been proved for this scheme is that the local error bound also locally depends on some *local sample density*: in order to do this, we first have to find an adequate and explicit definition for a *local fill distance*.

§3. Varying Sample Density

A first definition of a local fill distance of a sample set $h(\mathcal{X})$ at point $\boldsymbol{x}$ could simply be

$$h_{\mathcal{X}}(\boldsymbol{x}) = \min_{\boldsymbol{x}' \in \mathcal{X}} |\boldsymbol{x} - \boldsymbol{x}'|.$$

Ideally we would like to have a local error bound of the form

$$|f_{\mathcal{X}}(\boldsymbol{x}) - f(\boldsymbol{x})| \leq h_{\mathcal{X}}(\boldsymbol{x}).$$

With this, we cannot prove this bound. Especially, since the scheme is not truly interpolating (some samples are discarded), we know that the local fill distance can vanish at points where the error is nonzero.

Definition 6. *Instead of a true local fill distance, we use a* local fill distance envelope *of parameter P defined as*

$$\bar{h}_{\mathcal{X}}(\boldsymbol{x}) = \min\{h : \|\nabla h'\| \leq 1/P \quad \text{and} \quad h(\boldsymbol{x}) \geq h_{\mathcal{X}}(\boldsymbol{x}) \forall\ \boldsymbol{x}\}.$$

Theorem 7. *Given a wavelet system $(\phi_{j\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{J}}$ with $\lceil\alpha\rceil$ vanishing moments, there exist parameters (P,ρ) such that the conclusions of Th. 4 hold, and if f is α-Lipschitz, then*

$$|f_{\mathcal{X}}(\boldsymbol{x}) - f(\boldsymbol{x})| \leq M \times \bar{h}_{\mathcal{X}}(\boldsymbol{x})^{\alpha}\|f\|_{\alpha},$$

where again $\|f\|_{\alpha}$ is the α-Lipschitz norm of f, M does not depend on f or $\mathcal{X}$, and $\bar{h}$ is the local fill distance envelope of parameter P.

The proof of the theorem consists in showing that after allocation and subsystem selection, the remaining wavelets are locally at a resolution which is related to the local fill distance. This is exactly stated as follows: defining levels sets Λ_j by

$$\Lambda_j = \{\boldsymbol{x} : \bar{h}_{\mathcal{X}}(\boldsymbol{x}) \leq M2^{-j}\},$$

all wavelets $(j',\boldsymbol{k})$ whose center $\nu_{j'\boldsymbol{k}}$ is in Λ_j and for which $j' < j$ appear in the selected subsystem (1), i.e. $(j',\boldsymbol{k}) \in \mathcal{J}_S$.

Then, local error bounds are proven on the error $f_{\mathcal{X}} - f$ using the expansion of $f_{\mathcal{X}}$ on the subfamily of relocated wavelets of Def. 5.

§4. Conclusion

With Theorem 7, we can describe the asymptotic behaviour of our wavelet scheme fairly accurately. The approximation error locally depends on the sample density and on the local unknown function smoothness. The condition number of the system matrix to be solved also increases very slowly with the sample separation distance. Formally, we thus have an efficient approximation process for *any* input space dimension. As it is now, the process still has a drawback: as the dimension of the input space increases, the constants of the bounds above increase very sharply making it completely inpractical for dimensions above 3. The challenge is now to modify the scheme to have tighter bounds and to make it usable at higher dimensions.

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