

# 18.100A: Typed Lecture Notes

## Lecture 23: Pointwise and Uniform Convergence of Sequences of Functions

### Sequences of Function

#### Power Series

**Remark 1.** *Power series motivate the general discussion of sequences of functions.*

#### **Definition 2** (Power series)

A power series about  $x_0$  is a series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

#### **Theorem 3**

Suppose

$$R = \lim_{m \rightarrow \infty} |a_m|^{1/m}$$

exists, and let

$$p = \begin{cases} \frac{1}{R} & R > 0 \\ \infty & R = 0. \end{cases}$$

Then,  $\sum a_m (x - x_0)^m$  converges absolutely if  $|x - x_0| < p$  and diverges if  $|x - x_0| > p$ .

#### **Definition 4** (Radius of Convergence)

In the above theorem, we define  $p$  to be the **radius of convergence**.

**Proof:** We have

$$\lim_{n \rightarrow \infty} |a_m (x - x_0)^m|^{1/m} = R|x - x_0|,$$

and the theorem follows by the Root test. □

Suppose  $\sum a_m (x - x_0)^m$  is a power series with radius of convergence  $p$ . Furthermore, define  $f : (x_0 - p, x_0 + p) \rightarrow \mathbb{R}$  such that

$$f(x) := \sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Then,  $f$  is a limit of a sequence of functions

$$f(x) = \lim_{n \rightarrow \infty} f_n(x),$$

for  $x \in (x_0 - p, x_0 + p)$  and where

$$f_n(x) = \sum_{m=0}^n a_m (x - x_0)^m.$$

### Example 5

For example, we have

$$f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.$$

**Question 6.** *This concept begs a number of questions:*

1. Is  $f$  continuous?
2. Is  $f$  differentiable, and does  $f' = \lim_{n \rightarrow \infty} f'_n$ ?
3. If 1. is true, does

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n?$$

These questions will be the key motivator for the last section of this course.

### Pointwise and Uniform Convergence

We now consider a setting far more general than power series.

#### Definition 7 (Pointwise Convergence)

For  $n \in \mathbb{N}$ , let  $f_n : S \rightarrow \mathbb{R}$ . Let  $f : S \rightarrow \mathbb{R}$ . We say that  $\{f_n\}$  converges pointwise to  $f$  if for all  $x \in S$ ,

$$\lim_{n \rightarrow \infty} f_n(x) = f(x).$$

Let's consider some examples.

1. Let  $f_n(x) = x^n$  on  $[0,1]$ . Then,

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}.$$

Thus,  $\{f_n\}$  converges to the above pointwise function. Hence, notice that a sequence of continuous functions may not converge pointwise to a continuous function!

2. Let  $f_n(x) = \sum_{m=0}^n x^m$  for  $x \in (-1, 1)$ . Then,

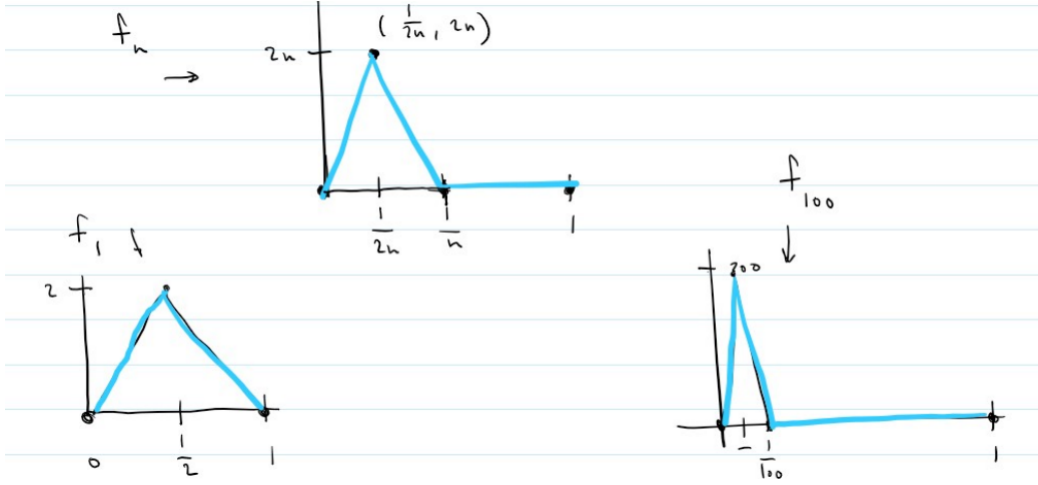
$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \sum_{m=0}^n x^m = \frac{1}{1-x}.$$

Hence, pointwise, this sequence converges to its power series (see the above example).

3. Let  $f_n(x) : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f_n(x) = \begin{cases} 4n^2 x & x \in [0, \frac{1}{2n}] \\ 4n - 4n^2 x & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases}.$$

We can picture this sequence (on the next page)



Then,  $\lim_{n \rightarrow \infty} f_n(0) = \lim_{n \rightarrow \infty} 0 = 0$ . Let  $x \in (0, 1]$ . Let  $N \in \mathbb{N}$  such that  $\frac{1}{N} < x$ . Then, for all  $n \geq N$ ,

$$f_n(x) = 0.$$

Therefore,

$$\{f_n(x)\} = f_1(x), \dots, f_{N-1}(x), 0, 0, 0, \dots$$

Hence,  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in [0, 1]$ . Thus,  $\{f_n\}$  converges pointwise to  $f(x) = 0$  on  $[0, 1]$ .

### Definition 8 (Uniform Convergence)

For  $n \in \mathbb{N}$ , let  $f_n : S \rightarrow \mathbb{R}$ , and let  $f : S \rightarrow \mathbb{R}$ . Then, we say  $f_n$  **converges to  $f$  uniformly** or **converges uniformly to  $f$**  if  $\forall \epsilon > 0 \exists M \in \mathbb{N}$  such that for all  $n \geq M \forall x \in S$ ,

$$|f_n(x) - f(x)| < \epsilon$$

### Theorem 9

If  $f_n : S \rightarrow \mathbb{R}$ ,  $f : S \rightarrow \mathbb{R}$ , and  $f_n \rightarrow f$  uniformly, then  $f_n \rightarrow f$  pointwise.

**Proof:** Let  $c \in S$  and let  $\epsilon > 0$ . Then,  $f_n \rightarrow f$  uniformly implies that there exists  $M_0 \in \mathbb{N}$  such that for all  $n \geq M, \forall x \in S, |f_n(x) - f(x)| < \epsilon$ . Choose  $M = M_0$ . Then,  $\forall n \geq M$ ,

$$|f_n(c) - f(c)| < \epsilon.$$

Thus,  $\lim_{n \rightarrow \infty} f_n(c) = f(c)$  for all  $c \in S$ , and therefore  $f_n \rightarrow f$  pointwise. □