

18.100A: Typed Lecture Notes

Lecture 6: The Uncountability of the Real Numbers

Theorem 1 (Triangle Inequality)

$\forall x, y \in \mathbb{R},$

$$|x + y| \leq |x| + |y|.$$

Proof: Let $x, y \in \mathbb{R}$. Then, $x + y \leq |x| + |y|$ and

$$(-x) + (-y) \leq |-x| + |-y| = |x| + |y|.$$

Therefore, $-(|x| + |y|) \leq x + y \leq |x| + |y|$. Hence,

$$|x + y| \leq |x| + |y|$$

by our previous theorem. □

Remark 2. We may denote the Triangle Inequality with \triangle -inequality as a shorthand.

Question 3. As we showed in Assignment 1, we know that \mathbb{Q} is countable. Is the set of real numbers countable?

Recall 4

Recall that a set A is countable if A is either finite or $|A| = |\mathbb{N}|$.

We can think of \mathbb{Q} as decimal expansions. In other words, we can think of a rational number x as being in the form

$$x = 10^k d_k + \cdots + 10d_1 + d_0 + 10^{-1}d_{-1} + \cdots + 10^{-M}d_{-M}$$

with $d_i \in \{0, 1, 2, 3, \dots, 9\}$. We may write

$$x = d_k d_{k-1} \dots d_1 d_0 \bullet d_{-1} \dots d_{-M}$$

where \bullet is the decimal point. The same can be said about real numbers if we allow for infinite decimal expansions.

Definition 5

Let $x \in (0, 1]$ and let $d_{-j} \in \{0, 1, \dots, 9\}$. We say that x is **represented** by the digits $\{d_{-j} \mid j \in \mathbb{N}\}$, i.e. $x = 0 \bullet d_{-1} d_{-2} \dots$, if

$$x = \sup\{10^{-1}d_{-1} + 10^{-2}d_{-2} + \cdots + 10^{-n}d_{-n} \mid n \in \mathbb{N}\}.$$

Here is an example: $.2500 = \sup\{2 \cdot 10^{-1}, 2 \cdot 10^{-1} + 5 \cdot 10^{-2}, 2 \cdot 10^{-1} + 5 \cdot 10^{-2} + 0 \cdot 10^{-3}, \dots\}$. Notice here that after a while the previous set becomes $\frac{1}{4}$ repeating. Hence, we have $.2500 = \sup\{\frac{1}{5}, \frac{1}{4}\} = \frac{1}{4}$.

Theorem 6

For every $x \in (0, 1]$, there exists a unique sequence of digits d_{-j} such that $x = 0 \bullet d_{-1}d_{-2} \dots$ and

$$0 \bullet d_{-1}d_{-2} \dots d_{-n} < x \leq 0 \bullet d_{-1}d_{-2} \dots d_{-n} + 10^{-n}.$$

Furthermore, for every set of digits $\{d_{-j} \mid j \in \mathbb{N}\}$, there exists a unique $x \in [0, 1]$ such that $x = 0 \bullet d_{-1} \dots$.

Notice however that the representative of $\frac{1}{2}$ is $0.4999 \dots$.

Theorem 7 (Cantor)

$(0, 1]$ is uncountable.

Proof: We will prove this through contradiction. Suppose that $(0, 1]$ is countable. Therefore, there exists a bijection $x : \mathbb{N} \rightarrow (0, 1]$. We now construct a $y \in (0, 1]$ such that y is not in the range of x . We write

$$x(n) = 0 \bullet d_{-1}^{(n)} d_{-2}^{(n)} \dots$$

These are not exponents! This is the set of digits for a given $n \in \mathbb{N}$. In other words, x takes in a natural number n and maps it to the sequence of digits $\{d_{-j}^{(n)} \mid n \in \mathbb{N}\}$. Let

$$e_{-j} = \begin{cases} 1, & d_{-j}^{(j)} \neq 1 \\ 2, & d_{-j}^{(j)} = 1 \end{cases}.$$

Let $y = 0 \bullet e_{-1}e_{-2} \dots$. Then, $\forall n \in \mathbb{N}$,

$$0 \bullet e_{-1}e_{-2} \dots e_{-n} \leq y < 0 \bullet e_{-1} \dots e_{-n} + 10^{-n}$$

since all e_{-j} s are positive. Thus, $0 \bullet e_{-1} \dots$ is the unique decimal expansion of y . However, for all $n \in \mathbb{N}$, $d_{-n}^{(n)} \neq e_{-n}$. Therefore, $\forall n$, $x(n) \neq y$. This is a contradiction, and thus $(0, 1]$ is uncountable. \square

So $(0, 1] \subset \mathbb{R}$ is uncountable!

Corollary 8

The set of real numbers, \mathbb{R} , is uncountable.

Sequences and Series

Remark 9. *Analysis is the study of limits.*

Sequences and Limits

Definition 10 (Sequence of Reals)

A **sequence** of real numbers is a function $x : \mathbb{N} \rightarrow \mathbb{R}$. We denote $x(n) = x_n$ and we denote the sequence by $\{x_n\}_{n=1}^{\infty}$, $\{x_n\}$, or x_1, x_2, \dots .

Definition 11

A sequence $\{x_n\}$ is **bounded** if $\exists B \geq 0$ such that $\forall n$, $|x_n| \leq B$.

One example of a bounded sequence is $x_n = \frac{1}{n}$, since $|\frac{1}{n}| \leq 1$ for all $n \in \mathbb{N}$. However, $x_n = n$ is not bounded.

Remark 12. A sequence is different from a set!

For example,

$$-1, 1, -1, 1, \dots = \{(-1)^n\}_{n=1}^{\infty},$$

while

$$\{(-1)^n \mid n \in \mathbb{N}\} = \{-1, 1\}.$$

Definition 13 (Sequence Convergence of Reals)

A sequence $\{x_n\}$ **converges** to $x \in \mathbb{R}$ if $\forall \epsilon > 0, \exists M \in \mathbb{N}$ such that $\forall n \geq M$,

$$|x_n - x| < \epsilon.$$

A sequence that converges is said to be **convergent**, and otherwise is said to be **divergent**. We can also define divergence as the negation of convergent.

Negation 14 (Not Convergent)

The sequence $\{x_n\}$ is **not convergent**, or **divergent** if $\exists \epsilon_0 > 0$ such that $\forall M \in \mathbb{N}, \exists n \geq M$ so that

$$|x_n - x| \geq \epsilon_0.$$

We now prove two theorems:

Theorem 15

If $\{x_n\}$ converges for x and y , then $x = y$. In other words, limits of convergent sequences of real numbers are unique.

Theorem 16

Let $x, y \in \mathbb{R}$. If $\forall \epsilon > 0, |x - y| < \epsilon$, then $x = y$.

Proof: We first prove the second theorem. Suppose that $x \neq y$. Then, $|x - y| > 0$. Hence, choosing $\epsilon = \frac{|x-y|}{2}$, we have

$$|x - y| \leq \frac{|x - y|}{2} \implies \frac{|x - y|}{2} < 0$$

which is a contradiction. □

Using this we prove the former theorem. Suppose x_n converges to x and to y . We will show that for all $\epsilon > 0$, $|x - y| < \epsilon$. Firstly, given $x_n \rightarrow x$, for $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$,

$$|x_n - x| < \frac{\epsilon}{2}.$$

Then, given $x_n \rightarrow y$, for $\epsilon > 0$ there exists an $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$,

$$|x_n - y| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $\epsilon > 0$ there exists an $N = \max\{N_1, N_2\}$ such that for all $n \geq N$,

$$|x - y| \leq |x - x_n| + |x_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the Triangle Inequality. Hence, for all $\epsilon > 0$, $|x - y| < \epsilon$. Therefore, $x = y$. \square

Notation 17

We write $x = \lim_{n \rightarrow \infty} x_n$ or $x_n \rightarrow x$.

Example 18

Given the sequence $x_n = c \forall n$, $\lim_{n \rightarrow \infty} x_n = c$.

Proof: Let $\epsilon > 0$ and $M = 1$. Thus, for all $n \geq 1$,

$$|x_n - c| = |c - c| = 0 < \epsilon.$$

■

Example 19

$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Proof: Let $\epsilon > 0$. Choose $M \in \mathbb{N}$ such that $M^{-1} > \epsilon^{-1}$. Hence, for all $n \geq M$, $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{M} \leq \epsilon$. ■