

# 18.100A: Typed Lecture Notes

## Lecture 18:

### Weierstrass's Example of a Continuous and Nowhere Differentiable Function

#### Theorem 1

If  $f : I \rightarrow \mathbb{R}$  is differentiable at  $c \in I$ , then  $f$  is continuous at  $c$ .

**Proof:** Since every point of  $I$  is a cluster point of  $I$ ,  $f$  is continuous at  $c \in I \iff \lim_{x \rightarrow c} f(x) = f(c)$ . Now,

$$\begin{aligned}\lim_{x \rightarrow c} f(x) &= \lim_{x \rightarrow c} (f(x) - f(c) + f(c)) \\ &= \lim_{x \rightarrow c} \left( (x - c) \frac{f(x) - f(c)}{x - c} + f(c) \right) \\ &= 0 \cdot f'(c) + f(c) = f(c).\end{aligned}$$

□

**Question 2.** *Is the converse true? Does  $f$  being continuous imply that  $f$  is differentiable?*

The answer, is **no**.

#### Example 3

Let  $f(x) = |x|$ . Then,  $f$  is not differentiable at 0.

**Proof:** We find a sequence  $x_n \rightarrow 0$  such that

$$\lim_{n \rightarrow \infty} \frac{f(x_n) - f(0)}{x_n - 0} \text{ does not exist.}$$

Let  $x_n = \frac{(-1)^n}{n}$ . Then,  $\lim_{n \rightarrow \infty} x_n = 0$ . However,

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{|(-1)^n/n|}{(-1)^n/n} = (-1)^n,$$

and  $\lim_{n \rightarrow \infty} (-1)^n$  does not exist. ■

**Question 4.** *If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then does there exist a  $c \in \mathbb{R}$  such that  $f$  is differentiable at  $c$ ?*

The answer is again **no**! This was shown by Weierstrass, aka the Godfather.

The basic idea is to build a continuous function that is a sum of highly oscillating functions.

**Remark 5.** *Note that we number the upcoming theorems so we may reference them a bit later in this lecture.*

**Theorem 6 (Theorem I)**

We will show the following

1.  $\forall x, y \in \mathbb{R}, |\cos x - \cos y| \leq |x - y|$ .
2. Let  $c \in \mathbb{R}$ . Then, for all  $K \in \mathbb{N}$ ,  $\exists y \in (c + \pi/K, c + 3\pi/K)$  such that

$$|\cos(Kc) - \cos(Ky)| \geq 1.$$

**Proof:**

1. In the proof of continuity of  $\sin x$ , we showed that  $\forall x, y \in \mathbb{R}, |\sin x - \sin y| \leq |x - y|$ . Thus,

$$|\cos x - \cos y| = |\sin(x + \pi/2) - \sin(y + \pi/2)| \leq |x - y|.$$

2. The function  $f(x) = \cos(Kx)$  is a  $\frac{2\pi}{K}$ -periodic function. In particular,  $([-1, 1] \setminus \cos(Kc)) \subset f(c + \pi/K, c + 3\pi/K)$ .

If  $\cos Kc \geq 0$ , then we choose  $y$  such that  $\cos(Ky) = -1$ . If  $\cos(Kc) < 0$ , then we choose  $y$  such that  $\cos(Ky) = 1$ . This completes the proof

□

**Theorem 7 (Theorem II)**

For all  $a, b, c \in \mathbb{R}$ ,

$$|a + b + c| \geq |a| - |b| - |c|.$$

**Proof:** We apply the Triangle Inequality twice:

$$|a| = |a + b + b + (-b) + (-c)| \leq |a + b + b| + |b + c| \leq |a + b + c| + |b| + |c|.$$

□

**Theorem 8 (Theorem III)**

We will show the following:

1.  $\forall x \in \mathbb{R}, \sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$  is absolutely convergent.
2. The function  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$  is bounded and continuous.

**Proof:**

1.  $\forall k, \left| \frac{\cos(160^k x)}{4^k} \right| \leq 4^{-k}$ . Hence, by the Comparison Test,

$$\sum_{k=0}^{\infty} \left| \frac{\cos(160^k x)}{4^k} \right| \text{ converges.}$$

2. For all  $x \in \mathbb{R}, |f(x)| \leq \sum_{k=0}^{\infty} \frac{|\cos(160^k x)|}{4^k} \leq \sum_{k=0}^{\infty} 4^{-k} = \frac{4}{3}$ . Therefore,  $f$  is bounded.

We now show that  $f$  is continuous over  $\mathbb{R}$ . Suppose  $c \in \mathbb{R}$  and  $x_n \rightarrow c$ . Note that  $\{|f(x_n) - f(c)|\}_n$  is bounded, and thus

$$\lim_{n \rightarrow \infty} |f(x_n) - f(c)| = 0 \iff \limsup_{n \rightarrow \infty} |f(x_n) - f(c)| = 0.$$

We claim that for all  $\epsilon > 0$ ,  $\limsup_{n \rightarrow \infty} |f(x_n) - f(c)| \leq \epsilon$ . Let  $\epsilon > 0$ . Choose  $M_0$  such that  $\sum_{k=M_0+1}^{\infty} 4^{-k} < \frac{\epsilon}{2}$ . Then,

$$\begin{aligned} \limsup_{n \rightarrow \infty} |f(x_n) - f(c)| &= \limsup_n \left| \sum_{k=0}^{M_0} \frac{\cos(160^k x_n)}{4^k} - \frac{\cos(160^k c)}{4^k} + \sum_{k=M_0+1}^{\infty} \frac{\cos(160^k x_n)}{4^k} - \frac{\cos(160^k c)}{4^k} \right| \\ &\leq \limsup_n \sum_{k=0}^{M_0} 4^{-k} |\cos(160^k x_n) - \cos(160^k c)| + \sum_{k=M_0+1}^{\infty} 4^{-k} (|\cos(160^k x_n)| + |\cos(160^k c)|) \\ &\leq \limsup_n \left( \sum_{k=0}^{M_0} 40^k \right) |x_n - c| + \epsilon = \epsilon. \end{aligned}$$

□

### Theorem 9 (Weierstrass)

The function  $f(x) = \sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$  is nowhere differentiable.

**Proof:** Let  $c \in \mathbb{R}$ . We will construct a sequence  $x_n \rightarrow c$  such that  $\left\{ \frac{f(x_n) - f(c)}{x_n - c} \right\}_n$  is unbounded. By Theorem I 2),  $\forall n \in \mathbb{N}$  there exists an  $x_n$  such that **a)**  $\frac{\pi}{160^n} < x_n - c < \frac{3\pi}{160^n}$  and **b)**  $|\cos(160^n c) - \cos(160^n x_n)| \geq 1$ .

By **a)**,  $x_n \neq 0 \forall n$  and  $|x_n - c| \leq \frac{3\pi}{160^n} \rightarrow 0$ . Let  $f_k(x) = \frac{\cos(160^k x)}{4^k}$  so  $f(x) = \sum f_k(x)$ . Let  $n \in \mathbb{N}$ . Thus, denote

$$\begin{aligned} f(c) - f(x_n) &= f_n(c) - f_n(x_n) + \sum_{k=0}^{n-1} (f_k(c) - f_k(x_n)) + \sum_{k=n}^{\infty} (f_k(c) - f_k(x_n)) \\ &:= a_n + b_n + c_n. \end{aligned}$$

Therefore, by Theorem II,

$$|f(c) - f(x_n)| \geq |a_n| - |b_n| - |c_n|.$$

By **b)**,  $|a_n| = 4^{-n} |\cos(160^n x_n) - \cos(160^n c)| \geq 4^{-n}$ . Furthermore, we have

$$|b_n| \leq \sum_{k=0}^{n-1} 4^{-k} |\cos(160^k c) - \cos(160^k x_n)| \leq \sum_{k=0}^{n-1} 4^{-k} \cdot 160^k |x_n - c| \leq \frac{3\pi}{160^n} \sum_{k=0}^{n-1} 40^k = \frac{3\pi}{160^n} \cdot \frac{40^n - 1}{39} \leq \frac{4^{-n+1}}{13}.$$

Finally, we have

$$|c_n| \leq \sum_{k=n+1}^{\infty} 4^{-k} (|\cos(160^k c)| + |\cos(160^k x_n)|) \leq 2 \sum_{k=n+1}^{\infty} 4^{-k} = 2 \cdot 4^{-n-1} \cdot \frac{4}{3} = 4^{-n} \frac{2}{3}.$$

Therefore, by the above inequalities, we have

$$|f(c) - f(x_n)| \geq 4^{-n} \left( 1 - \frac{4}{13} - \frac{2}{3} \right) = 4^{-n} \cdot \frac{1}{39}.$$

Therefore,

$$\frac{|f(c) - f(x_n)|}{|c - x_n|} \geq \frac{160^n}{3\pi} \cdot 4^{-n} \cdot \frac{1}{39} = \frac{40^n}{117\pi}.$$

Thus,  $\left\{ \frac{f(x_n) - f(c)}{x_n - c} \right\}_n$  is unbounded. □

**Remark 10.** In other words, this proof by Weierstrass shows that there exists a continuous function that is nowhere differentiable!