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"A Course in Operator Theory" by John Conway

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Overview

Motivation: $\mathbb{C}$

The complex numbers serve as the prototype for C*-algebras, a fundamental tool in Operator Theory.

- Magnitude $\longrightarrow$ Norms
- Complex conjugation $\longrightarrow$ Involutions

Norm

Given a vector space X, $\|\cdot\| : X \to [0, \infty)$ is a norm if it is positive definite, symmetric, and satisfies the triangle inequality

Involution

An involution is a map $a \to a^*$ from $\mathcal{A}$ into itself such that for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$,
(i) $(a^*)^* = a$; (ii) $(ab)^* = b^* a^*$; (iii) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$

The Definition

- **Normed space**: a vector space with a given norm $\|\cdot\|$
- **Banach Space**: a normed space that is Cauchy complete
- **Banach Algebra**: associative algebra on a Banach space (i.e., vector addition and scalar multiplication by complex numbers always make sense)
- **C*-Algebra**: Banach algebra with involution such that $\|a^*a\| = \|a\|^2$

C*-Algebra Examples

- $\mathbb{C}$
- $M_{n,n}(\mathbb{C})$: $n \times n$ matrices with complex entries. The involution of such a matrix is given by the conjugate transpose.
- One example of a C*-algebra is bounded continuous functions $f : X \to \mathbb{C}$ where $f^*(x) = \overline{f(x)}$. This space **has the identity**.
- $C_0(X)$, the algebra of continuous functions on local compact X that vanish at infinity, is abelian C*-algebra **without identity**

$*$-Homomorphisms

A $*$-homomorphism is a map $\rho : \mathcal{A} \to \mathcal{C}$ such that given $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$,

- $\rho(a + b) = \rho(a) + \rho(b)$
- $\rho(\alpha a) = \alpha\rho(a)$
- $\rho(ab) = \rho(a)\rho(b)$
- $\rho(a^*) = \rho(a)^*$

We can similarly define a $*$-isomorphism as a bijective $*$-homomorphism.

Proposition 1

If $\mathcal{A}$ has an identity and $\rho : \mathcal{A} \to \mathcal{C}$ is an algebraic homomorphism, then $\rho(1) = 1$.

Proof: Given $a \in \mathcal{A}$, $\rho(a) = \rho(1 \cdot a) = \rho(1)\rho(a)$, and thus $\rho(1) = 1$. $\square$

$*$-Homomorphism: Identity

What do we do if $\mathcal{A}$ does not have an identity?
One can show that we can simply "throw an identity in". In other words, we can use $*$-homomorphisms to easily adjoin an identity:

Remark

If $\mathcal{A}$ is C*-algebra without identity, then there exists a unique C*-algebra with identity $\mathcal{A}_1$ containing $\mathcal{A}$ as an ideal such that $\mathcal{A}_1/\mathcal{A}$ is one-dimensional.

Special Elements

When we look at the complex numbers under conjugation, there are some particularly interesting subsets:

1. $\mathbb{R}$, which are the only elements in $\mathbb{C}$ such that $\overline{\alpha} = \alpha$
2. the unit circle, where given $u \in \mathbb{C}$ such that u is on the circle, $\overline{u}$ is on the circle and $\|u\| = 1$.

These two examples motivate the following definitions:

- Hermitian: An element $a \in \mathcal{A}$ is Hermitian if $a^* = a$ (analogous to an element of $\mathbb{R}$ in $\mathbb{C}$). The set of Hermitian elements of $\mathcal{A}$ is denoted $\mathrm{Re}\mathcal{A}$
- Normal: An element $a \in \mathcal{A}$ is normal if $a^*a = aa^*$
- Unitary: if $\mathcal{A}$ has identity, $a \in \mathcal{A}$ is unitary if $aa^* = a^*a = 1$ (analogous to an element of the unit circle in $\mathbb{C}$)

Special Elements

We have some very useful propositions which follow:

Proposition 2

1. For all $a \in \mathcal{A}$, $a = x + iy$ for $x, y \in \text{Re}\mathcal{A}$
2. If u is unitary, then $\|u\| = 1$

More Propositions

Proposition 3

Let $\mathcal{A}$ be a C*-algebra, and let $h : \mathcal{A} \to \mathbb{C}$ be an algebraic homomorphism.

1. If $a \in \mathrm{Re}\mathcal{A}$, then $h(a) \in \mathbb{R}$
2. Given $a \in \mathcal{A}$, $h(a^*) = \overline{h(a)}$
3. Corollary: Every algebraic homomorphism from a C*-algebra into the real numbers is a $*$-homomorphism

Spectrum

Spectrum

Let $a \in \mathcal{A}$. Then, the spectrum of a, denoted $\sigma_{\mathcal{A}}(a)$, is the set

$$\sigma_{\mathcal{A}}(a) = \{\alpha \in \mathbb{C} \mid a - \alpha I \text{ is not invertible}\}$$

where I is the identity of $\mathcal{A}$.

Note that if $\mathcal{A}$ does not have an identity, we simply adjoin one. From functional analysis, one can show that the spectrum of a is the same as the set

$$\{h(a) \mid h : \mathcal{A} \to \mathbb{C} \text{ is a homomorphism}\}.$$

Corollary 4

Given $a \in \mathrm{Re}\mathcal{A}$, $\sigma_{\mathcal{A}}(a) \subset \mathbb{R}$.

Spectrum: Isomorphisms

Proposition 5

Let $\rho : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then, for $a \in \mathcal{A}$,

$$\sigma_{\mathcal{A}}(\rho(a)) \subseteq \sigma_{\mathcal{A}}(a).$$

Proof: We may assume $\mathcal{A}$ has identity. We will show that

$$\begin{aligned}
\sigma_{\mathcal{A}}(a)^c &= \{\alpha \in \mathbb{C} \mid a - \alpha I \text{ invertible}\} \\
&\subset \sigma_{\mathcal{A}}(\rho(a))^c \\
&= \{\alpha \in \mathbb{C} \mid \rho(a) - \alpha I \text{ invertible}\}.
\end{aligned}$$

Functional Calculus

The functional calculus is a generalization from functions on numbers to functions on operators.

Theorem 6

If $\mathcal{A}$ is any C*-algebra and a is a normal element of $\mathcal{A}$, then there is a $*$-monomorphism $f \to f(a)$ from $C(\sigma_{\mathcal{A}}(a))$ into $\mathcal{A}$.

Example 7

If X is a compact space and $g \in C(X)$, then $\sigma_{C(X)}(g) = g(X)$ and the functional calculus for g is given by $f \to f \circ g$ for all $f \in C(g(X))$.

Spectral Mapping Theorem

Theorem 10: Spectral Mapping Theorem

If $\mathcal{A}$ is a C^*-algebra, a is a normal element in $\mathcal{A}$, and $f \in C(\sigma_{\mathcal{A}}(a))$, then $\sigma_{\mathcal{A}}(f(a)) = f(\sigma_{\mathcal{A}}(a))$

References

John B. Conway (1999)
A Course in Operator Theory
Graduate studies in mathematics; v.21

The End