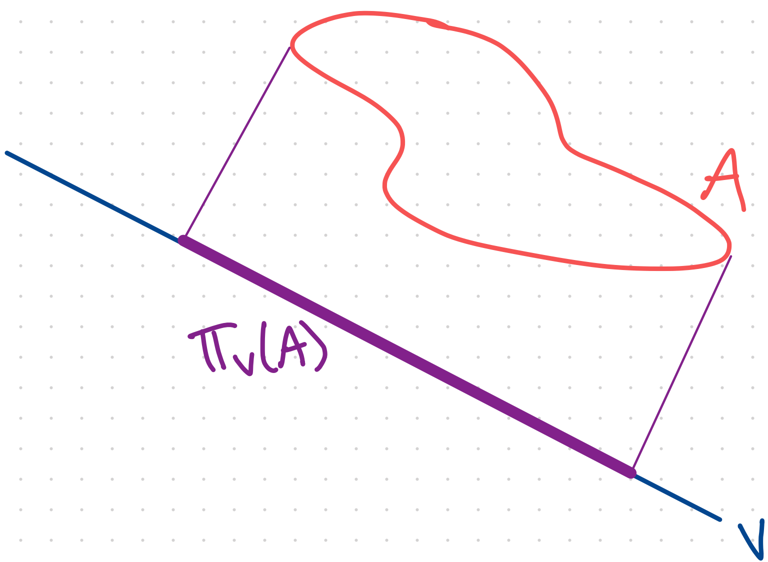
Exceptional Set Estimates for orthogonal projections

By: Paige Bright



Q: "How often is the shadow of a set small?"

Question 1: What is the relationship between the size of A and $\pi_v(A)$?

Tool: Hausdorff dimension!

Let $A \subseteq \mathbb{R}^2$, V be 1-dim subspace.

$\Longrightarrow$ Clearly,
$$\dim \pi_v(A) \leq \min\{1, \dim A\}.$$

Thm: [Marstrand's Projection Theorem]

For almost every $V \in G(2,1)$ (1-dim subspaces in $\mathbb{R}^2$),
$$\boxed{\dim \pi_v(A) = \min\{1, \dim A\}.}$$

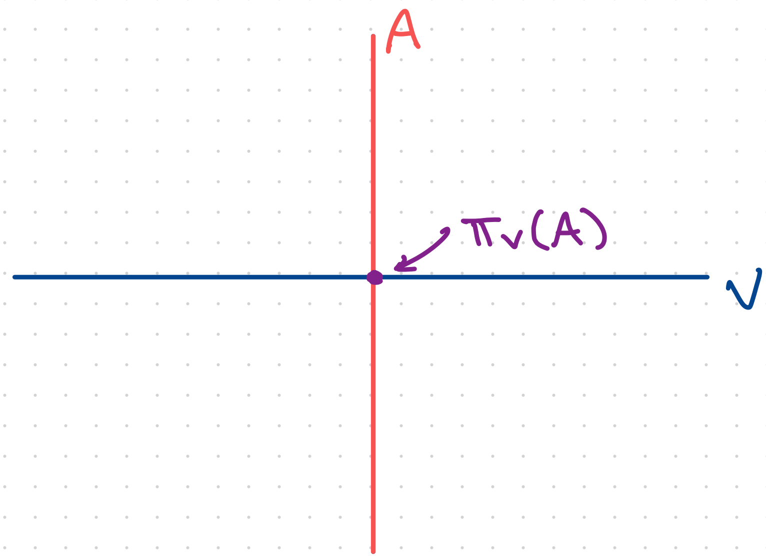
Thm: [Marstrand's Projection Theorem]

For almost every $V \in G(2,1)$ (1-dim subspaces in $\mathbb{R}^2$),

$$\dim \pi_V(A) = \min\{1, \dim A\}.$$

Example:

$A = \{x = 0\}$, $V = \{y = 0\}$



Thm: [Marstrand's Projection Theorem]
For almost every $V \in G(2,1)$ (1-dim subspaces in $\mathbb{R}^2$),
$$\dim \pi_V(A) = \min\{1, \dim A\}.$$

Question 2: When is the size of the projection even smaller?

Let $s < \min\{1, \dim A\}$ & define "the exceptional set of A":
$$E_s(A) = \{V \in G(2,1) \mid \dim \pi_V(A) < s\} \subseteq G(2,1).$$

Question 2': How can we bound $\dim E_s(A)$?

Rmk: We want our bound to be smaller than $\dim G(2,1) = 1$ to be nontrivial.

Question 2': How can we bound $\dim E_s(A)$?

Thm: Let $E_s(A) = \{V \in G(2,1) : \dim \pi_V(A) < s\}$. Then,

$$\dim E_s(A) \leq \begin{cases} 1 + s - \dim A & \text{(Falconer / Peres-Schlag)} \\ s & \text{(Kaufman)} \end{cases}$$

Notice that for $s < \min\{\dim A, 1\}$, $\Longrightarrow \dim E_s(A) < 1$. This implies Marstrand's Projection Theorem (as we can write $\{V \in G(2,1) : \dim \pi_V(A) < \min\{\dim A, 1\}\}$ as a countable union of (measure 0) exceptional sets).

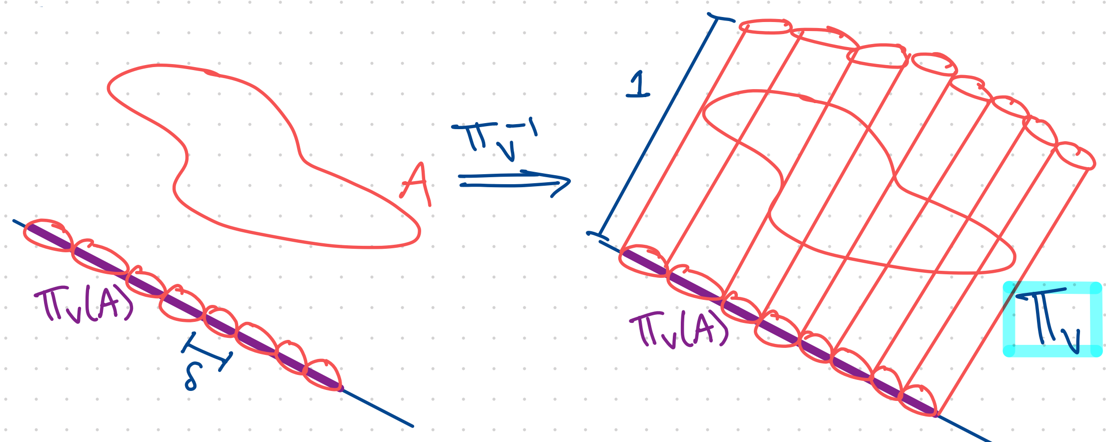
Outline of proofs (reproven by B.–Gan '22):

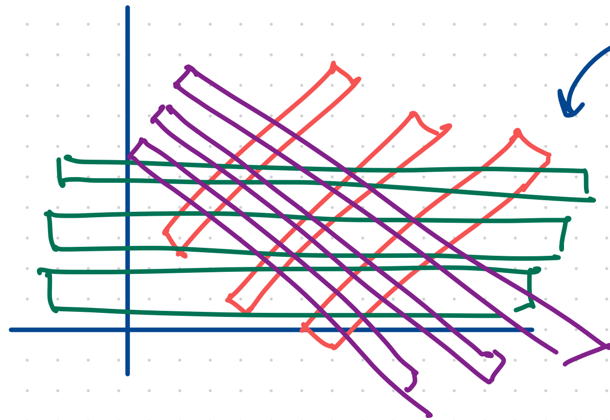
Heuristic: If we cover $E_s(A)$ by δ-balls, it takes $\sim \delta^{-\dim E_s(A)}$ balls. "t"

Similarly, $\sim \delta^{-\dim A}$ balls to cover A.

Additionally $\lessapprox \delta^{-s}$ balls to cover $\pi_v(A)$, $v \in E_s(A)$.

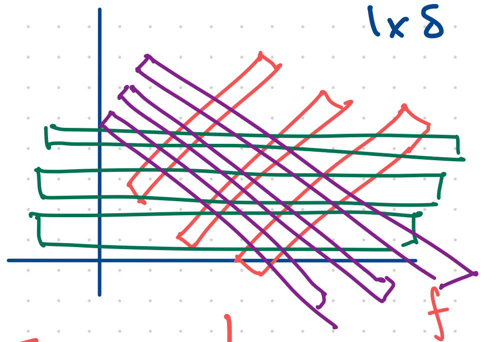
For every $v \in E_s(A)$, cover $\pi_v(A)$ by δ-balls ($\lessapprox \delta^{-s}$ many $\forall v$).



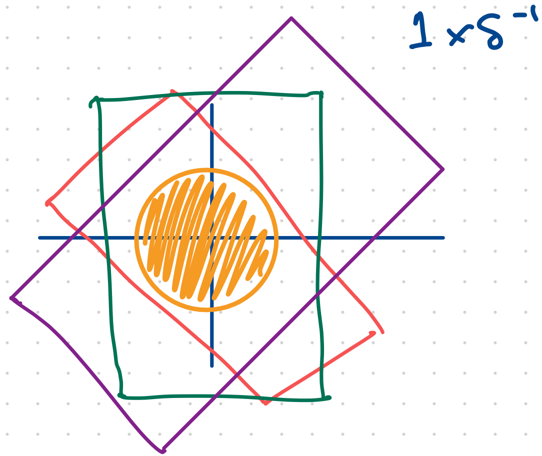


$\delta \times 1$ tubes

- Kaufman: Count the number of tubes (using (δ, s)-sets)
- Falconer: Consider the L^2-norm of a sum of indicator functions on the tubes (Fourier analysis?)
  - ↳ uses the high-low method



Fourier transform



WTS: $t \leq 1+s-a$

$$f_v = \sum_{T\in\mathbb{T}_v} \psi_T, \quad f = \sum_{v\in E_s(A)} f_v .$$

$$\delta^2 \delta^{-a} \delta^{-2t} \lesssim |A|\,\delta^{-2t} \leq \int_A |f|^2$$

$$\leq \int_{\mathbb{R}^2} |f|^2$$

$$= \int_{\mathbb{R}^2} |\hat{f}|^2$$

small

$$\lesssim \int_{\mathbb{R}^2} |\hat{f}_{high}|^2 + \cancel{\int_{\mathbb{R}^2} |\hat{f}_{low}|^2}$$

$$\lesssim \sum_{v\in E_s(A)} \sum_{T\in\mathbb{T}_v} \int |\psi_T|^2$$

$$\lesssim \delta^{-t}\cdot\delta^{-s}\cdot\delta \implies \delta^{-t} \lesssim \delta^{-1-s+a}$$

$\square$

You can generalize these to higher dimensions & codimensions!

Thm Let $A \subseteq \mathbb{R}^n$ Borel, and let $s < \min\{m, \dim A\}$. Define

$$E_s(A) := \{V \in G(n,m) \mid \dim \pi_V(A) < s\}.$$

Then

$$\dim E_s(A) \leq \begin{cases} m(n-m) + s - a & \text{(Falconer)} \\ m(n-m) + s - m & \text{(Kaufman)} \end{cases}$$

$$\Downarrow$$

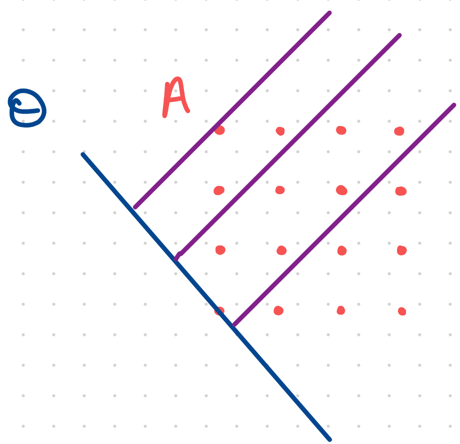
Thm [Marstrand Proj Thm]: For a.e. $V \in G(n,m)$,

$$\dim \pi_V(A) = \min\{m, \dim A\}.$$

In $\mathbb{R}^2$: $\dim E_s(A) \leq \begin{cases} 1 + s - \dim A & \text{(Falconer)} \\ s & \text{(Kaufman)} \end{cases}$

Can we do better? Yes!

To motivate the sharp statement, consider the following:
Let A be a uniform (finite) square lattice in $[0,1]^2$ & $0 \leq s \leq |A|$.
Consider $E_s(A) := \{\theta : |\pi_\theta(A)| < s\}$.
For all $\theta \in E_s(A)$, we can cover A by $\approx \frac{|A|}{s} =: r$ rich lines.



$\pi_\theta^{-1}(\pi_\theta(A))$ Thus, by Szemeredi-Trotter,

$$s \cdot \# E_s(A) \leq |r\text{-rich lines}| \lesssim \frac{|A|^2}{r^3} + \frac{|A|}{r}$$

$$\Longrightarrow \# E_s(A) \lesssim \frac{s^2}{|A|}.$$

This motivates the following continuum theorem, conjectured by Oberlin, and recently resolved by Ren–Wang '23:

$$\boxed{\text{Thm [Ren–Wang]: Let } A \subseteq \mathbb{R}^2, \text{ Borel. Then, for all } 0 \le s \le \min\{1, \dim A\}, \quad \dim(\{\theta \in S^1 : \dim \pi_\theta(A) < s\}) \le \max\{2s - \dim A, 0\}.}$$

Q: What if instead of considering <u>all</u> subspaces of $G(n,m)$, we <u>restricted</u> ourselves to a submanifold?

↪ "restricted projection problem"

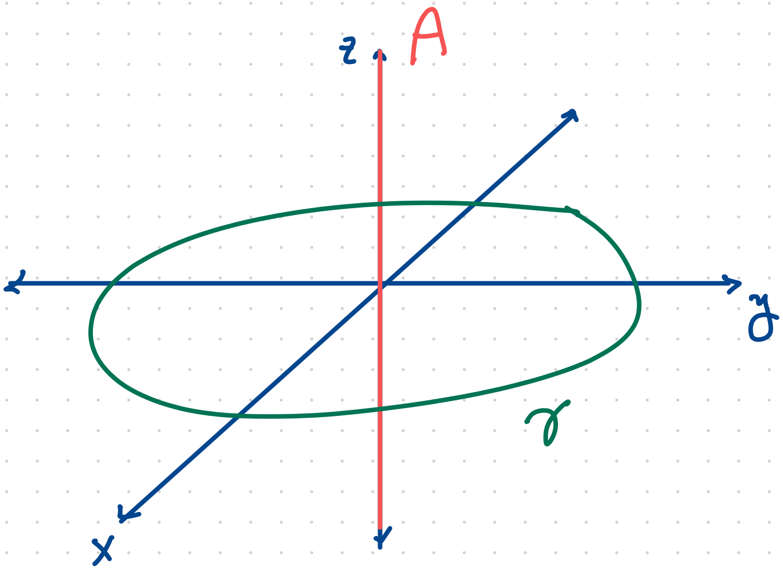
Example: Projection onto lines generated by a curve in $\mathbb{R}^3$.

Let $\gamma : [0,1] \to S^2$, C^2 curve, such that

$$\det(\gamma(\theta), \dot{\gamma}(\theta), \ddot{\gamma}(\theta)) \neq 0. \leftarrow \text{non degenerate}$$

Let $\rho_\theta : \mathbb{R}^3 \to \ell_\theta \cong \mathbb{R}$ be orthogonal proj onto line spanned by $\gamma(\theta)$, ℓ_θ.

Degenerate Example:



Projection of A onto <u>any</u> line through the origin in the xy-plane has dimension 0.

- $\gamma : [0,1] \to \mathbb{S}^2$, C^2, s.t. $\det(\gamma(\theta), \dot\gamma(\theta), \ddot\gamma(\theta)) \neq 0$.

- $\rho_\theta : \mathbb{R}^3 \to \ell_\theta \cong \mathbb{R}$

Thm: Let $A \subseteq \mathbb{R}^3$ Borel, γ nondegenerate

- For $0 \le s < \min\{\dim A, 1\}$,

$$\dim\{\theta : \dim \rho_\theta(A) < s\} \le \begin{cases} s & \text{Pramanik-Yang-Zahl} \\ 1 + \dfrac{s - \dim A}{2} & \text{Gan-Guth-Maldague} \end{cases}$$

$\Downarrow$

Thm: $\dim \rho_\theta(A) = \min\{\dim A, 1\}$ a.e. θ

You can also consider:

- (Restricted) Proj. of $\mathbb{R}^n$ onto k-planes:
  - $n=3$, $k=2$: Gan-Guo-Guth-Harris-Maldague-Wang
  - n arbitrary, $k=1$: Zahl
  - n & k arbitrary: Gan-Guo-Wang

Marstrand-type

- Projecting onto directions given by manifolds $\subseteq S^{n-1}$:
  - e.g. Jiayin Liu
- "When does the proj. have positive volume?"
  - GGGHMW and Harris
- Discrete/Finite Field version?
  - see B-Gan '23, Lund-Pham-Vinh

Other types of projection? Radial?

See talk: "Recent Developments in Radial Projections" !

Thank You!