Prime Number Theorem

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December 29, 2021

This manuscript contains notes for the proof of the prime number theorem that was presented in the Fall 2021 offering of MIT's 18.112 (*de facto* Complex Analysis), taught by Professor Alexei Borodin. It also contains some classical consequences of the prime number theorem that can be found many sources, such as A. J. Hilderbrand's lecture notes [3].

The notes culminate in a complex-analytic proof of the prime number theorem, which is stated below. This proof was discovered by D. J. Newman and is presented in D. Zagier's paper [2].

As usual, these notes are by no means complete, and any error in them is on my part. If you do find any mistakes, please let me know at chentouf@mit.edu.

1 Notation and preliminary results

If s is a complex number, let $\Re(s)$ denote its real part.

Definition 1 (Asymptotic Equivalence)**.** Given two functions f, g defined on the positive reals, we say that the two functions are asymptotically equivalent, and write $f \sim g$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Definition 2 (Big O notation)**.** Let f, g be two functions $\mathbb{R} \to \mathbb{R}_+$. We say that $f(x) \in O(g(x))$ if

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.$$

Whenever a sum or product is indexed by p, such as $\sum_p \frac{1}{p}$, this means that the sum/product is to be taken over all prime numbers. Similarly, $\sum_{p\le x}$ means the sum is to be taken over prime numbers less than or equal to x.

Definition 3. Define $\pi(x)$ as the prime number counting function, i.e.:

$$\pi(x) = \left|\{p \le x : p \text{ prime }\}\right|$$

Theorem 1 (Prime Number Theorem, Version 1)**.** *The asymptotic equivalence* $\pi(x) \sim \frac{x}{\log x}$ *holds.*

Recall the Riemann zeta function, which is defined as

$$\zeta(s) = \sum_{n\ge1} \frac{1}{n^s}$$

on the strip $\Re(s) > 1$. As we will see later, the zeta function can then be meromorphically extended to the complex plane, with exactly a simple pole at $s = 1$ of residue 1, and with trivial zeros at the negative even integers.

Define the two auxiliary functions

$$\Phi(s) = \sum_p \frac{\log p}{p^s}$$

and

$$\theta(x) = \sum_{p \le x} \log p.$$

Note that the series defining $\Phi(s)$ converges absolutely uniformly in any compact subset of $\Re(s) > 1$, and hence $\Phi(s)$ is holomorphic there. Through a series of steps, we will prove the asymptotic relation $\theta(x) \sim x$, which then immediately implies the prime number theorem.

Moreover, define the xi function as

$$\xi(s) = \pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s)$$

in the half-plane $\Re(s) > 1$. We recall a property of the xi function without proof.

Fact 1. *The function $\xi(s)$ has an analytic continuation into a meromorphic on the complex plane with only simple poles at $0, 1$, and*

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + F(s),$$

where $F(s)$ is entire.

Corollary 1. *The zeta function can then be meromorphically extended to the complex plane, with exactly a simple pole at $s = 1$ of residue 1, and with trivial zeros at the negative even integers.*

2 Steps towards the proof

Lemma 1. *The relation $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$ holds for $\Re(s) > 1$.*

Proof. Expand and use the Fundamental Theorem of Arithmetic to show that the LHS is the uniform limit of the partial products $\prod_{p \le x} \left(1 - \frac{1}{p^s}\right)^{-1}$. □

Lemma 2. *The function $\zeta(s) - \frac{1}{s-1}$ extends holomorphically to an entire function.*

Proof. Using Fact 1, and since $\frac{\pi^{s/2}}{\Gamma(s/2)}$ is entire, looking at the residues implies that $\zeta(s) - \frac{1}{s-1}$ extends to an entire function. □

Lemma 3. *We have the relation $\theta(x) \in O(x)$.*

Proof. If $n \in \mathbb{N}$, then

$$4^n = (1+1)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} > \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \ge \prod_{n < p \le 2n} p = e^{\theta(2n) - \theta(n)}.$$

Therefore

$$\theta(2n) - \theta(n) \le (2\log 2)n \tag{1}$$

Observe that for $\delta \le 1$, we have that $\theta(x+\delta) - \theta(x)$ is at most $\log(x+1)$. This allows us to bound $\theta(x)$ in terms of $\theta(\lceil x \rceil)$, and an application of the "Master Theorem" to Equation 1 implies that $\theta(x) \in O(x)$. □

Lemma 4. *In the closed half plane $\Re(s) \ge 1$, the zeta function is nonzero and $\Phi(s) - \frac{1}{s-1}$ is holomorphic.*

Proof. Recall that $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$ holds for $\Re(s) > 1$. By [1, Prop. 5.3.2], we have

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\frac{d}{ds}\left(1 - p^{-s}\right)^{-1}}{\left(1 - p^{-s}\right)^{-1}} = \sum_p \frac{-\log p}{p^s - 1} = -\Phi(s) - \sum_p \frac{\log p}{p^s(p^s - 1)}. \tag{2}$$

Since $\sum_p \frac{\log p}{p^s(p^s-1)}$ converges to a holomorphic function in $\Re(s) > \frac{1}{2}$, and using Corollary 1, we have that Φ extends meromorphically unto $\Re(s) > \frac{1}{2}$.

In particular, this implies that $\zeta(s)$ has no roots in $\Re(s) > 1$, as otherwise the right-hand-side would have a pole there, but $\Phi(s)$ is holomorphic in $\Re(s) > 1$ by a previous observation.

Note that

$$\operatorname*{Res}_{s=1} \Phi(s) = \lim_{\epsilon \to 0} \varepsilon \Phi(1+\epsilon) = 1.$$

That is, $\Phi(s)$ has a simple pole of residue 1 at $s = 1$, and by the analytic continuation of the zeta function and Equation 2, this is the only pole of Φ in $\Re(s) > \frac{1}{2}$. Therefore, $\Phi(s) - \frac{1}{s-1}$ has a holomorphic extension in $\Re(s) > \frac{1}{2}$.

Now assume that ζ has a zero of multiplicity μ at $s = 1 + i\alpha$. Since ζ is real on the real line, the Schwarz reflection principle implies that $1 - i\alpha$ is also a root of multiplicity μ. Let ν be the multiplicity of the root at $1 \pm 2i\alpha$.

Similarly, using Equation 2, we see that

$$\lim_{\epsilon \to 0} \epsilon \Phi(1+\epsilon \pm i\alpha) = \mu$$

and that

$$\lim_{\epsilon \to 0} \epsilon \Phi(1+\epsilon \pm 2i\alpha) = -\nu.$$

We can therefore compute:

$$\sum_{r=-2}^{2} \binom{4}{2+r} \Phi(1+\epsilon+ri\alpha) = \sum_{r=-2}^{2} \binom{4}{2+r} \sum_p \frac{\log p}{p^{1+\epsilon+ri\alpha}} = \sum_p \frac{\log p}{p^{1+\epsilon}} \left(p^{i\alpha/2} + p^{-i\alpha/2}\right)^4 \geq 0.$$

On the other hand, expanding the following expression yields

$$\lim_{\epsilon \to 0} \epsilon \sum_{r=-2}^{2} \binom{4}{2+r} \Phi(1+\epsilon+ri\alpha) 6 - 8\mu - 2\nu.$$

Combining the two results yields that $6 - 8\mu - 2\nu \geq 0$ and hence $4\mu + \nu \leq 3$. Since μ, ν are nonnegative integers (multiplicities of roots), we must have $\mu = 0$, and so $\zeta(s)$ has no roots on the line $\Re(s) = 1$. $\square$

Theorem 2 (Analytic Theorem)**.** *Let $f(t)$ be a bounded, locally integrable function over the non-negative reals. Suppose that the function*

$$g(z) = \int_0^\infty e^{-xt} f(t) dt$$

defined on $\Re(z) > 0$ extends homorphically to $Re(z) \geq 0$. Then $\int_0^\infty f(t)dt$ converges and equals $g(0)$.

Lemma 5. *The integral*

$$\int_1^\infty \frac{\theta(x) - x}{x^2} dx$$

converges.

Proof. Note that $\theta(x) = \sum_p \mathbb{1}_{x \geq p} \log p$, and so if $\Re(s) > 1$ then

$$s \int_1^\infty \frac{\theta(x)}{x^{s+1}} dx = \sum_p s \int_p^\infty \frac{dx}{x^{s+1}} \log p = \sum_p \frac{\log p}{p^s} = \Phi(s).$$

Making a change of variable $x = e^t$, we get that this is equal to

$$s \int_0^\infty e^{-st} \theta(e^t) dt.$$

Applying the analytic theorem to $f(t) = \theta(e^t)e^{-t} - 1$, which is clearly locally integrable and is bounded by Lemma 3, we get that

$$f(z) = \int_0^\infty e^{-zt}\Big(\theta(e^t)e^{-t} - 1\Big)dt = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}.$$

This is holomorphic by Lemma 4 and the poles at zero cancel by comparing residues, so the analytic theorem implies that

$$\int_0^\infty \Big(\theta(e^t)e^{-t} - 1\Big)dt < \infty.$$

Making a change of variables once again, we get that

$$\int_1^\infty \frac{\theta(x) - x}{x^2}dx$$

converges, as required. □

Lemma 6. *The asymptotic equivalence $\theta(x) \sim x$ holds.*

Proof. Assume, for the sake of contradiction, that there exists some $\lambda > 1$ such that $\theta(x) \geq \lambda x$ for infinitely many x. Then by Lemma 5:

$$\int_x^{\lambda x} \frac{\theta(t) - t}{t^2}dt \geq \int_x^{\lambda x} \frac{\lambda t - t}{t^2}dt = \int_1^{\lambda} \frac{\lambda - t}{t^2}dt > 0.$$

However, this contradicts the convergence of the integral from Lemma 5.

Similarly, we get a contradiction if we assume that there exists some $\lambda < 1$ such that $\theta(x) \leq \lambda x$ for infinitely many x. □

3 Proof of the PNT

Theorem 3 (Prime Number Theorem, Version 1)**.** *The asymptotic equivalence $\pi(x) \sim \frac{x}{\log x}$ holds.*

Proof. Note that

$$\theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x \tag{3}$$

However, we also have that for any $0 < \epsilon < 1$

$$\theta(x) = \sum_{p \leq x} \log p \geq \sum_{x^{1-\epsilon} \leq p \leq x} \log p \geq (1-\epsilon)\log x\big(\pi(x) - \pi(x^{1-\epsilon})\big). \tag{4}$$

Combining Eq. (3) and (4), we get that for any $0 < \epsilon < 1$,

$$\frac{\theta(x)}{\log x} \leq \pi(x) \leq \frac{\theta(x)}{(1-\epsilon)\log x} + O(x^{1-\epsilon}).$$

Now since $\theta(x) \sim x$, we conclude that $\pi(x) \sim \frac{x}{\log x}$. □

Definition 4. Let $\mathrm{Li}(x)$ be the logarithmic integral defined by $\int_2^x \frac{dt}{\log t}$.

Theorem 4 (Prime Number Theorem, Version 2)**.** *The asymptotic equivalence $\pi(x) \sim \mathrm{Li}(x)$ holds.*

Proof. It suffices to prove that $\mathrm{Li}(x) \sim \frac{x}{\log x}$. Using integration by parts:

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{(\log t)^2}.$$

Note that to prove $\mathrm{Li}(x) \sim \frac{x}{\log x}$, it suffices to show that

$$\lim_{x\to\infty} \frac{\log x}{x} \int_2^x \frac{dt}{(\log t)^2} = 0.$$

Applying l'Hôpital's rule, this is equal to

$$\lim_{x\to\infty} \left(\frac{1}{x} \int_2^x \frac{dt}{(\log t)^2} + \frac{1}{\log x} \right) = \lim_{x\to\infty} \frac{\int_2^x \frac{dt}{(\log t)^2}}{x}.$$

Once again, applying l'Hôpital's rule, we get that this is equal to

$$\lim_{x\to\infty} \frac{1}{(\log x)^2} = 0,$$

which concludes the proof. $\square$

In fact, $\mathrm{Li}(x)$ provides an even better approximation to $\pi(x)$ than $\frac{x}{\log x}$. It also turns out that $\mathrm{Li}(x)$ belongs to the class $O\big(\frac{x}{(\log x)^2}\big)$, althought this was not required for our proof.

4 Consequences of the prime number theorem

Theorem 5. *Let p_n be the n-th prime number. Then the the asymptotic equivalence $p_n \sim n \log n$ holds.*

Proof. The prime number theorem implies that $\pi(x) \log x \sim x$, and hence, by properties of asymptotic equivalence, we have that $\log \pi(x) + \log\log x \sim \log x$. However, $\lim_{x\to\infty} \frac{\log\log x}{\log x} = 0$, and so $\log \pi(x) \sim \log x$.

By the prime number theorem, we have that

$$n \sim \frac{p_n}{\log p_n},$$

and so $p_n \sim n \log p_n \sim n \log n$, as required. $\square$

A longer proof that contains a recurring idea in analytic number theory is also presented.

Proof. Fix $\epsilon > 0$. By the prime number theorem, we have that

$$n \sim \frac{p_n}{\log p_n}.$$

In particular, for large enough n, we have that $p_n^\epsilon \geq \log p_n$ and hence

$$p_n^{1-\epsilon} \leq n \leq p_n.$$

Taking logarithms, we see that

$$(1-\epsilon) \log p_n \leq \log n \leq \log p_n,$$

and so $\log p_n \sim \log n$. Therefore, we get that $p_n \sim n \log n$, as required. $\square$

Theorem 6. *For all $\epsilon > 0$, there exists N such that for all $n \geq N$, the interval $[n, (1+\epsilon)n]$ contains a prime number.*

Proof. The prime number theorem, and elementary properties of asymptotic equivalence, imply that

$$\frac{\pi\big((1+\epsilon)x\big)}{\pi(x)} \sim \frac{\frac{(1+\epsilon)x}{\log\left((1+\epsilon)x\right)}}{\frac{x}{\log x}} = \frac{(1+\epsilon)\log x}{\log\left((1+\epsilon)x\right)},$$

which by l'Hôpital's rule, tends to $1+\epsilon$. Hence, we have that $\lim_{x\to\infty} \frac{\pi\left((1+\epsilon)x\right)}{\pi(x)} = 1+\epsilon$, and so there exists N such that $\pi\big((1+\epsilon)x\big) > \pi(x)$ for all $x \geq N$. This directly implies the result. $\square$

Corollary 2. *Given any string of decimal digits $a_1 \dots a_k$, there exists infinitely many primes whose decimal representation begins with $a_1 \dots a_k$.*

Proof. Let $M = \overline{a_1 \dots a_k}$. The decimal representation of an integer n begins with $a_1 \dots a_k$ iff $10^k M \leq n < 10^k(M+1)$ for some $k \in \mathbb{N}$. Applying the previous theorem, we see that the interval $[10^k M, 10^k(M+1)]$ contains a prime for all large enough k. $\square$

References

[1] E. M. Stein and R. Shakarchi, *Complex Analysis,* Princeton University Press, 2003.

[2] D. Zagier, Newman's Short Proof of the Prime Number Theorem, *American Math. Monthly* **104**, 1999.

[3] A.J. Hildebrand, *Introduction to Analytic Number Theory,* 2005. Available at `https://faculty.math.illinois.edu/~hildebr/ant/main3.pdf`.