Computation of ϵ-equilibria in Separable Games

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Outline

- Motivation
- Previous work
  - Structural results (e.g. Karlin, Glicksberg 1950s)
  - SDP formulation of equilibrium for zero-sum polynomial games (Parrilo 2006)
- Background and definitions
- Theory and examples
- An algorithm

Continuous games

- Finite set of players $I = \{1, \ldots, n\}$. For player i, let:
  - the **pure strategy** space C_i be a compact metric space.
  - the **utility** or **payoff function** $u_i : \Pi_{j=1}^n C_j \to \mathbb{R}$ be continuous.
  - the **mixed strategy** space Δ_i be the set of Borel probability measures over C_i.
- Extend u_i to all of $\Pi_{j=1}^n \Delta_j$ by defining the utility to be the expected utility.
- Notation: $\sigma_i \in \Delta_i$ and $\sigma_{-i} \in \Pi_{j \neq i} \Delta_j$.

Equilibria

- An ϵ-**equilibrium** is a $\sigma \in \Pi_{j=1}^{n} \Delta_j$ such that for all i and $\tau_i \in \Delta_i$:

$$u_i(\tau_i, \sigma_{-i}) \leq u_i(\sigma_i, \sigma_{-i}) + \epsilon$$

  i.e. no player can unilaterally improve his payoff by more than ϵ.

- A **Nash equilibrium** is a 0-equilibrium.
- Theorem: Every continuous game has a Nash equilibrium (Glicksberg 1952).
- But this equilibrium may be arbitrarily complicated!

Separable games

- A continuous game is **separable** if it has payoffs:

$$u_i(s_1, \ldots, s_n) = \sum_{k=1}^{r} a_i^k f_1^k(s_1) \cdots f_n^k(s_n)$$

  where $a_i^k \in \mathbb{R}$ and $f_j^k : C_j \to \mathbb{R}$ is continuous.

- E.g. games with polynomial payoffs; finite games.
- For $\sigma_i \in \Delta_i$, define the **moments** $\nu_i^k = \int_{C_i} f_i^k \mathrm{d}\sigma_i$.
- Then:

$$u_i(\sigma_1, \ldots, \sigma_n) = \sum_{k=1}^{r} a_i^k \nu_1^k \cdots \nu_n^k$$

  so the payoffs are determined by the moments.

Finite-dimensional representations for separable games

- Theorem:

  Set of moments $\stackrel{def.}{=} \left\{(\nu_i^1, \ldots, \nu_i^r) | \sigma_i \in \Delta_i\right\}$

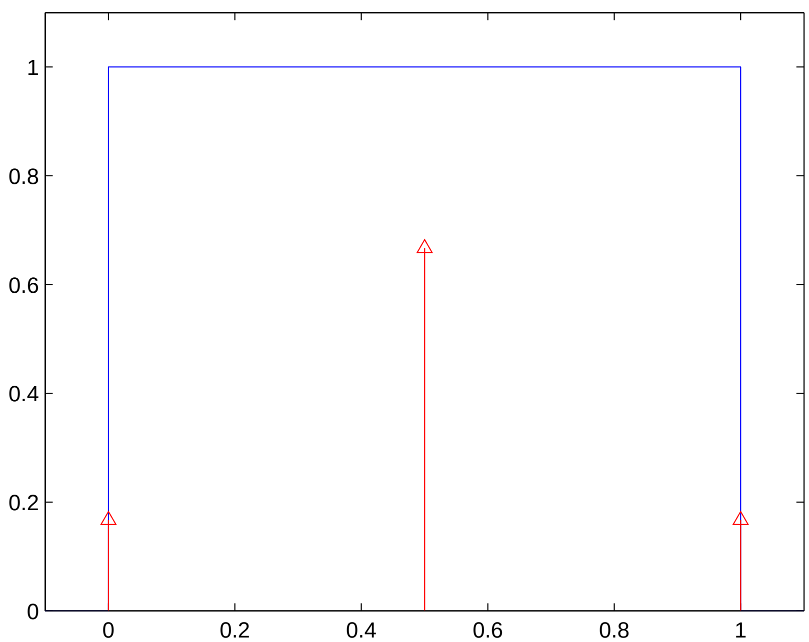
  $$= \left\{(\nu_i^1, \ldots, \nu_i^r) | \tau_i \in \Delta_i \text{ such that } |\operatorname{supp}(\tau_i)| \leq r + 1\right\}$$

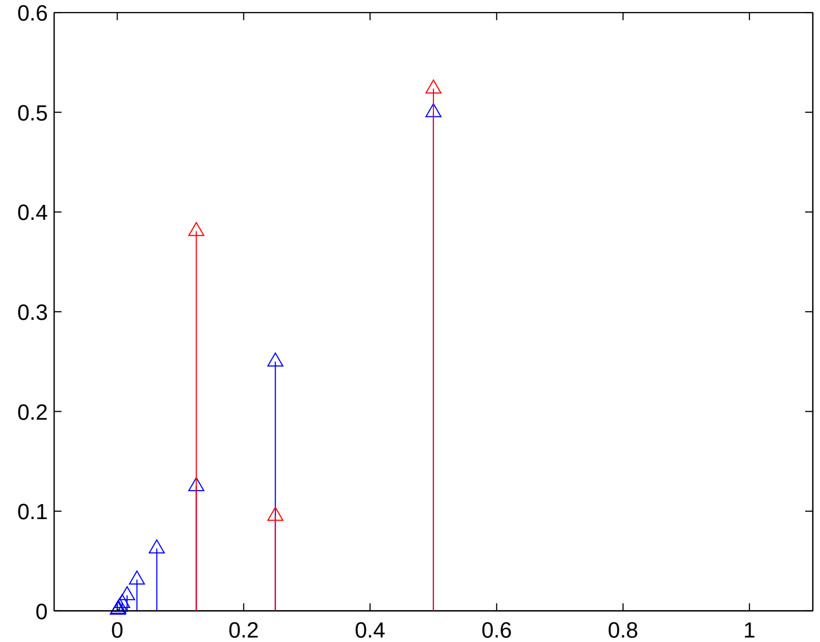
  Proof: separating hyperplanes, Carathéodory's thm.

- Any $\sigma_i \in \Delta_i$ has the same moments as a $\tau_i \in \Delta_i$ in which player i mixes among at most $r + 1$ strategies.
- The strategies σ_i and τ_i are **payoff equivalent**.
- A separable game has equilibria in which no player mixes among more than $r + 1$ strategies.

An example

$$C_1 = C_2 = [0,1]; \quad u_i(x,y) = a_i x y^2 + b_i x^2 y; \quad a_i, b_i \in \mathbb{R}$$





$\sigma_1 =$ uniform distribution

$$\tau_1 = \frac{1}{6}\delta(x) + \frac{2}{3}\delta\left(x - \frac{1}{2}\right) + \frac{1}{6}\delta(x-1)$$

$$\sigma_2 = \sum_{i=1}^{\infty} 2^{-i}\delta(y - 2^{-i})$$

$$\tau_2 = \frac{8}{21}\delta\left(y - \frac{1}{8}\right) + \frac{2}{21}\delta\left(y - \frac{1}{4}\right) + \frac{11}{21}\delta\left(y - \frac{1}{2}\right)$$

Classical results about separable games

Separable

$\Downarrow$

Mixed strategy spaces mod payoff equivalence relation are finite dimensional

$\Downarrow$	$\Downarrow$
Each mixed strategy is payoff equivalent to a finitely-supported mixed strategy	Each countably supported σ is payoff equivalent to a finitely supported τ such that $\mathrm{supp}(\tau) \subset \mathrm{supp}(\sigma)$

Some new results about separable games

Separable

$\Downarrow\Uparrow$

Mixed strategy spaces mod payoff equivalence relation are finite dimensional

$\Downarrow\not\Uparrow$

Each mixed strategy is payoff equivalent to a finitely-supported mixed strategy

$\Downarrow\Uparrow$

Each countably supported σ is payoff equivalent to a finitely supported τ such that $\mathrm{supp}(\tau) \subset \mathrm{supp}(\sigma)$

Proof ideas

- Extending a game from pure to mixed strategies yields multilinear payoffs.
- Modding out by payoff equivalence relation removes any superfluous structure introduced in this process without affecting multilinearity of the payoffs.
- Multilinear functions on finite-dimensional vector spaces are separable.
- To get counterexample in lower left, apply this procedure to a game whose pure strategy spaces are infinite-dimensional and whose payoffs are multilinear and non-degenerate.

Computing ϵ-equilibria for two-player separable games

- Assume $C_i = [-1, 1]$ and the utilities are Lipschitz.
- Discretize the game by choosing m equally spaced pure strategies for each player, call this set D_i.
- Choose m so that payoffs of the original game are always within ϵ of the payoffs obtained by rounding to the nearest point in D_i. By the Lipschitz assumption we may choose m proportional to $\frac{1}{\epsilon}$.
- Compute a Nash equilibrium of this finite game.
- This yields an ϵ-equilibrium of the separable game.

Will this work?

- In general computing an equilibrium of a finite game is not easy.
- But in this case the finite game has the same separable structure as the original game:

$$u_i(s_1, s_2) = \sum_{k=1}^{r} a_i^k f_1^k(s_1) f_2^k(s_2)$$

- In particular the finite game has an equilibrium in which each player mixes among at most $r + 1$ strategies, independent of the choice of $m \propto \frac{1}{\epsilon}$.

Computing an equilibrium of the finite game

- Choose a support: up to $r + 1$ strategies from the finite game for each player to play with positive probability.
- There exists an LP (size polynomial in m, r) to check whether this is the support of an equilibrium of the finite game (lose linearity with > 2 players).
- $$\# \text{ supports for each player} \leq \underbrace{\binom{m+r}{m-1}}_{\text{polynomial in } r \text{ for fixed } m}$$

Complexity of the algorithm

- The number of LPs and the time to solve each are both polynomial in r for fixed ϵ.
- So the algorithm is polynomial in r for fixed ϵ and similarly polynomial in $\frac{1}{\epsilon}$ for fixed r.
- A recent ϵ-equilibrium algorithm for finite games has similar $\frac{1}{\epsilon}$ dependence for fixed m, but is quasipolynomial in m for fixed ϵ (LMM 2003).
- Separability, combined with the continuous nature of the space and the Lipschitz condition make computing ϵ-equilibria easier!

Conclusions

- Separable games are games which abstractly resemble finite games, enabling one to:
  - Generalize structural results (e.g. r / rank)
  - Extend computational results

Future work

- Algorithms for computing other solution concepts in separable games
  - Correlated equilibria
  - Iterated elimination of dominated strategies

Correlated equilibria (in polynomial games)

- Main difficulty - not finite-dimensional
  - Finitely many joint moments do not determine conditional distributions
- Discretization algorithms
  - A priori discretization - Converges slowly
  - Adaptive discretization - Convergence is hard to prove, seems to be fast
- SDP relaxation algorithms
  - Converge, faster than above algorithms

Iterated elimination of strictly dominated strategies (in polynomial games)

- Replace iterative procedure with fixed point characterization (Dufwenberg & Stegeman 2002; Chen et al. 2005)
- Main difficulty - This yields a second-order condition, with quantifiers ranging over sets
- Results limited to cases in which these sets can be parametrized, e.g. games with intervals for strategy sets and quasiconcave utility functions