Exchangeable Equilibria

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Outline

Complete positivity

Definition

The set of **completely positive** $n \times n$ matrices is defined by

$$\mathrm{CP}_n = \operatorname{conv}\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \middle| x_1, \ldots, x_n \geq 0 \right\}$$

Definition

A matrix is **doubly nonnegative** if it is symmetric, elementwise nonnegative, and positive semidefinite.

$$\mathrm{DNN}_n = \left\{ X \in \mathbb{R}^{n \times n} \middle| X = X', X \geq 0, X \succeq 0 \right\}$$

Properties

Useful facts

- $\mathrm{CP}_n \subseteq \mathrm{DNN}_n$ for all n
- $\mathrm{CP}_n = \mathrm{DNN}_n$ if and only if $n \leq 4$ [Diananda, Horn]
- CP_n and DNN_n are closed convex cones

Big difference

- Optimization over DNN_n is computationally tractable
- Checking membership in CP_n is NP-hard

Exchangeability

Definition

A sequence of random variables $X_1, X_2, \ldots$ is **exchangeable** if permuting finitely many of the X_k doesn't affect its distribution.

Properties

i.i.d.

$\Rightarrow$ exchangeable

$\Rightarrow$ X_j, X_k marginal is symmetric, fixed for any $j \neq k$

$\Rightarrow$ identically distributed

Exchangeable but not independent examples

- Distribution of X_1 arbitrary, all $X_k = X_1$ almost surely
- Repeated flips of a coin with a random bias

de Finetti's theorem

Theorem (corollary of de Finetti's theorem)

A matrix P is the X_i, X_j marginal of an exchangeable sequence $X_1, X_2, \ldots$ taking values in $\{1, \ldots, n\}$ if and only if $P \in \mathrm{CP}_n$ and $\sum P_{ij} = 1$.

Non-example

No symmetric distribution of X_1, X_2, X_3 has marginal $\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$.

Proof.

- With probability one $X_i \neq X_j$ for all $i \neq j$.
- By the pigeonhole principle, $X_i = X_j$ for some $i \neq j$.

Conic programming

Definition

A **conic program** is an optimization problem over vectors x:

maximize	$f(x)$	[linear objective]
subject to	$g_i(x) = b_i, \quad i = 1, \ldots, m$	[linear constraints]
	$x \in K$	[convex cone constraint]

Examples

- Linear program: $K = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ for all } i\}$
- Semidefinite program: $K = \{X \in \mathbb{R}^{n \times n} \mid X = X', X \succeq 0\}$
- $K = \mathrm{DNN}_n$ reduces to semidefinite program
- $K = \mathrm{CP}_n$ is NP-hard, even for $m = 1$

Completely positive programming

Modeling

Many hard optimization problems can be written as conic programs with $K = \mathrm{CP}_n$ (completely positive programs or **CPPs**), e.g.:

- Quadratic programs with linear and 0-1 constraints [Burer]
- Stability number of a graph [De Klerk and Pasechnik]
- Chromatic number of a graph [Gvozdenović and Laurent]

LP and SDP relaxations

- Relax exchangeability to extendibility to a symmetric distribution on $X_1, \ldots, X_k$ to get LPs
- Relax CP_n to DNN_n to get an SDP
- Hierarchy of tighter SDP relaxations for CP_n [Parrilo]

Bimatrix games

Definition

A **bimatrix game** is one with:

- Two players
- Finite sets of strategies S_1, S_2
- Simultaneous moves (strategic/normal form)
- Utilities $u_i : S_1 \times S_2 \to \mathbb{R}$

Definition

A bimatrix game is **symmetric** if $S_1 = S_2$ and all $(s_1, s_2) \in S_1 \times S_2$ satisfy $u_1(s_1, s_2) = u_2(s_2, s_1)$.

Nash and correlated equilibria

The game of chicken

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4, 4)$	$(1, 5)$
Macho	$(5, 1)$	$(0, 0)$

Nash equilibria (self-enforcing independent distributions)

- (M, W) yields utilities $(5, 1)$; (W, M) yields $(1, 5)$
- $(\frac{1}{2}W + \frac{1}{2}M, \frac{1}{2}W + \frac{1}{2}M)$ yields expected utilities $(2\frac{1}{2}, 2\frac{1}{2})$

Correlated equilibria (self-enforcing joint distributions)

- $\frac{1}{2}(W, M) + \frac{1}{2}(M, W)$ yields $(3\frac{1}{2}, 3\frac{1}{2})$
- $\frac{1}{3}(W, W) + \frac{1}{3}(W, M) + \frac{1}{3}(M, W)$ yields $(3\frac{2}{3}, 3\frac{2}{3})$, etc.

Computational complexity of equilibria

Nash equilibria

- Exist (even symmetric ones)
- Can be viewed as pairs (π_1, π_2) or as products $\pi_1 \times \pi_2$
- Set of Nash equilibria given by polynomial inequalities
- PPAD-complete to compute one Nash equilibrium
- NP-hard to optimize over Nash equilibria

Correlated equilibria

- Joint probability distribution written as a matrix
- Nash equilibria $=$ rank 1 correlated equilibria
- Set of correlated equilibria given by linear inequalities
- Polynomial time to optimize over correlated equilibria

Motivation

Computation

- Computing “approximate” Nash equilibria in some sense
- Shrink correlated equilibrium set to get “closer” to Nash
- Add convex constraints satisfied by Nash equilibria but not all correlated equilibria?
- Want “natural” constraints expressible in terms of utilities
- Still want to be able to compute efficiently

Interpretation

- Do these constraints define correlated equilibria which are “reasonable” or “fair” in some sense?

Exchangeable equilibria

Definition

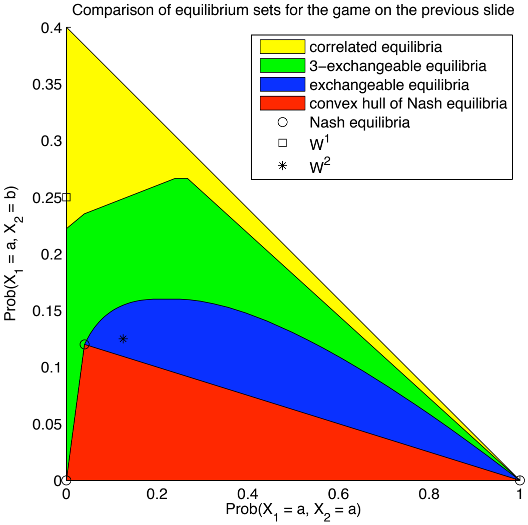
A **(symmetric) exchangeable equilibrium** is a correlated equilibrium which is completely positive.

Example

(u_1, u_2)	a	b	c
a	$(5,5)$	$(5,4)$	$(0,0)$
b	$(4,5)$	$(4,4)$	$(4,5)$
c	$(0,0)$	$(5,4)$	$(5,5)$

- conv(Nash equilibria) $\subsetneq$ Exchangeable equilibria
- Exchangeable equilibria $\subsetneq$ Correlated equilibria

Equilibria of the example



- Correlated equil. which is not exchangeable:

$$W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

- Exchangeable equil. not in conv(Nash):

$$W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

Interpretation

Definition

The *n*-**player extension** of a symmetric bimatrix game Γ is the *n*-player game in which each pair of players plays Γ and each player's utility is the sum of his utilities from these subgames.

Remark

It doesn't matter whether we allow the players to choose different strategies in each subgame.

Theorem

A matrix π is an exchangeable equilibrium of Γ if and only if it is the marginal of a symmetric correlated equilibrium of the n-player extension for all n.

Computation

Theorem

Can compute an exchangeable equilibrium in polynomial time.

Remark

This is surprising because checking complete positivity of a matrix is NP-hard. Our algorithm constructs a proof that its output is completely positive.

Approximation results

- Set of exchangeable equilibria is the feasible set of a CPP.
- Convex hull of Nash equilibria is the feasible set of a CPP.
- Get immedate LP and SDP relaxations for these.
- Can optimize over relaxations efficiently.

Concluding remarks

Complete positivity, exchangeability, conic programming

- Good tools to know
- Interesting open questions remain

Exchangeable equilibria

- Main contribution of this talk
- Intermediate between Nash and correlated equilibria
- Game theoretic interpretations
- Efficient computation

Future work

- Tighter computable relaxations
- Rounding to Nash equilibria