Convex Geometry, Extremal Measures, and Correlated Equilibria in Polynomial Games

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Outline

Game Theory – two slides only!

- Rush through definitions
- Introduce question
- Reduce to geometry
- Forget everything

The rest – slightly more relaxed

- Convex sets
- Extreme points
- Sets of probability distributions
- Finite-dimensional representations
- Example to resolve the question

Brief mention of game theory

Polynomial games

- Two players choose $x, y \in [-1, 1]$ and receive utilities

$$u_x(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x^i y^j \quad \text{and} \quad u_y(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x^i y^j$$

Nash equilibria

- Pairs $\sigma, \tau \in \Delta([-1,1])$ so $X \sim \sigma$ and $Y \sim \tau$ indep. satisfy

$$\mathbb{E}\, u_x(x, Y) \leq \mathbb{E}\, u_x(X, Y) \qquad \text{for all } x \in [-1, 1]$$
$$\mathbb{E}\, u_y(X, y) \leq \mathbb{E}\, u_y(X, Y) \qquad \text{for all } y \in [-1, 1]$$

- Independence, linearity: these only depend on σ and τ via

$$(\mathbb{E}_\sigma X, \ldots, \mathbb{E}_\sigma X^m, \mathbb{E}_\tau Y, \ldots, \mathbb{E}_\tau Y^n)$$

- Existence $\Rightarrow$ finitely supported Nash equilibria

Brief mention of game theory

Correlated equilibria

- $\mu \in \Delta([-1,1]^2)$ such that $(X, Y) \sim \mu$ makes

$$\mathbb{E}\, u_x(h(X), Y) \leq \mathbb{E}\, u_x(X, Y) \qquad \text{for all } h : [-1,1] \to [-1,1]$$
$$\mathbb{E}\, u_y(X, h(Y)) \leq \mathbb{E}\, u_y(X, Y) \qquad \text{for all } h : [-1,1] \to [-1,1]$$

- (σ, τ) is a Nash if and only if $\sigma \times \tau$ is a correlated equilibrium
- Convex relaxation of Nash equilibria
- Correlated equilibrium conditions via finite # of moments?
- Direct existence of finitely supported correlated equilibrium?

Example

- $u_x(x, y) = xy = -u_y(x, y)$
- Nash equilibria $\equiv$ pairs of zero-mean distributions
- Correlated equilibria $\equiv$ conditionally zero-mean distributions

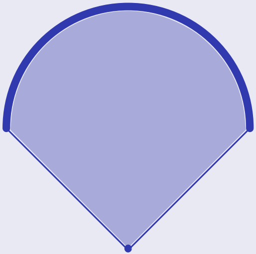
Extreme points

- K is **convex** means

$$\text{if } x, y \in K, p \in (0,1) \text{ then } px + (1-p)y \in K$$

- $z \in K$ is **extreme** (pictured in bold) means

$$\text{if } x, y \in K \text{ and } z = px + (1-p)y \text{ then } x = y = z$$



Probability and convexity

Sets of probability distributions

- $\Delta(S) = \{$(Borel) probability distributions on compact set $S\}$
- If S is finite then $\Delta(S)$ is a simplex
- Define convex combinations

$$p\mu + (1-p)\nu = \text{sample from } \mu \text{ or } \nu \text{ based on a } p\text{-biased coin}$$

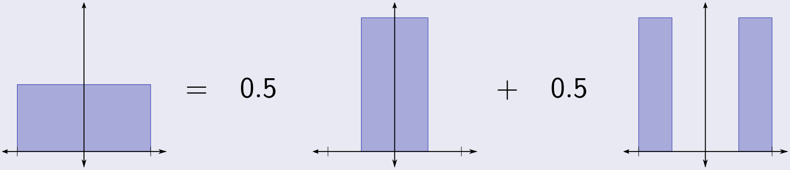
- Pointwise convex combination if μ and ν have densities
- **Support** of μ: smallest closed set C with $\mu(C) = 1$
    - Dirac distributions δ_x have $\text{supp}(\delta_x) = \{x\}$
    - $\text{supp}(p\mu + (1-p)\nu) = \text{supp}(\mu) \cup \text{supp}(\nu)$

Proposition

If $K \subseteq \Delta(S)$ has a unique measure with support contained in a set C, this measure is an extreme point of K.

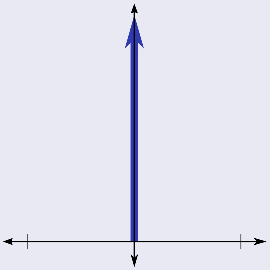
Zero-mean distributions

- $\{\mu \in \Delta([-1,1]) \mid \mathbb{E}_\mu X = 0\}$ is convex
- Non-extreme point

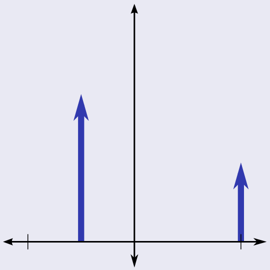


Zero-mean distributions

- $\{\mu \in \Delta([-1,1]) \mid \mathbb{E}_\mu X = 0\}$ is convex
- Extreme points all have support of size ≤ 2



δ_0



$\frac{y}{x+y}\delta_{-x} + \frac{x}{x+y}\delta_y$

for $x, y \in (0,1]$

Generalization: Finite-Dimensional Representability

Representability by moments

- Formalize describability by finitely many parameters
- A set R is **representable by (generalized) moments** if

$$R = \{\mu \in \Delta(S) \mid (\mathbb{E}_\mu f_1(X), \ldots, \mathbb{E}_\mu f_n(X)) \in Q\}$$

  (f_i bounded Borel measurable)
- Move questions about R into finite dimensions

Theorem

Extreme points of R have support of size at most $n + 1$.

Proof of Theorem

Theorem

Extreme points of R have support of size at most $n+1$ when

$$R = \{\mu \in \Delta(S) \mid (\mathbb{E}_\mu f_1(X), \ldots, \mathbb{E}_\mu f_n(X)) \in Y\}\,.$$

Proof.

- Let $\mu \in R$ have larger support so $S = \bigsqcup_{j=1}^{n+2} B_j$ with $\mu(B_j) > 0$
- For $c \in \mathbb{R}^{n+2}_{\geq 0}$ and $A \subseteq S$ define $\nu_c(A) := \sum_j c_j \mu(A \cap B_j)$
- $\nu_c \in R$ whenever c satisfies:
  - $\nu_c(S) := \sum_j c_j \mu(B_j) = 1$
  - $\mathbb{E}_\mu f_i(X) = \mathbb{E}_{\nu_c} f_i(X) := \sum_j c_j \mathbb{E}_\mu f_i(X) \mathbf{1}_{B_j}(X)$ for $i = 1, \ldots, n$
- $n+1$ linear equations in $n+2$ variables c_j
- $(1, \ldots, 1)$ in interior of line segment of feasible c
- $c \mapsto \nu_c$ injective, linear
- $\mu := \nu_{(1,\ldots,1)}$ in interior of line segment in R $\square$

A non-example

Conditional zero-mean distributions

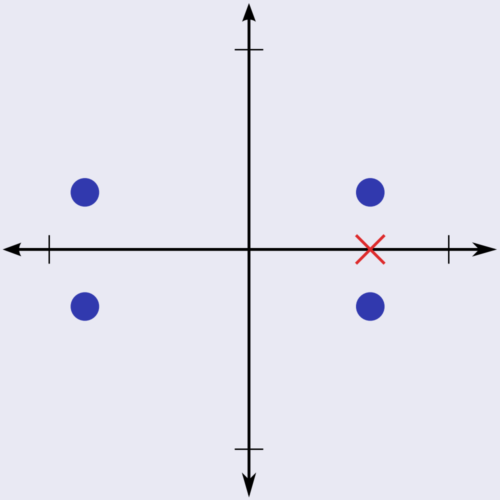
- Let $Z \subset \Delta([-1,1]^2)$ be the distributions with zero mean conditioned on any horizontal or vertical line
- Z is convex
- This looks like infinitely many linear constraints
- Z "should not" be representable by moments
- Proof: extreme points with arbitrarily large support

Constructing elements of Z

- Three steps:
  1. Take a distribution assigning equal mass on both sides of the axis to each line
  2. Weight by density $|xy|^{-1}$
  3. Renormalize
- This construction lets us focus on support only

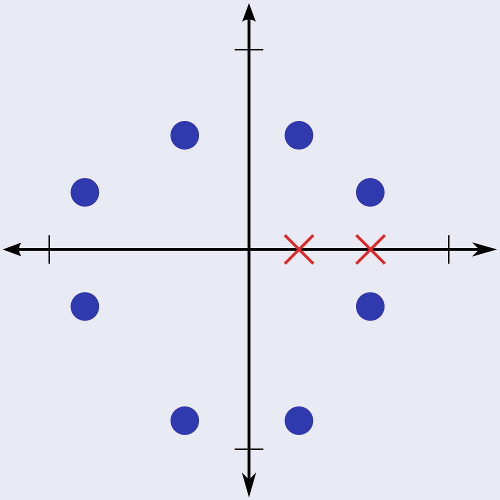
Extreme conditional zero-mean distributions

Example #1



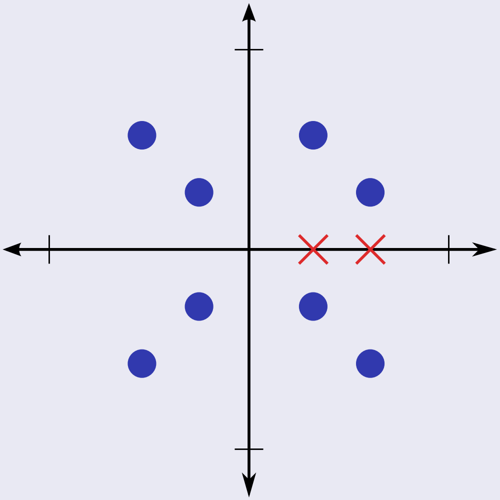
Extreme conditional zero-mean distributions

Non-example



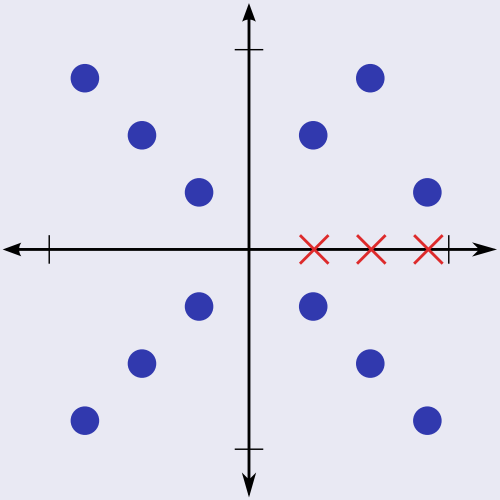
Extreme conditional zero-mean distributions

Example #2



Extreme conditional zero-mean distributions

Example #3



Extreme conditional zero-mean distributions

Theorem

The set of conditional zero-mean distributions is not representable by moments.

Proof.

- Select a finite set $T \subset (0, 1]$
- Select a map $g : T \to T$
- As t ranges over T place equal mass at points: $(t, g(t)), (-t, t), (-t, -t), (t, -t)$
- Weight by $|xy|^{-1}$ and normalize
- If g is a permutation result will by conditionally zero-mean
- If g consists of a single cycle result will be extreme
- Support size is $4|T|$ □

Extreme distributions with infinite support

Generalizing the finite construction

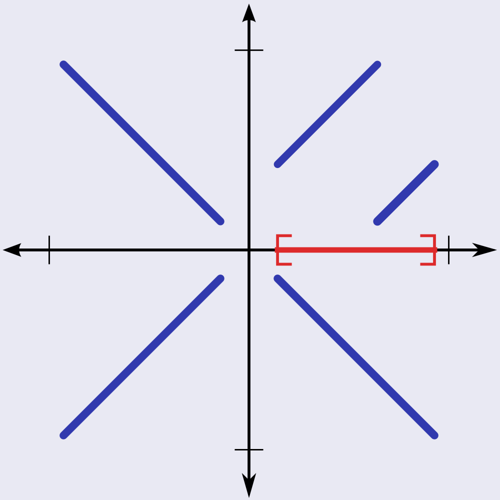
	Finite case	General case
T	finite subset of $(0,1]$	subset of $[\epsilon, 1]$ endowed with measure λ
$g : T \to T$	permutation	measure-preserving: $A \subseteq T$: $\lambda\left(g^{-1}(A)\right) = \lambda(A)$
	single cycle	ergodic: if $A = g^{-1}(A)$ then $\lambda(A) = 0$ or $\lambda(A^c) = 0$

An ergodic transformation

- $T = [0,1)$ with $\lambda =$ uniform distribution
- $g_\alpha(x) = x + \alpha \mod 1 = x + \alpha - \lfloor x + \alpha \rfloor$ measure-preserving
- g_α ergodic $\Leftrightarrow$ α irrational

Extreme distributions with infinite support

Example #4



Wrap-up

Conclusions

- Nash equilibria representable by moments, known since 1950's
- Outer approximations of correlated equilibria by moments using SDP (master's thesis)
- Correlated equilibria not representable by moments
  - Odd – for finite games correlated equilibria are "simpler"

Future work

- Explicit inner approximation of correlated equilibria which is representable by moments
- Provably efficient algorithms for computing correlated equilibria of polynomial games