Computation and Characterization of Equilibria in Polynomial Games

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Goals

- Characterize equilibria of infinite games
- Compute equilibria of infinite games

Outline

- What is a polynomial game?
- Computing Nash eq. of zero-sum polynomial games
- Correlated equilibria in finite games
- Defining correlated equilibria in polynomial games
- Computing correlated eq. of polynomial games

Polynomial games

- Definition [Drescher, Karlin, Shapley 1950s]
  - n players, strategic form
  - Set of strategies $S_i = [-1, 1] \subset \mathbb{R}$ for each player
  - Utilities $u_i : [-1, 1]^n \to \mathbb{R}$ are polynomials
- Properties
  - Finitely supported equilibria
  - Can compute Nash equilibria in zero-sum case [Parrilo 2006]
  - Can compute correlated equilibria in general case [SPO 2007]

How can we solve zero-sum polynomial games?

	Finite Games	Poly. Games
Nash eq. (zero-sum)	LP	??
Correlated equilibria	LP	??

Equilibria of zero-sum poly. games

- $u_x(x,y) = \sum_{j,k} a_{jk} x^j y^k = -u_y(x,y)$
- A minimax strategy τ for player y solves

$$\begin{array}{lll} \min & \beta & \\ \text{s.t.} & \tau & \text{is a prob. measure on } [-1,1] \\ & \tau_k = \int_{-1}^{1} y^k d\tau & \text{for } k \leq [y\text{-degree of } u_x] \\ & \sum_{j,k} a_{jk} x^j \tau_k \leq \beta & \text{for all } x \in [-1,1] \end{array}$$

- Must describe polynomials nonnegative on $[-1,1]$ as well as moments of measures on $[-1,1]$

Sums of squares + SDP

- A polynomial $p(x)$ is ≥ 0 for all $x \in \mathbb{R}$ iff it is a sum of squares of polynomials q_k (SOS)

$$p(x) = \sum q_k^2(x) \text{ for all } x \in \mathbb{R}$$

- A polynomial $p(x)$ is ≥ 0 for all $x \in [-1, 1]$ iff there are SOS polynomials $s(x), t(x)$ such that

$$p(x) = s(x) + (1 - x^2)t(x)$$

- Coefficients of SOS polynomials can be described in a semidefinite program (SDP)

Moments of measures + SDP

- For all polynomials p, must have

$$\int p^2(x)d\tau(x) \geq 0 \quad \text{and} \quad \int (1-x^2)p^2(x)d\tau(x) \geq 0$$

- $\tau_0, \ldots, \tau_{2m}$ are the moments of a measure τ on $[-1,1]$ (i.e. $\tau_k = \int y^k d\tau$) iff

$$\begin{bmatrix} \tau_0 & \tau_1 & \tau_2 \\ \tau_1 & \tau_2 & \tau_3 \\ \tau_2 & \tau_3 & \tau_4 \end{bmatrix} \succeq 0, \begin{bmatrix} \tau_0 - \tau_2 & \tau_1 - \tau_3 \\ \tau_1 - \tau_3 & \tau_2 - \tau_4 \end{bmatrix} \succeq 0 \quad (m = 2 \text{ case})$$

- Moments of measures on $[-1,1]$ can be described in a semidefinite program

Example

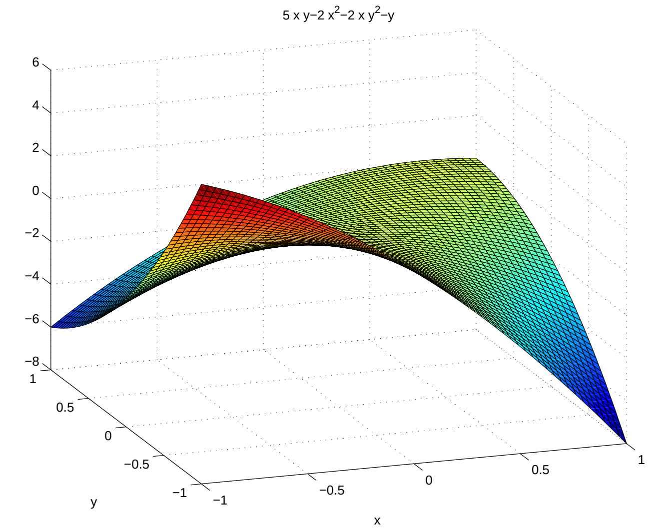
Payoffs:

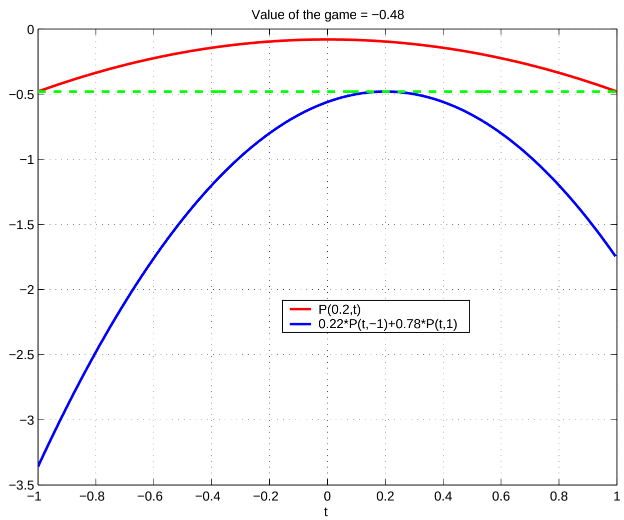
$$u_x(x,y) = -u_y(x,y)$$
$$= 5xy - 2x^2 - 2xy^2 - y$$

Value: -0.48

Optimal mixed strategies:

- P1 always picks $x = 0.2$
- P2 plays $y = 1$ with probability 0.78, and $y = -1$ with probability 0.22.





Chicken and correlated equilibria

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4, 4)$	$(1, 5)$
Macho	$(5, 1)$	$(0, 0)$

- Nash equilibria (self-enforcing independent distrib.)
  - (M, W) yields utilities $(5, 1)$; (W, M) yields $(1, 5)$
  - $\left(\frac{1}{2}W + \frac{1}{2}M, \frac{1}{2}W + \frac{1}{2}M\right)$ yields expected utility $(2\frac{1}{2}, 2\frac{1}{2})$
- Correlated equilibria (self-enforcing joint distrib.)
  - e.g. $\frac{1}{2}(W, M) + \frac{1}{2}(M, W)$ yields $(3\frac{1}{2}, 3\frac{1}{2})$
  - $\frac{1}{3}(W, W) + \frac{1}{3}(W, M) + \frac{1}{3}(M, W)$ yields $(3\frac{2}{3}, 3\frac{2}{3})$

Correlated equilibria in finite games

- $u_i(t_i, s_{-i}) - u_i(s)$ is change in player i's utility when strategy t_i replaces s_i in $s = (s_1, \ldots, s_n)$
- A prob. distribution π is a **correlated equilibrium** if

$$\sum_{\{s:\ s_i = r_i\}} [u_i(t_i, s_{-i}) - u_i(s)]\pi(s) \leq 0$$

  for all players i and all strategies $r_i, t_i \in S_i$
- Linear ineq. in variables $\pi(s) \Rightarrow$ linear program

How can we compute correlated equilibria in polynomial games?

	Finite Games	Poly. Games
Nash eq. (zero-sum)	LP	SDP
Correlated equilibria	LP	??

Computing CE in poly. games: Naive attempt (LP)

- Intended as a benchmark to judge other techniques
- Ignore polynomial structure
- Restrict strategies to finite sets $\tilde{S}_i \subset S_i$
- Compute exact correlated eq. of approximate game
- This is a sequence of LPs which converges (slowly!) to the set of correlated equilibria as the discretization gets finer.

Defining CE in infinite games

- Definition in literature:

$$\int [u_i(\zeta_i(s_i), s_{-i}) - u_i(s)] d\pi \leq 0$$

  for all i and all (measurable) departure functions ζ_i

- Equivalent to above def. if strategy sets are finite
- Quantifier ranging over large set of functions
- Is there a characterization which looks more like the finite case and doesn't have this problem?

An instructive failed attempt

- The following "characterization" fails:

$$\int_{\{s:\ s_i = r_i\}} [u_i(t_i, s_{-i}) - u_i(s)]\pi(s) \leq 0$$

for all players i and all strategies $r_i, t_i \in S_i$

- Holds for *any* continuous probability distribution π
- This condition is much weaker than correlated equilibrium

New equivalent definitions of CE

- These conditions are equivalent to the departure function definition
- For all i, all $t_i \in S_i$, and all $-1 \leq a_i \leq b_i \leq 1$,

$$\int_{\{a_i \leq s_i \leq b_i\}} [u_i(t_i, s_{-i}) - u_i(s)] d\pi \leq 0$$

- For all i, all $t_i \in S_i$, and all polynomials p,

$$\int [u_i(t_i, s_{-i}) - u_i(s)] p^2(s_i) d\pi \leq 0$$

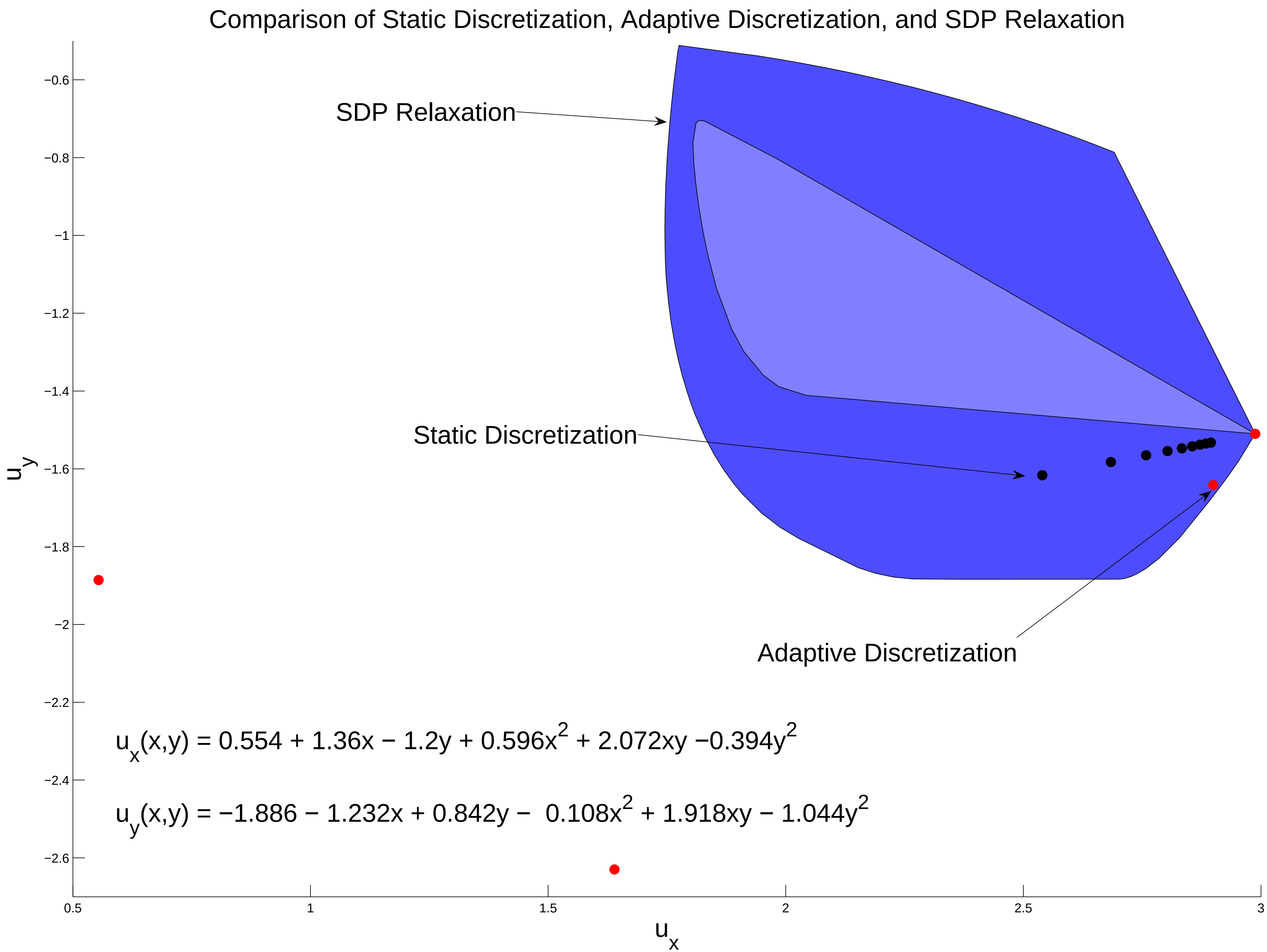
Computing CE in poly. games (SDP)

- No discretization
- Sequence of SDP constraints to describe moments of measures π on $[-1,1]^n$
- For fixed d, use SDP to express

$$\int [u_i(t_i, s_{-i}) - u_i(s)] p^2(s_i) d\pi \leq 0$$

for all i, $t_i \in [-1,1]$, and polys. p of degree $\leq d$

- Get a nested sequence of SDPs converging to the set of correlated equilibria!



Comparison of Static Discretization, Adaptive Discretization, and SDP Relaxation
SDP Relaxation
Static Discretization
Adaptive Discretization
$u_x(x,y) = 0.554 + 1.36x - 1.2y + 0.596x^2 + 2.072xy - 0.394y^2$
$u_y(x,y) = -1.886 - 1.232x + 0.842y - 0.108x^2 + 1.918xy - 1.044y^2$
u_y
u_x
−0.6
−0.8
−1
−1.2
−1.4
−1.6
−1.8
−2
−2.2
−2.4
−2.6
0.5
1
1.5
2
2.5
3

Conclusions

- New characterizations of correlated equilibria in infinite games
- First algorithms for computing correlated equilibria in any class of infinite games

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